Mathematical modeling of electromagnetic wave propagation in heterogeneous lossy coaxial cables with variable cross section

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Abstract

In this work, we focus on the time-domain simulation of the propagation of electromagnetic waves in non-homogeneous lossy coaxial cables. The full 3D Maxwell equations, that described the propagation of current and electric potential in such cables, are classically not tackled directly, but instead a 1D scalar model known as the telegraphist's model is used. We aim at justifying, by means of asymptotic analysis, a time-domain "homogenized" telegraphist's model. This model, which includes a non-local in time operator, is obtained via asymptotic analysis, for a lossy coaxial cable whose cross-section is not homogeneous.

Keywords: Coaxial cables, telegraphist's model, asymptotic analysis, Maxwell equations
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1. Introduction and motivation. Objectives of the paper.

Although of a rather fundamental nature, the present work has been motivated by an industrial application in the framework of a collaboration between the Project team POEMS of INRIA and the laboratory LIST of CEA Saclay. The general objective of this collaboration has been the numerical simulation of piezo-electric sensors used for non destructive testing

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with ultrasonic waves. This led us to develop in [7] a specific finite element approach. One important aspect of the work is the modeling of the electromagnetic supply process : an electric field is generated by an electric generator and transmitted to the sensor via a coaxial electric cable: see Figure 1 (left and middle pictures).

The presence of the coaxial cable was neglected in [7]. However, as pointed

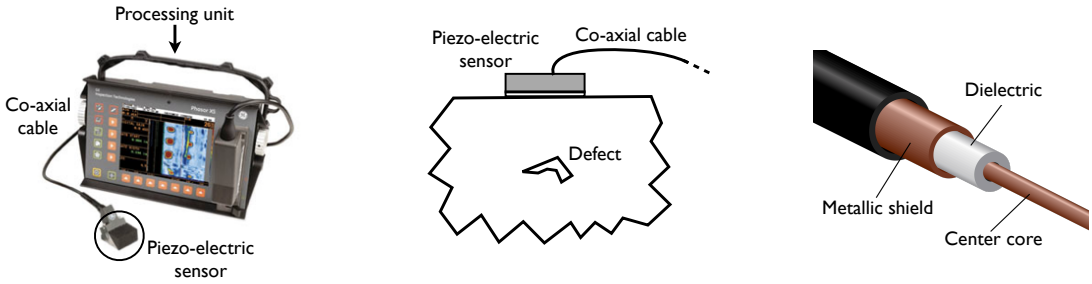


Figure 1: Left and center: Piezoelectric sensors, middle: coaxial cable.

out in [12], it can have an influence on the signal that is transmitted to the sensor as well on the recorded signal, especially if it is quite long, which can be the case in airplanes for instance. A cable being a structure whose transverse dimensions are much smaller than the longitudinal one, one would like to use a simplified 1D model. In such a situation, electrical engineers use the so-called telegraphist's equations where the electric unknowns are reduced to an electric potential $V(x_3, t)$ and an electric current $I(x_3, t)$, where x_3 denotes the abscissa along the cable and t is time. The telegraphist's equations are generally used (see [13] for instance) in the case of a "perfect" coaxial cable: this corresponds to the case (illustrated on Figure 1, right picture) where the domain of propagation (namely the insulator) is a cylindrical annulus, filled with a homogeneous dielectric (possibly conducting) medium, which surrounds a perfectly conducting cylinder (the center core) and is surrounded by another cylindrical perfectly conducting medium (the metallic shield). The equations governing $V(x_3, t)$ and electric current $I(x_3, t)$ are

$$\begin{cases} C\,\dfrac{\partial V}{\partial t} + G\,V - \dfrac{\partial I}{\partial x_3} = 0, \\ L\,\dfrac{\partial I}{\partial t} + R\,I - \dfrac{\partial V}{\partial x_3} = 0, \end{cases} \tag{1}$$

where the capacitance C, the inductance L, the conductance G and the resistance R can be expressed in terms of the interior and exterior radii r_1 and r_2 of the annulus and of the physical characteristics of the dielectric material, in particular the electric permittivity ε and the magnetic permeability μ. For instance, the coefficients C and L are given by

$$C = 2\pi \left(\ln \frac{r_2}{r_1} \right)^{-1} \varepsilon \ , \quad L = \frac{1}{2\pi} \left(\ln \frac{r_2}{r_1} \right) \mu \ .$$

This telegraphist's equation can be obtained from a low frequency analysis of the equations for the fundamental mode of the cable (see [13] or [5]).

A natural question is: what does the approximate model (1) become in a more general and realistic situation namely when each cross section is heterogeneous (physical characteristics of the medium vary in the cross section) and the cable has a variable cross section and material properties along the cable (translation invariance and modal representations are lost).

This is the question (perhaps considered as completely understood by electrical engineers, see for instance [6]) that has motivated us to look at Maxwell's equations in such "generalized" coaxial cables. Our objective was to derive a 1D effective model (let's say a generalized telegraphist's equation) via an asymptotic analysis with respect to the (small) transverse dimensions of the cable, in the spirit of what has been done in mechanics for the justification of plate, shell or beam models (see [2], [3], [14]) but has apparently not been done in electromagnetism. Above the intrinsic interest of such a study, we think that a profound understanding of the passage 3D-1D is a necessary preliminary step towards understanding the junction of several cables (see [9] or [10] for the acoustic case) or the connexion of a cable with another 2D or 3D structure (which we encounter, for instance, in the case of piezo-electric devices for the connexion between coaxial cables and thin electrodes).

In this first paper, we have chosen to restrict ourselves to formal asymptotic analysis and to the study in the time domain, of the effective 1D model. In section 2, we describe the geometry of a coaxial cable and the family of problems, depending of a small parameter $\delta > 0$, of which we make an asymptotic analysis, first in the frequency domain. In section 3, we explain the structure of the leading terms in the formal expansions. This leads us to the generalized telegraphist's equations (48) in the frequency domain, in which

the effective (or homogenized) coefficients, which generalize (C, L, R, G) in (1) are given by formulas (37)-(40). Section 4 concerns preparatory material before coming back to time domain. We establish in section 5 the generalized telegraphist's equations: the conjugated presence of electromagnetic losses and heterogeneity of the cross-section induces memory effects via time convolution terms in the equations.

2. The mathematical model for coaxial cables

To derive our effective model (see section 3), we consider a family of problems posed in domains, called cables, that depend on a small geometric parameter $\delta > 0$ representing the transverse dimensions of the cables. Of course, a given cable corresponds to a given value of δ but the effective model will be constructed by an asymptotic analysis in δ. These domains will be obtained by scaling with respect to appropriate transverse variables of a "normalized" reference cable. This normalized infinite Ω is defined as the union of its cross-sections (see Figure 2, left picture)

$$\Omega = \underset{z\in\mathbb{R}}{\cup} S_z$$

with S_z a connected bounded and Lipschitz domain included in the plane $x_3 = z$. In addition, we assume that for all $z \in \mathbb{R}$, S_z includes the origin: $\mathbf{0} \in S_z$, which, in some sense, means that we consider straight cables. We also assume that $\partial\Omega$ is a smooth enough (Lipschitz) manifold along which the normal unit vector $\mathbf{n}$ satisfies

Hypothesis 2.1.

$$\exists\, \nu \in \,]\,0,1\,[\quad \textit{such that} \quad |\,\mathbf{n}\cdot\mathbf{e}_3\,| \le \nu < 1, \quad \textit{a. e. on } \partial\Omega.$$

Finally, we shall concentrate on the case of coaxial cables, namely cables whose cross-sections are not simply connected. For simplicity, we shall restrict ourselves to the case where each cross section contains only one hole (see Figure 2, middle picture)

Hypothesis 2.2. *For all $z \in \mathbb{R}$, $S_z = \mathcal{O}_z \setminus \overline{T_z}$, where $\mathcal{O}_z$ and T_z (the "hole", occupied by a perfect conductor) are simply connected, Lipschitz, open sets included in the plane $x_3 = z$ with $\overline{T_z} \subset\subset \mathcal{O}_z$, and meas $T_z \ge a_* > 0, \forall\, z \in \mathbb{R}$. In this case, the boundary of S_z has two connected components, the exterior (∂S_z^+) and the interior (∂S_z^-) one $\partial S_z^+ \equiv \partial\mathcal{O}_z, \quad \partial S_z^- \equiv \partial T_z$.*

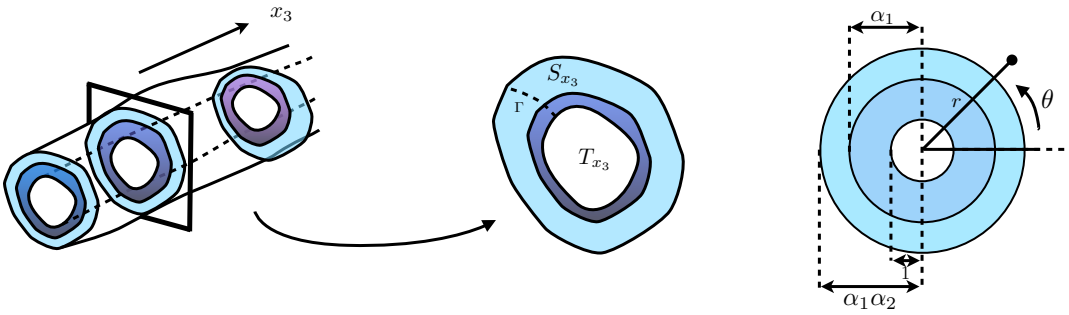


Figure 2: Left: domain of interest, center: slice of the domain, right: specific situation consider in section 5.3.

We consider a family of (thin) domains parametrized by a strictly positive scalar δ and defined by $\Omega^\delta = \mathcal{G}_\delta\big(\Omega\big)$, where $\mathcal{G}_\delta : (x_1, x_2, x_3) \longrightarrow (\delta x_1, \delta x_2, x_3)$.

Remark 2.1. Abuse of notation. *In the sequel, we shall use systematically the passage from Ω to Ω^δ via the change of variable $\mathcal{G}_\delta$. To avoid heavy notation, we shall use the same letters $(\mathbf{x}, ...)$ for the coordinates in Ω or Ω^δ, the meaning being fixed by the context. In the same way, we shall use the same notation for the differential operators in both variables (**grad**, **rot**, ...).*

Example 2.1. *Consider the case where $\mathcal{O}_z$ and T_z are star-like domains with respect to the origin. If (r, θ) are the polar coordinates in the (x_1, x_2)-plane, there exist two Lipschitz continuous functions $0 < \rho_-(x_3, \theta) < \rho_+(x_3, \theta)$, 2π periodic in θ, such that*

$$\Omega = \big\{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \;\; / \;\; \rho_-(x_3, \theta) < r < \rho_+(x_3, \theta) \,\big\}\,.$$

On $\partial\Omega^\delta$, it is useful to use a non unit outward normal vector $\mathbf{n}^\delta$ which has a unitary projection on the plane (x_1, x_2). With some elementary geometry, $\mathbf{n}^\delta$ can be considered of the form

$$\begin{aligned} &\text{a. e. } \mathcal{G}_\delta(\mathbf{x}) \in \partial\Omega^\delta, \quad \mathbf{n}^\delta \circ \mathcal{G}_\delta(\mathbf{x}) = \big(n_1(\mathbf{x}), n_2(\mathbf{x}), 0\big)^t + \delta\; g(\mathbf{x})\; \mathbf{e}_3, \\ &\mathbf{n}_T = (n_1, n_2)^t : \partial\Omega \to \mathbb{R}^2,\; |\mathbf{n}_T| = 1, \quad g : \partial\Omega \to \mathbb{R}. \end{aligned} \tag{2}$$

Remark 2.2. *In the case of example 2.1,along each part of the boundary*

$$n_\pm(r, \theta) = \left(\cos\theta - \frac{\partial\rho_\pm}{\partial\theta}\frac{\sin\theta}{\rho_\pm}\,,\, \sin\theta + \frac{\partial\rho_\pm}{\partial\theta}\frac{\cos\theta}{\rho_\pm} \right)^t, \quad g_\pm(r, \theta) = -\frac{\partial\rho_\pm}{\partial x_3}.$$

We shall assume that the permittivity $\varepsilon^\delta(\mathbf{x})$, the permeability $\mu^\delta(\mathbf{x})$, the electric and magnetic conductivity $\sigma_e^\delta(\mathbf{x})$ and $\sigma_m^\delta(\mathbf{x})$ are obtained by a scaling in $\mathbf{x}_T = (x_1, x_2)$, of fixed distributions over the domain Ω:

$$\varepsilon^\delta = \varepsilon \circ \mathcal{G}_\delta^{-1}, \quad \mu^\delta = \mu \circ \mathcal{G}_\delta^{-1}, \quad \sigma_e^\delta = \sigma_e \circ \mathcal{G}_\delta^{-1}, \quad \sigma_m^\delta = \sigma_m \circ \mathcal{G}_\delta^{-1},$$

where $(\varepsilon, \mu, \sigma_e, \sigma_m)$ satisfy the usual boundedness and positivity properties:

$$\begin{aligned} &0 < \varepsilon_- \le \varepsilon(\mathbf{x}) \le \varepsilon_+, \qquad 0 < \mu_- \le \mu(\mathbf{x}) \le \mu_+ , \qquad \mathbf{x} \in \Omega, \\ &0 \le \sigma_{e,-} \le \sigma_e(\mathbf{x}) \le \sigma_{e,+}, \quad 0 \le \sigma_{m,-} \le \sigma_m(\mathbf{x}) \le \sigma_{m,+} , \quad \mathbf{x} \in \Omega. \end{aligned} \tag{3}$$

Example 2.2. Perfectly cylindrical cables. *These correspond to the case where the problem is invariant under translation along the x_3 axis. This means that Ω is a cylindrical domain, $\Omega = S \times \mathbb{R}$, and that all coefficients are independent of x_3 : $\varepsilon(x_1, x_2, x_3) = \varepsilon(x_1, x_2)$, $\mu(x_1, x_2, x_3) = \mu(x_1, x_2)$, ...*

Example 2.3. Cables with homogeneous cross sections. *These correspond to the situation where all coefficients only depend on the x_3 variable : $\varepsilon(x_1, x_2, x_3) = \varepsilon(x_3)$, $\mu(x_1, x_2, x_3) = \mu(x_3)$, ...*

The equations that govern electromagnetic waves are Maxwell's equations

$$\begin{cases} \varepsilon^\delta \dfrac{\partial}{\partial t} E^\delta + \sigma_e^\delta E^\delta - \boldsymbol{\nabla} \times H^\delta = \mathbf{j}^\delta, & \text{in } \Omega^\delta \times \mathbb{R}^+, \\ \mu^\delta \dfrac{\partial}{\partial t} H^\delta + \sigma_m^\delta H^\delta + \boldsymbol{\nabla} \times E^\delta = \mathbf{0}, & \text{in } \Omega^\delta \times \mathbb{R}^+, \end{cases} \tag{4}$$

where $\boldsymbol{\nabla}\times$ is the 3D curl operator. We use perfectly conducting boundary conditions and zero initial conditions:

$$E^\delta \times \mathbf{n}^\delta = \mathbf{0} \quad \text{on } \partial\Omega^\delta \times \mathbb{R}^+, \tag{5}$$

$$E^\delta(\mathbf{x}, 0) = \mathbf{0}, \quad H^\delta(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{x} \in \Omega^\delta. \tag{6}$$

For the sake of simplicity, we shall assume that the source term, namely the current density $\mathbf{j}^\delta$ is such that

$$\mathbf{j}^\delta = \mathbf{j} \circ \mathcal{G}_\delta^{-1}, \quad \mathbf{j} = (\, \mathbf{j}_T, 0\,)^t, \quad \mathbf{j}_T = (\, j_1, j_2\,)^t, \quad \frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} = 0, \quad \mathbf{x} \in \Omega. \tag{7}$$

It appears judicious to introduce the tangential components

$$E_T^\delta = (E_1^\delta, E_2^\delta)^t, \quad H_T^\delta = (H_1^\delta, H_2^\delta)^t,$$

of the electric and magnetic fields as well as the longitudinal components of these fields: E_3^δ and H_3^δ. We can rewrite the equations (4) with these new unknowns. To do so, we shall use, for all scalar functions u and 2D vector fields $\mathbf{v}$ with two components v_1 and v_2, the following notations:

$$\mathbf{grad}\, u \equiv \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\Big)^t, \quad \mathbf{rot}\; u \equiv \Big(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}\Big)^t,$$
$$\text{div}\; \mathbf{v} \equiv \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}, \quad \text{rot}\; \mathbf{v} \equiv \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2},$$

and, for any $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$ (possibly complex valued),

$$\mathbf{v} \cdot \mathbf{u} \equiv v_1\, u_1 + v_2\, u_2, \qquad \mathbf{v} \times \mathbf{u} \equiv v_1\, u_2 - v_2\, u_1, \qquad \mathbf{e}_3 \times \mathbf{v} \equiv (-v_2, v_1)^t,$$

The reader will observe in particular that, for any scalar functions u and $\widetilde{u}$,

$$\mathbf{rot}\; u \cdot \mathbf{grad}\, \widetilde{u} = -\,\mathbf{rot}\; \widetilde{u} \cdot \mathbf{grad}\, u \tag{8}$$

$$\mathbf{e}_3 \times \mathbf{grad}\, u = -\mathbf{rot}\; u, \qquad \mathbf{grad}\, u \times \mathbf{v} = -\,\mathbf{v}\; \cdot \mathbf{rot}\, u\; . \tag{9}$$

We shall also use the Green's formula (σ_{x_3} is the lineic measure along ∂S_{x_3})

$$\big(\mathbf{rot}\; u, \mathbf{v}\big)_{L^2(S_{x_3})} \;=\; \big(u, \text{rot}\; \mathbf{v}\big)_{L^2(S_{x_3})} + \int_{\partial S_{x_3}} u\, \overline{(\mathbf{v} \times \mathbf{n}_T)}\, d\sigma_{x_3}. \tag{10}$$

Remark 2.3. *Formula (10) is valid for $(u, \mathbf{v})$ in $H^1(S_{x_3}) \times H(rot, S_{x_3})$ and the boundary integral must be (as usual) understood as a duality product.*

The equations (4) can be rewritten in $\Omega^\delta \times \mathbb{R}^+$ as

$$\begin{cases} \varepsilon^\delta\, \dfrac{\partial}{\partial t} E_T^\delta + \sigma_e^\delta\, E_T^\delta - \dfrac{\partial}{\partial x_3}\, \mathbf{e}_3 \times H_T^\delta - \mathbf{rot}\; H_3^\delta = \mathbf{j}_T \circ \mathcal{G}_\delta^{-1}, \\ \varepsilon^\delta\, \dfrac{\partial}{\partial t} E_3^\delta + \sigma_e^\delta\, E_3^\delta - \text{rot}\; H_T^\delta = 0, \\ \mu^\delta\, \dfrac{\partial}{\partial t} H_T^\delta + \sigma_m^\delta\, H_T^\delta + \dfrac{\partial}{\partial x_3}\, \mathbf{e}_3 \times E_T^\delta + \mathbf{rot}\; E_3^\delta = \mathbf{0}, \\ \mu^\delta\, \dfrac{\partial}{\partial t} H_3^\delta + \sigma_m^\delta\, H_3^\delta + \text{rot}\; E_T^\delta = 0. \end{cases} \tag{11}$$

Moreover, using (2), the boundary conditions (5) on $\partial\Omega^\delta \times \mathbb{R}^+$ become

$$E_T^\delta \times (\mathbf{n}_T \circ \mathcal{G}_\delta^{-1}) = \mathbf{0}, \quad E_3^\delta\, (\mathbf{n}_T \circ \mathcal{G}_\delta^{-1}) - \delta\, (g \circ \mathcal{G}_\delta^{-1})\, E_T^\delta = 0, \tag{12}$$

Taking the divergence of the equations (11) and using (7), we get a set of "hidden" divergence equations in $\Omega^\delta \times \mathbb{R}^+$:

$$\begin{aligned} &\mathrm{div}\,\Big(\varepsilon^\delta \frac{\partial}{\partial t} E_T^\delta + \sigma_e^\delta\, E_T^\delta\Big) \;+\; \frac{\partial}{\partial x_3}\Big(\varepsilon^\delta \frac{\partial}{\partial t} E_3^\delta + \sigma_e^\delta\, E_3^\delta\Big) = 0, \\ &\mathrm{div}\,\Big(\mu^\delta \frac{\partial}{\partial t} H_T^\delta + \sigma_m^\delta\, H_T^\delta\Big) \;+\; \frac{\partial}{\partial x_3}\Big(\mu^\delta \frac{\partial}{\partial t} H_3^\delta + \sigma_m^\delta\, H_3^\delta\Big) = 0. \end{aligned} \tag{13}$$

Moreover, from (14), (13) and (12) it is classical to derive an additional boundary condition for the magnetic field (see for instance [5]):

$$H_T^\delta \cdot (\mathbf{n}_T \circ \mathcal{G}_\delta^{-1}) + \delta\,(g \circ \mathcal{G}_\delta^{-1})\, H_3^\delta = 0, \quad \text{on } \partial\Omega^\delta \times \mathbb{R}^+.$$

We want to describe the behavior of (E^δ, H^δ) when δ tends to 0. For this, it is useful to apply a change of variables in order to work in a fixed geometry. Doing so, the parameter δ will appear in the coefficients of the governing equations. We introduce the rescaled fields $(\widetilde{E}_T^\delta, \widetilde{E}_3^\delta, \widetilde{H}_T^\delta, \widetilde{H}_3^\delta)$ defined by:

$$E_T^\delta = \widetilde{E}_T^\delta \circ \mathcal{G}_\delta^{-1}, \quad E_3^\delta = \widetilde{E}_3^\delta \circ \mathcal{G}_\delta^{-1}, \quad H_T^\delta = \widetilde{H}_T^\delta \circ \mathcal{G}_\delta^{-1}, \quad H_3^\delta = \widetilde{H}_3^\delta \circ \mathcal{G}_\delta^{-1}.$$

We can write from (11, 12) the equations for $(\widetilde{E}_T^\delta, \widetilde{E}_3^\delta, \widetilde{H}_T^\delta, \widetilde{H}_3^\delta)$ in the reference domain Ω using the abuse of notation explained in remark 2.1 (the δ^{-1} terms simply come from derivatives in $\mathbf{x}_T = (x_1, x_2)$):

$$\begin{cases} \varepsilon \dfrac{\partial}{\partial t} \widetilde{E}_T^\delta + \sigma_e\, \widetilde{E}_T^\delta, - \dfrac{\partial}{\partial x_3}\, \mathbf{e}_3 \times \widetilde{H}_T^\delta - \delta^{-1}\, \mathbf{rot}\, \widetilde{H}_3^\delta = \mathbf{j}_T, & \text{in } \Omega \times \mathbb{R}^+, \\ \varepsilon \dfrac{\partial}{\partial t} \widetilde{E}_3^\delta + \sigma_e\, \widetilde{E}_3^\delta - \delta^{-1}\, \mathrm{rot}\, \widetilde{H}_T^\delta = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \mu \dfrac{\partial}{\partial t} \widetilde{H}_T^\delta + \sigma_m\, \widetilde{H}_T^\delta + \dfrac{\partial}{\partial x_3}\, \mathbf{e}_3 \times \widetilde{E}_T^\delta + \delta^{-1}\, \mathbf{rot}\, \widetilde{E}_3 = \mathbf{0}, & \text{in } \Omega \times \mathbb{R}^+, \\ \mu \dfrac{\partial}{\partial t} \widetilde{H}_3^\delta + \sigma_m\, \widetilde{H}_3^\delta + \delta^{-1}\, \mathrm{rot}\, \widetilde{E}_T^\delta = 0, & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{14}$$

while from the divergence equations (13), we have, in $\Omega \times \mathbb{R}^+$

$$\begin{cases} \delta^{-1}\, \mathrm{div}\,\Big(\varepsilon \dfrac{\partial}{\partial t} \widetilde{E}_T^\delta + \sigma_e\, \widetilde{E}_T^\delta\Big) \;+\; \dfrac{\partial}{\partial x_3}\Big(\varepsilon \dfrac{\partial}{\partial t} \widetilde{E}_3^\delta + \sigma_e\, \widetilde{E}_3^\delta\Big) = 0, \\ \delta^{-1}\, \mathrm{div}\,\Big(\mu \dfrac{\partial}{\partial t} \widetilde{H}_T^\delta + \sigma_m\, \widetilde{H}_T^\delta\Big) \;+\; \dfrac{\partial}{\partial x_3}\Big(\mu \dfrac{\partial}{\partial t} \widetilde{H}_3^\delta + \sigma_m\, \widetilde{H}_3^\delta\Big) = 0, \end{cases} \tag{15}$$

(14) and (15) are completed by zero initial conditions and boundary conditions easily deduced from (12), on $\partial\Omega \times \mathbb{R}^+$, we find

$$\widetilde{E}_T^\delta \times \mathbf{n}_T = \mathbf{0}, \quad \widetilde{E}_3^\delta\, \mathbf{n}_T - \delta\, g\, \widetilde{E}_T^\delta = 0, \quad \widetilde{H}_T^\delta \cdot \mathbf{n}_T + \delta\, g\, \widetilde{H}_3^\delta = 0. \tag{16}$$

3. Formal derivation of the limit problem in the frequency domain

It appears useful to work in the frequency domain, using the time Fourier transform of the unknown functions (all extended by 0 for negative time)

$$\widetilde{E}_T^\delta(\mathbf{x},t) \;\longrightarrow\; \widehat{E}_T^\delta(\mathbf{x},\omega), ... \quad \text{and so on.}$$

As we work with zero initial conditions, from (14), (15) and (16), we get :

$$\begin{cases} \big(i\omega\,\varepsilon+\sigma_e\big)\,\widehat{E}_T^\delta - \dfrac{\partial}{\partial x_3}\,\mathbf{e}_3\times\widehat{H}_T^\delta - \delta^{-1}\,\mathbf{rot}\;\widehat{H}_3^\delta = \widehat{\mathbf{j}}_T, & \text{in } \Omega,\\ \big(i\omega\,\varepsilon+\sigma_e\big)\,\widehat{E}_3^\delta - \delta^{-1}\,\mathrm{rot}\;\widehat{H}_T^\delta = 0, & \text{in } \Omega,\\ \big(i\omega\,\mu+\sigma_m\big)\,\widehat{H}_T^\delta + \dfrac{\partial}{\partial x_3}\,\mathbf{e}_3\times\widehat{E}_T^\delta + \delta^{-1}\,\mathbf{rot}\;\widehat{E}_3 = \mathbf{0}, & \text{in } \Omega,\\ \big(i\omega\,\mu+\sigma_m\big)\,\widehat{H}_3^\delta + \delta^{-1}\,\mathrm{rot}\;\widehat{E}_T^\delta = 0, & \text{in } \Omega, \end{cases} \tag{17}$$

while from (15) we get

$$\begin{cases} \delta^{-1}\,\mathrm{div}\,\big(\,(i\omega\,\varepsilon+\sigma_e)\,\widehat{E}_T^\delta\big) + \;\dfrac{\partial}{\partial x_3}\big(\,(i\omega\,\varepsilon+\sigma_e)\,\widehat{E}_3^\delta\,\big) = 0, & \text{in } \Omega\\ \delta^{-1}\,\mathrm{div}\,\big(\,(i\omega\,\mu+\sigma_m)\,\widehat{H}_T^\delta\big) + \;\dfrac{\partial}{\partial x_3}\big(\,(i\omega\,\mu+\sigma_m)\,\widehat{H}_3^\delta\,\big) = 0, & \text{in } \Omega. \end{cases} \tag{18}$$

The boundary conditions keep the same form

$$\widehat{E}_T^\delta\times\mathbf{n}_T = \mathbf{0}, \quad \widehat{E}_3^\delta\,\mathbf{n}_T - \delta\,g\,\widehat{E}_T^\delta = 0, \quad \widehat{H}_T^\delta\cdot\mathbf{n}_T + \delta\,g\,\widehat{H}_3^\delta = 0, \quad \text{on } \partial\Omega. \tag{19}$$

We look for a formal power series expansion in δ of the electromagnetic field

$$\begin{aligned} \widehat{E}_P^\delta(\mathbf{x},\omega) &= \widehat{E}_P^0(\mathbf{x},\omega) + \delta\;\widehat{E}_P^1(\mathbf{x},\omega) + \cdots, \qquad P = 3 \text{ or } T,\\ \widehat{H}_P^\delta(\mathbf{x},\omega) &= \widehat{H}_P^0(\mathbf{x},\omega) + \delta\;\widehat{H}_P^1(\mathbf{x},\omega) + \cdots, \qquad P = 3 \text{ or } T. \end{aligned} \tag{20}$$

We now adopt the usual methodology which consists in inserting (20) in (17) and identifying the series term by term, which begins by the δ^{-1} terms.

3.1. Identification of the δ^{-1} terms : transverse polarization and "1D-structure" of the limit fields.

From (17), (18) and (19) we deduce the equations

$$\mathbf{rot}\;\widehat{H}_3^0 = \mathbf{rot}\;\widehat{E}_3^0 = \mathbf{0}, \qquad \mathrm{rot}\;\widehat{H}_T^0 = \mathrm{rot}\;\widehat{E}_T^0 = 0, \quad \text{in } \Omega, \tag{21}$$

$$\text{div}\,\big(\,(i\omega\,\varepsilon+\sigma_e)\,\widehat{E}_T^0\,\big)=0,\qquad \text{div}\,\big(\,(i\omega\,\mu+\sigma_m)\,\widehat{H}_T^0\,\big)=0,\quad \text{in }\Omega. \tag{22}$$

$$\widehat{E}_T^0\times\mathbf{n}_T=\mathbf{0},\quad \widehat{E}_3^0=0,\quad \widehat{H}_T^0\cdot\mathbf{n}_T=0,\quad \text{on }\partial\Omega. \tag{23}$$

We see that these equations only involve the leading terms of the expansions (20). However, they are not sufficient, to determine completely the (formal) limit field $(\widehat{E}^0,\widehat{H}^0)$. For this, we shall need also the δ^0 terms. For the moment, we exploit equations (21) to (23).

First note that the first equation of (21) means that $\widehat{E}_3^0$ and $\widehat{H}_3^0$ are functions of x_3 only. Using (23), we deduce

$$\widehat{E}_3^0(\mathbf{x},\omega)=0,\quad \mathbf{x}\in\Omega.$$

We shall show in section 3.2 that also $\widehat{H}_3^0=0$, that is: the limit field $(\widehat{E}^0,\widehat{H}^0)$ is transversely polarized.

Next, we are interested in the transverse electric field $\widehat{E}_T^0$. It is natural to introduce, for each $(x_3,\omega)\in\mathbb{R}^2$, the following space of 2D vector fields (below $\varepsilon\equiv\varepsilon(\cdot,x_3)$ and the same for σ_e):

$$\mathcal{E}_{stat}(x_3,\omega)\equiv\Big\{\,\mathbf{v}\in\mathbf{R}_0(x_3)\;/\;\text{div}\,\Big(\,\big(\,i\omega\,\varepsilon+\sigma_e\,\big)\,\mathbf{v}\,\Big)=0,\ \mathbf{v}\times\mathbf{n}_T=0\ \text{on}\ \partial S_{x_3}\,\Big\}$$

$$\mathbf{R}_0(x_3)\equiv\Big\{\,\mathbf{v}\in L^2\big(S_{x_3}\big)^2\quad/\quad\text{rot}\,\mathbf{v}=0\,\Big\}$$

Indeed, equations (21) to (23) imply that,

$$\forall\,(x_3,\omega)\in\mathbb{R}^2,\quad \widehat{E}_T^0(\cdot,x_3,\omega)\in\mathcal{E}_{stat}(x_3,\omega) \tag{24}$$

A fundamental point is that, thanks to hypothesis 2.2, $\mathcal{E}_{stat}(x_3,\omega)$ is a space of dimension 1. This is a consequence of a well known fundamental theorem on vector fields (see for instance [1],[5],[11]) that we state here in 2D:

Proposition 3.1. *Let $\mathcal{O}$ be a bounded Lipschitz domain of $\mathbb{R}^2$. Let $\mathbf{v}\in L^2(\mathcal{O})^2$, such that* $\text{rot}\,\mathbf{v}=0$. *Assuming one of the following assumptions :*

(i) *$\mathcal{O}$ is simply connected,* $\qquad$ (ii) $\mathbf{v}\times\mathbf{n}_T=0$ *on $\partial\mathcal{O}$,*

there exists $\varphi\in H^1(\mathcal{O})$ such that $\mathbf{v}=\mathbf{grad}\,\varphi$ in $\mathcal{O}$.

Lemma 3.1. *For all* $(x_3, \omega) \in \mathbb{R}^2$,

$$\mathcal{E}_{stat}(x_3, \omega) = \text{span}\, \{\, \mathbf{grad}\, \widehat{\varphi}_s(\cdot, x_3, \omega)\,\},$$

where $\widehat{\varphi}_s \equiv \widehat{\varphi}_s(\cdot, x_3, \omega)$ *is the unique function solution of the 2D boundary value problem* ((x_3, ω) *is a parameter)*

$$\begin{cases} \text{div}\, \Big(\big(i\omega\, \varepsilon(\cdot, x_3) + \sigma_e(\cdot, x_3) \big)\, \mathbf{grad}\, \widehat{\varphi}_s(\cdot, x_3, \omega) \Big) = 0, \qquad \text{in } S_{x_3}, \\ \widehat{\varphi}_s(\cdot, x_3, \omega) = 0 \quad \text{on } \partial S^+_{x_3}, \qquad \widehat{\varphi}_s(\cdot, x_3, \omega) = 1 \quad \text{on } \partial S^-_{x_3}. \end{cases} \tag{25}$$

Proof This is a classical result that can be found in [5],[1],[11] in the case of real constant coefficients. For the reader's convenience, we adapt the proof to variable and complex coefficients. The existence and uniqueness of $\widehat{\varphi}_s(\cdot, x_3, \omega)$ solution of (25) is a consequence of Lax-Milgram's theorem. We need to show now that every function $\mathbf{v} \in \mathbf{R}_0(S_{x_3})$ such that

$$\text{div}\, \Big(\big(i\omega\, \varepsilon(\cdot, x_3) + \sigma_e(\cdot, x_3) \big)\, \mathbf{v} \Big) = 0 \text{ in } S_{x_3}, \quad \mathbf{v} \times \mathbf{n}_T = 0 \text{ on } \partial S_{x_3}, \tag{26}$$

is proportional to $\mathbf{grad}\, \widehat{\varphi}_s(\cdot, x_3, \omega)$. By proposition 3.1 (case (ii)), we know that $\mathbf{v} = \mathbf{grad}\, \varphi$ for some $\varphi \in H^1(S_{x_3})$. From the second equality in (26), we deduce that $\text{div}\, \big((\, i\omega\, \varepsilon(\cdot, x_3) + \sigma_e(\cdot, x_3)) \, \mathbf{grad}\, \varphi \,\big) = 0$ and from the third one that φ is constant on each connected component of ∂S_{x_3}, which means that there exists $(\phi_-, \phi_+) \in \mathbb{C}^2$ such that $\varphi = \phi_\pm$ on $\partial S^\pm_{x_3}$. From the uniqueness for the boundary value problem (25), we deduce that

$$\varphi = \phi_+ + (\phi_- - \phi_+)\, \widehat{\varphi}_s(\cdot, x_3, \omega), \quad \text{i. e. } \mathbf{v} = (\phi_- - \phi_+)\, \mathbf{grad}\, \widehat{\varphi}_s(\cdot, x_3, \omega)$$

which concludes the proof. □

From (24) and lemma 3.1, we deduce that, for each ω, there exists a function $x_3 \to \widehat{V}(x_3, \omega) \in \mathbb{C}$ such that:

$$\widehat{E}^0_T(\cdot, x_3, \omega) = \widehat{V}(x_3, \omega)\, \mathbf{grad}\, \widehat{\varphi}_s(\cdot, x_3, \omega). \tag{27}$$

The formula (27) reveals the hidden "1D-structure" of the limit problem: the function $\widehat{\varphi}_s(\cdot, x_3, \omega)$ can be pre-computed by solving $2D$-problems in each cross section (independently of the source, x_3 and ω are simply parameters) and the only unknown is the 1D function $V(x_3, \omega)$.

Remark 3.1. *Two particular cases deserve some comments:*

- *In the perfectly cylindrical case of Example 2.2, the function* $\widehat{\varphi}_s(\mathbf{x},\omega)$ *is obviously independent of* x_3 *:* $\widehat{\varphi}_s(\mathbf{x},\omega) \equiv \widehat{\varphi}_s(\mathbf{x}_T,\omega)$ *and (27) becomes*

$$\widehat{E}_T^0(\cdot,x_3,\omega) = \widehat{V}(x_3,\omega)\,\mathbf{grad}\,\widehat{\varphi}_s(\cdot,\omega).$$

  In other words, at the limit $\delta \to 0$*, one has separation of variables in* (x_1,x_2) *and* x_3*.*

- *If, in each cross section,* ε *and* σ_e *are proportional, that is to say if there exists a function* $\alpha(x_3)$ *such that*

$$\sigma_e(\cdot,x_3) = \alpha(x_3)\,\varepsilon(\cdot,x_3), \qquad \text{in } \Omega, \tag{28}$$

  one sees that the function $\widehat{\varphi}_s(\cdot,x_3,\omega)$ *is real valued and independent of* ω*:* $\widehat{\varphi}_s(\cdot,x_3,\omega) \equiv \varphi_s^\infty(\cdot,x_3)$ *solution of*

$$\left\{ \begin{array}{ll} \mathrm{div}\,\big(\,\varepsilon(\cdot,x_3)\,\mathbf{grad}\,\varphi_s^\infty(\cdot,x_3)\,\big) = 0, & \text{in } S_{x_3}, \\ \varphi_s^\infty(\cdot,x_3) = 0 \quad \text{on } \partial S_{x_3}^+, \qquad \varphi_s^\infty(\cdot,x_3) = 1 \quad \text{on } \partial S_{x_3}^-. \end{array} \right. \tag{29}$$

Next, we look at the transverse magnetic field $\widehat{H}_T^0$. Looking at equations (21) to (23), we are led to introduce, for each $(x_3,\omega) \in \mathbb{R}^2$, the following space of 2D vector fields (below μ holds for $\mu(\cdot,x_3)$ and the same for σ_m):

$$\mathcal{H}_{stat}(x_3,\omega) \equiv \Big\{\, \mathbf{v} \in \mathbf{R}_0\big(S_{x_3}\big) \,/\, \mathrm{div}\,\big(\,(\,i\omega\,\mu+\sigma_m\,)\,\mathbf{v}\,\big) = 0,\ \mathbf{v}\cdot\mathbf{n}_T = 0 \text{ on } \partial S_{x_3} \Big\}$$

so that equations (21) to (23) imply that

$$\forall\ (x_3,\omega) \in \mathbb{R}^2, \quad \widehat{H}_T^0(\cdot,x_3,\omega) \in \mathcal{H}_{stat}(x_3,\omega)$$

We observe that, $\mathcal{H}_{stat}(x_3,\omega)$ only differs from $\mathcal{E}_{stat}(x_3,\omega)$ by the boundary condition on ∂S_{x_3}. To describe this space, it is useful to introduce an artificial "cut", namely a (possibly curved) line $\Gamma \subset S_{x_3}$ (see Figure 2, right picture) that connects the interior boundary $\partial S_{x_3}^-$ to the exterior one $\partial S_{x_3}^+$ in such a way that the domain

$$S_{x_3}^\Gamma \equiv S_{x_3} \setminus \Gamma \quad \text{ is simply connected.}$$

We shall define an orientation of Γ by introducing along Γ the unit tangent vector $\boldsymbol{\tau}$ that points from the exterior boundary $\partial S^+_{x_3}$ towards the interior boundary $\partial S^-_{x_3}$. Accordingly we define along Γ the unit normal vector:

$$\mathbf{n}_T := \mathbf{e}_3 \times \boldsymbol{\tau}.$$

We also need to define the jump across Γ, denoted $[u]_\Gamma$, of a function u which is sufficiently smooth from each side of Γ. This is done accordingly with the orientation of the vector $\mathbf{n}_T$, namely:

$$\forall\, \mathbf{x}_T \in \Gamma, \quad \big[u\big]_\Gamma(\mathbf{x}_T) \equiv \lim_{\eta \searrow 0} \big(u(\mathbf{x}_T + \eta\, \mathbf{n}_T) - u(\mathbf{x}_T - \eta\, \mathbf{n}_T) \big)$$

We shall also define by τ the curvilinear abscissa along Γ which is coherent with the orientation of Γ with the tangent vector $\boldsymbol{\tau}$. Doing so, one has

$$\mathbf{grad}\, u \times \mathbf{n}_T = -\frac{\partial u}{\partial \tau}.$$

Finally, if $\psi \in H^1(S^\Gamma_{x_3})$, we denote $\widetilde{\mathbf{grad}}\, \psi$ the vector in $L^2(S_{x_3})^2$ such that

$$\widetilde{\mathbf{grad}}\, \psi = \mathbf{grad}\, \psi \text{ in } S^\Gamma_{x_3} \quad (\text{but } \widetilde{\mathbf{grad}}\, \psi \neq \mathbf{grad}\, \psi \text{ in } S_{x_3}!),$$

moreover we define $\widetilde{\mathbf{rot}}\, \psi = -\mathbf{e}_3 \times \widetilde{\mathbf{grad}}\, \psi$. The description of $\mathcal{H}_{stat}(x_3, \omega)$ will be based on the following result.

Lemma 3.2. *For any (x_3, ω), there is a unique $\widehat{\psi}_s(\cdot, x_3, \omega)$ so that*

$$\begin{cases} \widehat{\psi}_s(\cdot, x_3, \omega) \in H^1(S^\Gamma_{x_3}) \text{ and } \displaystyle\int_{S_{x_3}} \widehat{\psi}_s(\mathbf{x}_T, x_3, \omega)\, d\mathbf{x}_T = 0, \\ \mathrm{div}\, \Big(\big(i\omega\, \mu(\cdot, x_3) + \sigma_m(\cdot, x_3) \big)\, \mathbf{grad}\, \widehat{\psi}_s(\cdot, x_3, \omega) \Big) = 0, \quad \text{in } S^\Gamma_{x_3}, \\ \Big[\big(i\omega\, \mu(\cdot, x_3) + \sigma_m(\cdot, x_3) \big) \dfrac{\partial \widehat{\psi}_s(\cdot, x_3, \omega)}{\partial n} \Big]_\Gamma = 0, \quad \big[\, \widehat{\psi}_s(\cdot, x_3, \omega) \,\big]_\Gamma = 1, \\ \dfrac{\partial \widehat{\psi}_s}{\partial n}(\cdot, x_3, \omega) = 0, \quad \text{on } \partial S_{x_3}. \end{cases} \tag{30}$$

Proof This result is a slight variant of a result from [5] (theorem 3, p. 24). We give a proof for the reader's convenience. One easily checks that the weak

formulation of (30) is (denoting $\widehat{\psi}_s = \psi_s(x_3,\omega)$, $\mu = \mu(x_3,\omega)$, etc ...)

$$\left\{\begin{array}{l} \text{Find } \widehat{\psi}_s \in H^1(S^\Gamma_{x_3}) \text{ such that } \big[\,\widehat{\psi}_s\,\big]_\Gamma = 1,\ \displaystyle\int_{S_{x_3}} \widehat{\psi}_s\, d\mathbf{x}_T = 0 \text{ and} \\ \forall\ \psi \in H^1(S_{x_3}), \quad \displaystyle\int_{S_{x_3}} \big(\, i\omega\,\mu + \sigma_m\,\big)\,\mathbf{grad}\,\widehat{\psi}_s \cdot \mathbf{grad}\,\psi\ d\mathbf{x}_T = 0. \end{array}\right.$$

Writing $\widehat{\psi}_s = \psi^* + \widehat{\theta}_s$ for a given $\psi^* \in H^1(S^\Gamma_{x_3})$ such that $\big[\,\psi^*\,\big]_\Gamma = 1$ and with mean value 0, the new unknown $\widehat{\theta}_s$ is solution of:

$$\left\{\begin{array}{l} \text{Find } \widehat{\theta}_s \in H^1(S_{x_3}) \text{ such that } \displaystyle\int_{S_{x_3}} \widehat{\theta}_s\, d\mathbf{x}_T = 0 \text{ and} \\ \forall\ \psi \in H^1(S_{x_3}), \quad \displaystyle\int_{S_{x_3}} \big(\, i\omega\,\mu + \sigma_m\,\big)\,\mathbf{grad}\,\widehat{\theta}_s \cdot \mathbf{grad}\,\psi\ d\mathbf{x}_T = \\ \qquad\qquad -\displaystyle\int_{S_{x_3}} \big(\, i\omega\,\mu + \sigma_m\,\big)(\cdot,x_3)\,\widetilde{\mathbf{grad}}\,\psi^* \cdot \mathbf{grad}\,\psi\ d\mathbf{x}_T. \end{array}\right.$$

One concludes using Poincaré-Wirtinger inequality (S_{x_3} is connected) and Lax-Milgram's lemma. □

Lemma 3.3. *For all* $(x_3,\omega) \in \mathbb{R}^2$,

$$\mathcal{H}_{stat}(x_3,\omega) = \text{span}\,\big\{\,\widetilde{\mathbf{grad}}\,\widehat{\psi}_s(\cdot,x_3,\omega)\,\big\}.$$

Proof We want to show that any function $\mathbf{v} \in \mathbf{R}_0\big(S_{x_3}\big)$ such that

$$\mathbf{div}\Big(\,\big(\,i\omega\,\mu(\cdot,x_3) + \sigma_m(\cdot,x_3)\,\big)\,\mathbf{v}\,\Big) = 0 \text{ in } S_{x_3}, \quad \mathbf{v}\cdot\mathbf{n}_T = 0 \text{ on } \partial S_{x_3} \tag{31}$$

is proportional to $\widetilde{\mathbf{grad}}\,\widehat{\psi}_s(\cdot,x_3,\omega)$. Since $S^\Gamma_{x_3}$ is by construction simply connected, we can apply lemma 3.1 (case (i)) to deduce that $\mathbf{v} = \mathbf{grad}\,\psi$ for some $\psi \in H^1(S^\Gamma_{x_3})$. Moreover, $\mathrm{rot}\,\mathbf{v} = 0$ in S_{x_3}, we know that the tangential component of $\mathbf{v}$ does not jump across Γ, namely the jump of ψ is constant along Γ: there exists a scalar $\phi \in \mathbb{R}$ such that $[\,\psi\,]_\Gamma = \phi$. As $[\,\widehat{\psi}_s\,] = 1$ and as $\mathbf{grad}\,\widehat{\psi}_s$ satisfies (31) we have $\mathbf{v} = \phi\,\mathbf{grad}\,\widehat{\psi}_s$. □

As we have defined $\widehat{V}(x_3,\omega)$, we choose to define $\widehat{I}(x_3,\omega)$ by

$$\widehat{H}^0_T(\cdot,x_3,\omega) = \widehat{I}(x_3,\omega)\,\widetilde{\mathbf{grad}}\,\widehat{\psi}_s(\cdot,x_3,\omega). \tag{32}$$

Remark 3.2. *Analogously to remark 3.1*

- *In the perfectly cylindrical case of Example 2.2, the function* $\widehat{\psi}_s(\cdot, x_3, \omega)$ *is obviously independent of* x_3*:* $\widehat{\psi}_s(\cdot, x_3, \omega) \equiv \widehat{\psi}_s(\cdot, \omega)$ *and one has asymptotic separation of variables when* $\delta \to 0$*:*

$$\widehat{H}_T^0(\cdot, x_3, \omega) = \widehat{I}(x_3, \omega)\, \widetilde{\mathbf{grad}}\, \widehat{\psi}_s(\cdot, \omega).$$

- *If, in each cross section* μ *and* σ_m *are proportional (in the sense of (28)), the function* $\widehat{\psi}_s(\cdot, x_3, \omega)$ *is real valued and independent of* ω*:* $\widehat{\psi}_s(\cdot, x_3, \omega) \equiv \psi_s^\infty(\cdot, x_3)$ *solution of*

$$\left\{ \begin{array}{l} \mathrm{div}\left(\mu(\cdot, x_3)\, \mathbf{grad}\, \psi_s^\infty \right) = 0, \quad \text{in } S_{x_3}^\Gamma, \quad \left[\psi_s^\infty \right]_\Gamma = 1, \\ \dfrac{\partial \psi_s^\infty(\cdot, x_3)}{\partial n} = 0, \quad \text{on } \partial S_{x_3}, \quad \left[\mu(\cdot, x_3)\, \dfrac{\partial \psi_s^\infty}{\partial n} \right]_\Gamma = 0. \end{array} \right. \tag{33}$$

Remark 3.3. *In the case of cables with homogeneous cross-sections (Example 2.3), there is a particularly simple relationship between the two spaces* $\mathcal{E}_{stat}(x_3) \equiv \mathcal{E}_{stat}(x_3, \omega)$ *and* $\mathcal{H}_{stat}(x_3) \equiv \mathcal{H}_{stat}(x_3, \omega)$ *(note that they are independent of* ω *by remarks 3.1 and 3.2). The vector fields of one space are obtained by rotating the ones of the other space by* $\pi/2$

$$\mathcal{H}_{stat}(x_3) = \left\{ \mathbf{e_3} \times \mathbf{v},\ /\ \mathbf{v} \in \mathcal{E}_{stat}(x_3) \right\}.$$

It remains, in order to determine the formal limit fields $(\widehat{E}_T^0, \widehat{H}_T^0)$, to find the equations for $\widehat{V}$ and $\widehat{I}$. This is the object of the next section.

3.2. Identification of the δ^0 *terms : derivation of the 1D effective model.*
Using the hypothesis (20) and identification of the term in power of δ^0 in (14), we obtain the equations :

$$\left\{ \begin{array}{ll} (i\varepsilon\,\omega + \sigma_e)\, \widehat{E}_T^0 - \dfrac{\partial}{\partial x_3}\, \mathbf{e}_3 \times \widehat{H}_T^0 - \mathbf{rot}\ \widehat{H}_3^1 = \widehat{\mathbf{j}}_T, & \text{in } \Omega, \\ (i\varepsilon\,\omega + \sigma_e)\, \widehat{E}_3^0 - \mathrm{rot}\ \widehat{H}_T^1 = 0, & \text{in } \Omega, \\ (i\mu\,\omega + \sigma_m)\, \widehat{H}_T^0 + \dfrac{\partial}{\partial x_3}\, \mathbf{e}_3 \times \widehat{E}_T^0 + \mathbf{rot}\ \widehat{E}_3^1 = \mathbf{0}, & \text{in } \Omega, \\ (i\mu\,\omega + \sigma_m)\, \widehat{H}_3^0 + \mathrm{rot}\ \widehat{E}_T^1 = 0, & \text{in } \Omega. \end{array} \right. \tag{34}$$

These equations are completed by boundary conditions for $\widehat{E}_T^1$ and $\widehat{E}_3^1$:

$$\widehat{E}_T^1 \times \mathbf{n}_T = 0, \quad \widehat{E}_3^1\, \mathbf{n}_T - g\, \widehat{E}_T^0 = 0, \quad \text{on } \partial\Omega, \tag{35}$$

that are obtained by identification of terms in δ in (19).

We are now in position to prove that, as announced in the previous section, the longitudinal magnetic field $\widehat{H}_3^0$ vanishes. We already know (see section 3.1) that this longitudinal field does not depend on x_1 and x_2. Integrating the last equation of (34) over S_{x_3}, we get by Green's formula

$$\widehat{H}_3^0(x_3,\omega) \int_{S_{x_3}} \big(i\omega\,\mu + \sigma_m\big)(\mathbf{x}_T, x_3)\, d\mathbf{x}_T = -\int_{\partial S_{x_3}} \widehat{E}_T^1 \times \mathbf{n}_T\, d\sigma_{x_3} = 0,$$

where we have used (35). Since $\int_{S_{x_3}} \big(i\omega\,\mu + \sigma_m\big)(\mathbf{x}_T, x_3)\, d\mathbf{x}_T \neq 0$,

$$\forall\, \omega \in \mathbb{R}, \quad \widehat{H}_3^0(\mathbf{x},\omega) = 0, \quad \mathbf{x} \in \Omega.$$

The remaining of this section will be rather formal and justified only under appropriate regularity assumptions about the data of the problem (source, coefficients). However everything can be made rigorous by proceeding in the weak sense (an example of which is given in the proof of (iii) lemma 3.4) and using appropriate regularization processes. Details can be found in [8].

To get the equations in $\widehat{V}$ and $\widehat{I}$ defined by (27, 32), we use the first and third equations of (34). These equations relate $\widehat{E}_T^0$ and $\widehat{H}_T^0$ to $\widehat{E}_3^1$ and $\widehat{H}_3^1$ which are not known for the moment. To get rid of them, we take, in each S_{x_3}, the L^2 scalar product of these two equations respectively with $\mathbf{grad}\,\widehat{\varphi}_s(\mathbf{x},\omega)$ and $\mathbf{grad}\,\widehat{\psi}_s(\mathbf{x},\omega)$. We obtain (modulo some obvious notation, in particular we write $\widehat{\varphi}_s$ instead of $\widehat{\varphi}_s(\mathbf{x},\omega)$, etc ...), for all $x_3 \in \mathbb{R}$.

$$\left\{\begin{array}{rcl} \Big(i\omega\,\widehat{C} + \widehat{G}\Big)\,\widehat{V} & - & \big(\dfrac{\partial}{\partial x_3}\,\mathbf{e}_3 \times \widehat{H}_T^0, \mathbf{grad}\,\widehat{\varphi}_s\big)_{L^2(S_{x_3})} \\ & - & \big(\mathbf{rot}\ \widehat{H}_3^1, \mathbf{grad}\,\widehat{\varphi}_s\big)_{L^2(S_{x_3})} = \big(\widehat{\mathbf{j}}_T, \mathbf{grad}\,\widehat{\varphi}_s\big)_{L^2(S_{x_3})}, \\ \Big(i\omega\,\widehat{L} + \widehat{R}\Big)\,\widehat{I} & + & \big(\dfrac{\partial}{\partial x_3}\,\mathbf{e}_3 \times \widehat{E}_T^0, \widetilde{\mathbf{grad}}\,\widehat{\psi}_s\big)_{L^2(S_{x_3})} \\ & + & \big(\mathbf{rot}\ \widehat{E}_3^1, \widetilde{\mathbf{grad}}\,\widehat{\psi}_s\big)_{L^2(S_{x_3})} = 0, \end{array}\right. \tag{36}$$

where the homogenized coefficients $\big(\widehat{C},\widehat{L},\widehat{G},\widehat{R}\big)$ depend on x_3 and ω:

$$\widehat{C}(x_3,\omega)\equiv\int_{S_{x_3}}\varepsilon(\mathbf{x}_T,x_3)\,\big|\mathbf{grad}\,\widehat{\varphi}_s(\mathbf{x}_T,x_3,\omega)\big|^2\,d\mathbf{x}_T, \tag{37}$$

$$\widehat{L}(x_3,\omega)\equiv\int_{S_{x_3}}\mu(\mathbf{x}_T,x_3)\,\big|\widetilde{\mathbf{grad}}\,\widehat{\psi}_s(\mathbf{x}_T,x_3,\omega)\big|^2\,d\mathbf{x}_T, \tag{38}$$

$$\widehat{G}(x_3,\omega)\equiv\int_{S_{x_3}}\sigma_e(\mathbf{x}_T,x_3)\,\big|\mathbf{grad}\,\widehat{\varphi}_s(\mathbf{x}_T,x_3,\omega)\big|^2\,d\mathbf{x}_T, \tag{39}$$

$$\widehat{R}(x_3,\omega)\equiv\int_{S_{x_3}}\sigma_m(\mathbf{x}_T,x_3)\,\big|\widetilde{\mathbf{grad}}\,\widehat{\psi}_s(\mathbf{x}_T,x_3,\omega)\big|^2\,d\mathbf{x}_T. \tag{40}$$

Next, we treat the remaining terms in the first equation of (36). Using (32) together with (9), we have

$$\begin{aligned}\Big(\frac{\partial}{\partial x_3}\,\mathbf{e}_3\times\widehat{H}^0_T,\widetilde{\mathbf{grad}}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})} = &-\Big(\widetilde{\mathbf{rot}}\;\widehat{\psi}_s,\mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})}\;\frac{\partial\widehat{I}}{\partial x_3}\\ &-\Big(\frac{\partial}{\partial x_3}\widetilde{\mathbf{rot}}\,\widehat{\psi}_s\,,\mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})}\;\widehat{I}\,.\end{aligned} \tag{41}$$

Moreover, using Green's formula (10), we have

$$\Big(\mathbf{rot}\;\widehat{H}^1_3,\mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})}=\int_{\partial S_{x_3}}\widehat{H}^1_3\;\overline{\big(\,\mathbf{grad}\,\widehat{\varphi}_s\times\mathbf{n}_T\,\big)}\,d\sigma_{x_3}=0, \tag{42}$$

since $\mathbf{grad}\,\widehat{\varphi}_s\times\mathbf{n}_T=0$ along ∂S_{x_3} and rot $\mathbf{grad}\,\widehat{\varphi}_s=0$. In the same way, for the second equation of (36), using (27), we get

$$\begin{aligned}\Big(\frac{\partial}{\partial x_3}\,\mathbf{e}_3\times\widehat{E}^0_T,\widetilde{\mathbf{grad}}\,\widehat{\psi}_s\Big)_{L^2(S_{x_3})} = &-\Big(\mathbf{rot}\;\widehat{\varphi}_s,\widetilde{\mathbf{grad}}\,\widehat{\psi}_s\Big)_{L^2(S_{x_3})}\;\frac{\partial\widehat{V}}{\partial x_3}\\ &-\Big(\frac{\partial}{\partial x_3}\mathbf{rot}\;\widehat{\varphi}_s\,,\widetilde{\mathbf{grad}}\,\widehat{\psi}_s\Big)_{L^2(S_{x_3})}\;\widehat{V}\,.\end{aligned} \tag{43}$$

On the other hand, by Green's formula

$$\Big(\mathbf{rot}\;\widehat{E}^1_3,\widetilde{\mathbf{grad}}\,\widehat{\psi}_s\Big)_{L^2(S_{x_3})}=\int_{\partial S_{x_3}}\widehat{E}^1_3\;\overline{\widetilde{\mathbf{grad}}\,\widehat{\psi}_s\times\mathbf{n}_T}\,d\sigma_{x_3}.$$

Using (9) and the second equation of (35), we get rid of $\widehat{E}_3^1$ and obtain

$$\Big(\mathbf{rot}\ \widehat{E}_3^1, \widetilde{\mathbf{grad}}\,\widehat{\psi}_s\Big)_{L^2(S_{x_3})} = -\int_{\partial S_{x_3}} g\,\widehat{E}_T^0 \cdot \widetilde{\mathbf{rot}}\ \widehat{\psi}_s\ d\sigma_{x_3}$$

$$= -\Big(\int_{\partial S_{x_3}} g\,\mathbf{grad}\,\widehat{\varphi}_s \cdot \widetilde{\overline{\mathbf{rot}\ \widehat{\psi}_s}}\ \Big)\ \widehat{V}\ d\sigma_{x_3} \quad (44)$$

where we have used again (27).

Finally, we obtain formally the equations governing the variations of $\widehat{V}$ and $\widehat{I}$ by substituting (41, 42, 43, 44) into (36):

$$\left\{\begin{array}{l} \Big(i\omega\,\widehat{C}+\widehat{G}\Big)\,\widehat{V} \ +\ \Big(\widetilde{\mathbf{rot}}\ \widehat{\psi}_s, \mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})}\ \dfrac{\partial \widehat{I}}{\partial x_3} \\ \qquad +\ \Big(\dfrac{\partial}{\partial x_3}\widetilde{\mathbf{rot}}\ \widehat{\psi}_s, \mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})}\ \widehat{I} \ =\ \Big(\widehat{\mathbf{j}}_T, \mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})}, \\ \Big(i\omega\,\widehat{L}+\widehat{R}\Big)\,\widehat{I} \ -\ \Big(\mathbf{rot}\ \widehat{\varphi}_s, \widetilde{\mathbf{grad}}\,\widehat{\psi}_s\Big)_{L^2(S_{x_3})}\ \dfrac{\partial \widehat{V}}{\partial x_3} \\ \qquad -\ \Big(\dfrac{\partial}{\partial x_3}\mathbf{rot}\ \widehat{\varphi}_s\,, \widetilde{\mathbf{grad}}\,\widehat{\psi}_s\Big)_{L^2(S_{x_3})}\ \widehat{V} \\ \qquad\quad -\ \Big(\displaystyle\int_{\partial S_{x_3}} g\,\mathbf{grad}\,\widehat{\varphi}_s \cdot \widetilde{\overline{\mathbf{grad}\,\widehat{\psi}_s}}\ d\sigma_{x_3}\Big)\ \widehat{V} = \ 0. \end{array}\right. \quad (45)$$

We obtain further simplification thanks to the following lemma:

Lemma 3.4. *One has the following identities (writing again $\widehat{\varphi}_s$ instead of $\widehat{\varphi}_s(x_3,\omega)$, etc ...)*

$$\begin{array}{ll} (i) & \Big(\widetilde{\mathbf{rot}}\ \widehat{\psi}_s, \mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})} = -\Big(\mathbf{rot}\ \widehat{\varphi}_s, \widetilde{\mathbf{grad}}\,\widehat{\psi}_s\Big)_{L^2(S_{x_3})} = 1, \\ (ii) & \Big(\dfrac{\partial}{\partial x_3}\mathbf{rot}\ \widehat{\varphi}_s, \widetilde{\mathbf{grad}}\,\widehat{\psi}_s\Big)_{L^2(S_{x_3})} + \Big(\displaystyle\int_{\partial S_{x_3}} g\,\mathbf{grad}\,\widehat{\varphi}_s \cdot \widetilde{\overline{\mathbf{rot}\ \widehat{\psi}_s}}\ \Big) = 0, \\ (iii) & \Big(\dfrac{\partial}{\partial x_3}\widetilde{\mathbf{rot}}\ \widehat{\psi}_s, \mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})} = 0. \end{array} \quad (46)$$

Proof (i) We only prove the first equality, the second one easily follows by (8). We first note that since Γ has (2D) measure 0, we can write

$$\Big(\widetilde{\mathbf{rot}}\ \widehat{\psi}_s, \mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})} = \int_{S^\Gamma_{x_3}} \mathbf{rot}\ \widehat{\psi}_s \cdot \overline{\mathbf{grad}\,\widehat{\varphi}_s}\ d\mathbf{x}_T.$$

Next we apply Green's formula (10, (ii)) with $u = \widehat{\varphi}_s, \mathbf{v} = \mathbf{rot}\ \widehat{\psi}_s$ and $S^\Gamma_{x_3}$ instead of S_{x_3}. Of course we have to cope with the fact that, due to the presence of the cut Γ, the domain $S^\Gamma_{x_3}$ is no longer locally on one side of its boundary. Using rot $\mathbf{grad}\,\widehat{\varphi}_s = 0$, the reader will verify that

$$\Big(\widetilde{\mathbf{rot}}\ \widehat{\psi}_s, \mathbf{grad}\,\widehat{\varphi}_s\Big)_{L^2(S_{x_3})} \quad = \quad \int_{\partial S_{x_3}} \widehat{\psi}_s\, \overline{\mathbf{grad}\,\widehat{\varphi}_s \times \mathbf{n}_T}\, d\sigma_{x_3} \ - \ \int_\Gamma \overline{\frac{\partial \widehat{\varphi}_s}{\partial \tau}}\, d\gamma.$$

The first term is 0 because $\widehat{\varphi}_s$ is constant on $S^\pm_{x_3}$, moreover $\int_\Gamma \overline{\frac{\partial \widehat{\varphi}_s}{\partial \tau}}\, d\gamma = 1$, since Γ joins $S^-_{x_3}$ where $\widehat{\varphi}_s = 0$ to $S^+_{x_3}$ where $\widehat{\varphi}_s = 1$.

(ii) When computing the left hand side of (46)(ii), we can this time apply Green's formula in all S_{x_3} because $\widehat{\varphi}_s$, and thus its x_3-derivative, belongs to $H^1(S_{x_3})$. We get, using rot $\mathbf{grad}\,\widehat{\varphi}_s = 0$ and the second identity in (9)

$$\Big(\frac{\partial}{\partial x_3}\mathbf{rot}\ \widehat{\varphi}_s\,, \widetilde{\mathbf{grad}}\ \widehat{\psi}_s\Big)_{L^2(S_{x_3})}$$

$$= \int_{\partial S_{x_3}} \frac{\partial \widehat{\varphi}_s}{\partial x_3}\ \overline{\widetilde{\mathbf{grad}}\ \widehat{\psi}_s \times \mathbf{n}_T}\, d\sigma_{x_3} = -\int_{\partial S_{x_3}} \frac{\partial \widehat{\varphi}_s}{\partial x_3}\ \mathbf{n}_T \cdot\ \overline{\widetilde{\mathbf{rot}}\ \widehat{\psi}_s}\, d\sigma_{x_3}. \quad (47)$$

Considering $\widehat{\varphi}_s$ as a function of $\mathbf{x} \in \Omega$, then $\widehat{\varphi}_s$ is constant on $\partial\Omega$, which means that, on $\partial\Omega$, the 3D gradient is orthogonal to $\partial\Omega$ or, equivalently, the 3D vector product between $(\,\mathbf{grad}\,\widehat{\varphi}_s, \partial_{x_3}\widehat{\varphi}_s)^t$ and the normal vector to $\partial\Omega$, namely $\mathbf{n} = (\mathbf{n}_T, g)^t$ (see (2) with $\delta = 1$) is identically 0. Projecting this equation on the (x_1, x_2) plane, we get

$$\frac{\partial \widehat{\varphi}_s}{\partial x_3}\ \mathbf{n}_T = g\ \mathbf{grad}\,\widehat{\varphi}_s,$$

substituting this equality into (47) leads to (46)(ii).

(iii) For proving the last equality, it appears useful from the technical point

of view to work in a weak sense and to exploit (46)(i). Let $\chi = \chi(x_3)$ be a 1D test function (C^∞ with compact support), we multiply (46)(i) by $\chi'(x_3)$ and then integrate over x_3. We obtain

$$\int_{-\infty}^{+\infty} \int_{S_{x_3}} \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ \chi'(x_3)\ d\mathbf{x}_T\, dx_3 \equiv \int_\Omega \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ \ \chi'\ d\mathbf{x} = 0.$$

We then use the 3D Green's formula in Ω to obtain, denoting $d\Sigma$ the surface measure on $\partial\Omega$ (we use again that $\mathbf{n} = (\mathbf{n}_T, g)^t$ is normal to $\partial\Omega$ which implies that the third component of the outgoing unit normal vector to $\partial\Omega$ is $g/|\mathbf{n}|$)

$$-\int_\Omega \frac{\partial}{\partial x_3} \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ \ \chi\ d\mathbf{x}$$

$$-\int_\Omega \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \frac{\partial}{\partial x_3} \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ \ \chi\ d\mathbf{x} + \int_{\partial\Omega} \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ \frac{g}{|\mathbf{n}|}\ \ \chi\ d\Sigma = 0.$$

Finally, as $d\sigma_{x_3}$ denoted the lineic measure along ∂S_{x_3}, one easily checks the formula

$$d\Sigma = |\mathbf{n}|\ d\sigma_{x_3}\ dx_3$$

so that the previous equality becomes

$$\int_{-\infty}^{+\infty} \left(\int_{S_{x_3}} \frac{\partial}{\partial x_3} \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ \ d\mathbf{x}_T + \int_{S_{x_3}} \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \frac{\partial}{\partial x_3} \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ \ d\mathbf{x}_T \right.$$
$$\left. - \int_{\partial S_{x_3}} \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ g\, d\sigma_{x_3} \right)\ \ \chi(x_3)\ dx_3 = 0,$$

that is to say, the above equality being valid for any test function χ,

$$\int_{S_{x_3}} \frac{\partial}{\partial x_3} \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ \ d\mathbf{x}_T +$$

$$\int_{S_{x_3}} \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \frac{\partial}{\partial x_3} \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ \ d\mathbf{x}_T\ - \int_{\partial S_{x_3}} \widetilde{\mathbf{rot}}\, \widehat{\psi}_s \cdot \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ g\, d\sigma_{x_3} = 0.$$

Using (8) for the second term, this gives

$$\Big(\frac{\partial}{\partial x_3} \widetilde{\mathbf{rot}}\ \widehat{\psi}_s, \mathbf{grad}\, \widehat{\varphi}_s \Big)_{L^2(S_{x_3})}$$

$$= \Big(\widetilde{\mathbf{grad}}\, \widehat{\psi}_s\, , \frac{\partial}{\partial x_3} \mathbf{rot}\ \widehat{\varphi}_s \Big)_{L^2(S_{x_3})} + \int_{\partial S_{x_3}} \widetilde{\mathbf{rot}}\ \widehat{\psi}_s \cdot \overline{\mathbf{grad}\, \widehat{\varphi}_s}\ g\, d\sigma_{x_3}$$

which implies (46)(iii) by (46)(ii). □

Finally using the identities (46) in (45), we obtain the homogenized 1D model in the frequency domain, for all $x_3 \in \mathbb{R}$

$$\begin{cases} \Big(i\omega\, \widehat{C}(x_3,\omega) + \widehat{G}(x_3,\omega) \Big)\, \widehat{V}(x_3,\omega) \ + \ \dfrac{\partial \widehat{I}}{\partial x_3}(x_3,\omega) \ = \ \widehat{j}(x_3,\omega), \\ \Big(i\omega\, \widehat{L}(x_3,\omega) + \widehat{R}(x_3,\omega) \Big)\, \widehat{I}(x_3,\omega) \ + \ \dfrac{\partial \widehat{V}}{\partial x_3}(x_3,\omega) \ = \ 0, \end{cases} \tag{48}$$

where the homogenized coefficients $\big(\widehat{C}(x_3,\omega), \widehat{L}(x_3,\omega), \widehat{G}(x_3,\omega), \widehat{R}(x_3,\omega)\big)$ are given by (37)-(40) and where the source term $\widehat{j}(x_3,\omega)$ is given by

$$\widehat{j}(x_3,\omega) = \big(\widehat{\mathbf{j}}_T(x_3,\omega), \mathbf{grad}\, \widehat{\varphi}_s(x_3,\omega) \big)_{L^2(S_{x_3})}\, .$$

4. Properties of the homogenized coefficients

We study the behavior, for large $|\omega|$ of $\widehat{\varphi}_s(\cdot, x_3, \omega)$ and $\widehat{\psi}_s(\cdot, x_3, \omega)$. The next two lemmas are stated without their proof, which is quite straightforward.

Lemma 4.1. *One has the decomposition*

$$\widehat{\varphi}_s(\cdot, x_3, \omega) = \varphi_s^\infty(\cdot, x_3) + \widehat{\varphi}_r(\cdot, x_3, \omega) \quad \text{in } H^1(S_{x_3}), \tag{49}$$

where the real valued function $\varphi_s^\infty(\cdot, x_3)$ is the unique solution of (29) and the "rest" $\widehat{\varphi}_r(\cdot, x_3, \omega)$ satisfies:

$$\forall\, x_3 \in \mathbb{R}, \quad \| \mathbf{grad}\, \widehat{\varphi}_r(\cdot, x_3, \omega) \|_{L^2(S_{x_3})} \ \le \ C(x_3)\, |\omega|^{-1},$$

where $C(x_3)$ is a strictly positive function of x_3 independent of ω.

Remark that $\widehat{\varphi}_r \equiv \widehat{\varphi}_r(\cdot, x_3, \omega) \in H^1(S_{x_3})$ defined by (49) satisfies

$$\begin{cases} \text{div}\,\big((i\omega\varepsilon + \sigma_e)\,\mathbf{grad}\,\widehat{\varphi}_r\big) = -\,\text{div}\,\big(\,\sigma_e\,\mathbf{grad}\,\varphi_s^\infty\,\big) & \text{in } S_{x_3}, \\ \widehat{\varphi}_r = 0, & \text{on } \partial S_{x_3}. \end{cases} \tag{50}$$

Lemma 4.2. *One has the decomposition*

$$\widehat{\psi}_s(\cdot, x_3, \omega) = \psi_s^\infty(\cdot, x_3) + \widehat{\psi}_r(\cdot, x_3, \omega) \quad \text{in } H^1(S_{x_3}^\Gamma), \tag{51}$$

where the real valued function $\psi_s^\infty(\cdot, x_3)$ is the unique solution of (33) and the "rest" $\widehat{\psi}_r(\cdot, x_3, \omega)$ satisfies:

$$\forall\, x_3 \in \mathbb{R}, \quad \|\mathbf{grad}\,\widehat{\psi}_r(\cdot, x_3, \omega)\|_{L^2(S_{x_3})} \;\leq\; C(x_3)\;|\omega|^{-1},$$

where $C(x_3)$ is a strictly positive function of x_3 independent of ω.

Note that $\widehat{\psi}_r \equiv \widehat{\psi}_r(\cdot, x_3, \omega) \in H^1(S_{x_3})$ satisfies $\displaystyle\int_{S_{x_3}} \widehat{\psi}_r\, d\mathbf{x}_T = 0$ and

$$\begin{cases} \text{div}\,\big((i\omega\mu + \sigma_m)\,\mathbf{grad}\,\widehat{\psi}_r\big) = -\text{div}\,\big(\sigma_m\,\widetilde{\mathbf{grad}}\,\psi_s^\infty\,\big) & \text{in } S_{x_3}, \\ \dfrac{\partial \widehat{\psi}_r}{\partial \mathbf{n}} = 0, & \text{on } \partial S_{x_3}. \end{cases} \tag{52}$$

We are now in position to state the main theorem of this section

Lemma 4.3. *The homogenized coefficients defined by* (37)-(40) *satisfy*

$$\begin{aligned} &(i) \quad i\omega\, C(x_3, \omega) + G(x_3, \omega) = i\omega\, C^\infty(x_3) + G^\infty(x_3) + \widehat{k}_e(x_3, \omega), \\ &(ii) \quad i\omega\, L(x_3, \omega) + R(x_3, \omega) = i\omega\, L^\infty(x_3) + R^\infty(x_3) + \widehat{k}_m(x_3, \omega), \end{aligned} \tag{53}$$

where the high frequency limits $\big(\,C^\infty(x_3), L^\infty(x_3), G^\infty(x_3), R^\infty(x_3)\,\big)$ are

$$\begin{aligned} C^\infty(x_3) &\equiv \int_{S_{x_3}} \varepsilon(\mathbf{x}_T, x_3)\;\big|\mathbf{grad}\,\varphi_s^\infty(\mathbf{x}_T, x_3)\big|^2\, d\mathbf{x}_T, \\ L^\infty(x_3) &\equiv \int_{S_{x_3}} \mu(\mathbf{x}_T, x_3)\;\big|\widetilde{\mathbf{grad}}\,\psi_s^\infty(\mathbf{x}_T, x_3)\big|^2\, d\mathbf{x}_T, \\ G^\infty(x_3) &\equiv \int_{S_{x_3}} \sigma_e(\mathbf{x}_T, x_3)\;\big|\mathbf{grad}\,\varphi_s^\infty(x, x_3)\big|^2\, d\mathbf{x}_T, \\ R^\infty(x_3) &\equiv \int_{S_{x_3}} \sigma_m(\mathbf{x}_T, x_3)\;\big|\widetilde{\mathbf{grad}}\,\psi_s^\infty(\mathbf{x}_T, x_3)\big|^2\, d\mathbf{x}_T, \end{aligned} \tag{54}$$

and where the functions $\widehat{k}_e(x_3,\omega)$ *and* $\widehat{k}_m(x_3,\omega)$ *are given by*

$$(i)\quad \widehat{k}_e(x_3,\omega) = \int_{S_{x_3}} \sigma_e(\mathbf{x}_T,x_3)\,\mathbf{grad}\,\varphi_s^\infty(\mathbf{x}_T,x_3)\cdot\mathbf{grad}\,\widehat{\varphi}_r(\mathbf{x}_T,x_3,\omega)\,d\mathbf{x}_T,$$

$$(ii)\quad \widehat{k}_m(x_3,\omega) = \int_{S_{x_3}} \sigma_m(\mathbf{x}_T,x_3)\,\widetilde{\mathbf{grad}}\,\psi_s^\infty(\mathbf{x}_T,x_3)\cdot\mathbf{grad}\,\widehat{\psi}_r(\mathbf{x}_T,x_3,\omega)\,d\mathbf{x}_T, \tag{55}$$

and satisfy, as functions of the frequency:

$$\forall\, x_3\in\mathbb{R},\quad \big(\,\widehat{k}_e(x_3,\cdot),\widehat{k}_m(x_3,\cdot)\,\big)\in L^2(\mathbb{R})^2. \tag{56}$$

Proof The proof follows from a simple calculation resulting from substitution of (49) into (37)-(40). We shall omit the proofs of (53) and (55) which are relatively straighforward. To prove (56), it suffices to prove that $\omega\longrightarrow\widehat{k}_e(x_3,\omega)$ is bounded and decays fast enough for large $|\omega|$. For boundedness, we first use Cauchy-Schartz inequality to get

$$|\,\widehat{k}_e(x_3,\omega)\,| \;\le\; \Big(\int_{S_{x_3}}\sigma_e\,|\,\mathbf{grad}\,\widehat{\varphi}_r\,|^2\,d\mathbf{x}_T\Big)^{\frac12}\,\Big(\int_{S_{x_3}}\sigma_e\,|\,\mathbf{grad}\,\varphi_s^\infty\,|^2\,d\mathbf{x}_T\Big)^{\frac12} \tag{57}$$

and to notice that, from the real part of the following equality

$$\int_{S_{x_3}}(i\omega\varepsilon+\sigma_e)\,|\,\mathbf{grad}\,\widehat{\varphi}_r\,|^2\,d\mathbf{x}_T = -\int_{S_{x_3}}\sigma_e\,\mathbf{grad}\,\varphi_s^\infty\cdot\overline{\mathbf{grad}\,\widehat{\varphi}_r}\,d\mathbf{x}_T. \tag{58}$$

which obtained using (50), and Cauchy-Schwartz inequality, we also have

$$\Big(\int_{S_{x_3}}\sigma_e\,|\,\mathbf{grad}\,\widehat{\varphi}_r\,|^2\,d\mathbf{x}_T\Big)^{\frac12} \;\le\; \Big(\int_{S_{x_3}}\sigma_e\,|\,\mathbf{grad}\,\varphi_s^\infty\,|^2\,d\mathbf{x}_T\Big)^{\frac12}$$

Thus (57) yields : $|\,\widehat{k}_e(x_3,\omega)\,|^2 \;\le\; \displaystyle\int_{S_{x_3}}\sigma_e\,|\,\mathbf{grad}\,\varphi_s^\infty\,|^2\,d\mathbf{x}_T.$

Next, for large ω, we deduce from (57), (58) and the range of variability of σ_e/ε that $|\,\widehat{k}_e(x_3,\omega)\,|\le C(x_3)\,/\,|\omega|$. This completes the proof. □

Remark 4.1. *It is clear on (54) that the homogenized coefficients* $C^\infty(x_3)$ *or* $L^\infty(x_3)$ *are real and positive. It is less clear, although most likely thanks*

to the non-degeneracy property (3) of ε and μ, that they do not degenerate, namely that they are bounded functions of x_3 that are bounded from below by a strictly positive constant. In section 5, to analyze the limit problem in a standard L^2 framework, we shall be led to make such an assumption, namely

Hypothesis 4.1. *There exists $0 < C_-^\infty < C_+^\infty$ and $0 < L_-^\infty < L_+^\infty$ such that*

$$0 < C_-^\infty \leq C^\infty(x_3) \leq C_+^\infty, \quad 0 < L_-^\infty \leq L^\infty(x_3) \leq L_+^\infty, \quad \forall\, x_3 \in \mathbb{R}.$$

Our conjecture is that such assumptions are satisfied "in general", as soon as the holes T_z do not degenerate (cf hypothesis 2.2). The question of finding explicit conditions on ε, μ and the domain Ω that would ensure hypothesis 4.1 is an interesting but delicate issue out of the scope of this paper.

Remark 4.2. *One can check that if the function $\varepsilon(\cdot, x_3)$ and $\sigma_e(\cdot, x_3)$ (respectively $\mu(\cdot, x_3)$ and $\sigma_m(\cdot, x_3)$) are proportional (as in remark 3.1) then $\widehat{k}_e$ and $\widehat{k}_m$ vanish.*

The next property whose proof is obvious (by simply taking the real parts of the equalities (53) and using the definition (39, 40) of $\widehat{G}(x_3, \omega)$ and $\widehat{R}(x_3, \omega)$ as well as inequalities (3)), plays an essential role in the analysis on the limit evolution problem in time domain.

Lemma 4.4. *With $\sigma_{e,-}$ and $\sigma_{m,-}$ given by (3), one has for any (x_3, ω)*

$$G^\infty(x_3) + \mathcal{R}e\left(\widehat{k}_e(x_3,\omega)\right) \;\geq\; \sigma_{e,-} \,||\, \mathbf{grad}\, \widehat{\varphi}_s(\cdot, x_3, \omega) \,||^2_{L^2(S_{x_3})},$$
$$R^\infty(x_3) + \mathcal{R}e\left(\widehat{k}_m(x_3,\omega)\right) \;\geq\; \sigma_{m,-} \,||\, \widetilde{\mathbf{grad}}\, \widehat{\psi}_s(\cdot, x_3, \omega) \,||^2_{L^2(S_{x_3})}.$$

5. The homogenized 1D problem in time-domain

Thanks to lemma 4.3, we can rewrite the homogenized 1D problem in frequency domain (48) as

$$\begin{cases} i\omega\, C^\infty(x_3)\, \widehat{V} + G^\infty(x_3)\, \widehat{V} + \widehat{k}_e(x_3,\omega)\, \widehat{V} + \dfrac{\partial \widehat{I}}{\partial x_3} = \widehat{j} \quad x_3 \in \mathbb{R}, \\ i\omega\, L^\infty(x_3)\, \widehat{I} + R^\infty(x_3)\, \widehat{I} + \widehat{k}_m(x_3,\omega)\, \widehat{I} + \dfrac{\partial \widehat{V}}{\partial x_3} = 0 \quad x_3 \in \mathbb{R}. \end{cases} \tag{59}$$

To come back to time domain, it is natural to introduce the inverse Fourier transforms (in time) of the functions $\widehat{k}_e(x_3, \omega)$ and $\widehat{k}_m(x_3, \omega)$.

5.1. The kernels $k_e(x_3,t)$ and $k_m(x_3,t)$.
Let us introduce by definition the functions:

$$\forall\, x_3 \in \mathbb{R}, \quad \big(\, k_e(x_3,\cdot), k_m(x_3,\cdot)\,\big) \in L^2(\mathbb{R})^2,$$

as the inverse Fourier transforms (in time) of $\widehat{k}_e(x_3,\omega)$ and $\widehat{k}_m(x_3,\omega)$

$$\forall\, x_3 \in \mathbb{R}, \quad \big(k_e(x_3,\cdot), k_m(x_3,\cdot)\big) \equiv \mathcal{F}^{-1}\big(\,\widehat{k}_e(x_3,\cdot), \widehat{k}_m(x_3,\cdot)\,\big).$$

We are going to give below a characterization, directly in time domain, of the functions $k_e(x_3,t)$ and $k_m(x_3,t)$ that will demonstrate in particular that, as expected, these are causal functions. This characterization will also be useful from both theoretical and computational points of view. We first introduce, for each x_3, the bounded operator in $H_0^1(S_{x_3})$:

$$\mathbf{A}_e(x_3) \in \mathcal{L}\,\big(\,\mathbf{V}_e(x_3)\,\big), \quad \mathbf{V}_e(x_3) \equiv H_0^1(S_{x_3})$$

such that, for any $\varphi_e \in \mathbf{V}_e$, $\Phi_e \equiv \mathbf{A}_e(x_3)\,\varphi_e$ if and only if

$$\begin{cases} \operatorname{div}\big(\varepsilon\,\mathbf{grad}\,\Phi_e\big) = -\operatorname{div}\big(\sigma_e\,\mathbf{grad}\,\varphi_e\big) & \text{in } S_{x_3}, \\ \Phi_e = 0, & \text{on } \partial S_{x_3}. \end{cases}$$

The reader will note that the fact that $\mathbf{A}_e(x_3)$ is well defined and bounded is a trivial consequence of the Lax-Milgram lemma, Poincaré's inequality and assumptions (3). Moreover, if $\mathbf{V}_e$ is equipped with the scalar product

$$\big(\,\varphi_e,\varphi_e'\big)_{\mathbf{V}_e(x_3)} := \int_{S_{x_3}} \varepsilon(\mathbf{x}_T,x_3)\;\mathbf{grad}\,\varphi_e(\mathbf{x}_T)\cdot\overline{\mathbf{grad}\,\varphi_e'(\mathbf{x}_T)}\;d\mathbf{x}_T$$

then $\mathbf{A}_e(x_3)$ is positive and self-adjoint since for any $\big(\,\varphi_e,\varphi_e'\big) \in \mathbf{V}_e(x_3)^2$

$$\big(\,\mathbf{A}_e(x_3)\,\varphi_e,\varphi_e'\big)_{\mathbf{V}_e(x_3)} := \int_{S_{x_3}} \sigma_e(\mathbf{x}_T,x_3)\;\mathbf{grad}\,\varphi_e(\mathbf{x}_T)\cdot\overline{\mathbf{grad}\,\varphi_e'(\mathbf{x}_T)}\;d\mathbf{x}_T$$

In the same way, we introduce, for each x_3, the bounded operator

$$\mathbf{A}_m(x_3) \in \mathcal{L}\big(\,\mathbf{V}_m(x_3)\,\big), \quad \mathbf{V}_m(x_3) = \Big\{\,\varphi \in H^1(S_{x_3}) \;/\; \int_{S_{x_3}} \varphi\;dx = 0\,\Big\}$$

such that, for any $\varphi_m \in \mathbf{V}_m(x_3)$, $\Phi_m \equiv \mathbf{A}_m(x_3)\,\varphi_m$ if and only if

$$\begin{cases} \operatorname{div}\big(\mu\,\mathbf{grad}\,\Phi_m\big) = -\operatorname{div}\big(\sigma_m\,\mathbf{grad}\,\varphi_m\big) & \text{in } S_{x_3}, \\ \dfrac{\partial\,\Phi_m}{\partial n} = 0, & \text{on } \partial S_{x_3}. \end{cases}$$

Note that $\mathbf{A}_m(x_3)$ is well defined and bounded as a consequence of Lax-Milgram's lemma, Poincaré-Wirtinger's inequality and assumptions (4.1). Moreover, if $\mathbf{V}_m(x_3)$ is equipped with the scalar product

$$\big(\,\varphi_m, \varphi'_m\big)_{\mathbf{V}_m(x_3)} := \int_{S_{x_3}} \mu(\mathbf{x}_T, x_3)\;\mathbf{grad}\,\varphi_m(\mathbf{x}_T)\cdot\overline{\mathbf{grad}\,\varphi'_m(\mathbf{x}_T)}\;d\mathbf{x}_T.$$

then $\mathbf{A}_m(x_3)$ is positive and self-adjoint since for any $\big(\,\varphi_m, \varphi'_m\big) \in \mathbf{V}_m(x_3)^2$

$$\big(\,\mathbf{A}_m(x_3)\,\varphi_m, \varphi'_m\big)_{\mathbf{V}_m(x_3)} := \int_{S_{x_3}} \sigma_m(\mathbf{x}_T, x_3)\;\mathbf{grad}\,\varphi_m(\mathbf{x}_T)\cdot\overline{\mathbf{grad}\,\varphi'_m(\mathbf{x}_T)}\;d\mathbf{x}_T$$

We also introduce $\big(\varphi_{r,0}(\cdot, x_3), \psi_{r,0}(\cdot, x_3)\big) \in \mathbf{V}_e(x_3)\times\mathbf{V}_m(x_3)$ as the respective solutions of

$$\begin{cases} \operatorname{div}\big(\varepsilon\,\mathbf{grad}\,\varphi_{r,0}\big) = -\operatorname{div}\big(\sigma_e\,\mathbf{grad}\,\varphi_s^\infty\big)\,, & \text{in } S_{x_3}, \\ \varphi_{r,0} = 0\,, & \text{on } \partial S_{x_3}, \end{cases}$$

$$\begin{cases} \operatorname{div}\big(\mu\,\mathbf{grad}\,\psi_{r,0}\big) = -\operatorname{div}\big(\sigma_m\,\widetilde{\mathbf{grad}}\,\psi_s^\infty\big), & \text{in } S_{x_3}\,, \\ \dfrac{\partial\,\psi_{r,0}}{\partial n} = 0\,, & \text{on } \partial S_{x_3}. \end{cases} \tag{60}$$

Finally, for each $x_3 \in \mathbb{R}$, we introduce the functions:

$$t \to \varphi_r(\cdot, x_3, t) \in L^\infty\big(\mathbb{R}, \mathbf{V}_e(x_3)\big), \qquad t \to \psi_r(\cdot, x_3, t) \in L^\infty\big(\mathbb{R}, \mathbf{V}_m(x_3)\big),$$

as the causal functions which are, for $t > 0$ the unique solutions of the evolution problems (x_3 playing the role of a parameter):

$$\begin{cases} \dfrac{d\varphi_r}{dt}\,(\cdot, x_3, t) + \mathbf{A}_e(x_3)\;\varphi_r(\cdot, x_3, t) = 0 \quad \text{for } t > 0, \\ \varphi_r(\cdot, x_3, 0) = \varphi_{r,0}(\cdot, x_3), \end{cases}$$

$$\begin{cases} \dfrac{d\psi_r}{dt}\,(\cdot, x_3, t) + \mathbf{A}_m(x_3)\;\psi_r(\cdot, x_3, t) = 0 \quad \text{for } t > 0, \\ \psi_r(\cdot, x_3, 0) = \psi_{r,0}(\cdot, x_3). \end{cases} \tag{61}$$

Since the operators $\mathbf{A}_e(x_3)$ and $\mathbf{A}_m(x_3)$ are bounded, we know that, for $t > 0$:

$$\begin{aligned} \varphi_r(\cdot, x_3, t) &= e^{-t\,\mathbf{A}_e(x_3)}\ \varphi_{r,0}(\cdot, x_3) \in C^\infty\big(\,\mathbb{R}^+, \mathbf{V}_e(x_3)\,\big), \\ \psi_r(\cdot, x_3, t) &= e^{-t\,\mathbf{A}_m(x_3)}\ \psi_{r,0}(\cdot, x_3) \in C^\infty\big(\,\mathbb{R}^+, \mathbf{V}_m(x_3)\,\big), \end{aligned} \tag{62}$$

and since $\mathbf{A}_e(x_3)$ and $\mathbf{A}_m(x_3)$ are positive self-adjoint, we know that $\varphi_r(\cdot, x_3, t)$ and $\psi_r(\cdot, x_3, t)$ are bounded functions of time and more precisely that

$$t \longrightarrow \|\,\varphi_r(\cdot, x_3, t)\,\|_{\mathbf{V}_e(x_3)} \quad \text{and} \quad t \longrightarrow \|\,\psi_r(\cdot, x_3, t)\,\|_{\mathbf{V}_m(x_3)}$$

are decreasing functions. We state the main result of this paragraph:

Lemma 5.1. *The functions $\varphi_r(\mathbf{x}_T, x_3, t)$ and $\psi_r(\mathbf{x}_T, x_3, t)$ are the inverse Fourier transform in time of the functions $\widehat{\varphi}_r(\mathbf{x}_T, x_3, \omega)$ and $\widehat{\psi}_r(\mathbf{x}_T, x_3, \omega)$ and the functions $k_e(x_3, t)$ and $k_m(x_3, t)$ are given by*

$$\begin{cases} k_e(x_3, t) = \displaystyle\int_{S_{x_3}} \sigma_e(\mathbf{x}_T, x_3)\ \mathbf{grad}\,\varphi_r(\mathbf{x}_T, x_3, t) \cdot \mathbf{grad}\,\varphi_s^\infty(x, x_3)\ d\mathbf{x}_T, \\ \\ k_m(x_3, t) = \displaystyle\int_{S_{x_3}} \sigma_m(\mathbf{x}_T, x_3)\ \mathbf{grad}\,\psi_r(\mathbf{x}_T, x_3, t) \cdot \widetilde{\mathbf{grad}}\,\psi_s^\infty(\mathbf{x}_T, x_3)\ d\mathbf{x}_T. \end{cases} \tag{63}$$

In particular, $k_e(x_3, t)$ and $k_m(x_3, t)$ are causal, C^∞ in time and satisfy

$$\begin{aligned} k_e(x_3, 0) &= -\int_{S_{x_3}} \varepsilon(\mathbf{x}_T, x_3)\ \big|\,\mathbf{grad}\,\varphi_{r,0}(\mathbf{x}_T)\,\big|^2\ d\mathbf{x}_T \le 0, \\ k_m(x_3, 0) &= -\int_{S_{x_3}} \mu(\mathbf{x}_T, x_3)\ \big|\,\mathbf{grad}\,\psi_{r,0}(x_T)\,\big|^2\ d\mathbf{x}_T \le 0. \end{aligned} \tag{64}$$

Proof We only give the proof $\varphi_r(\cdot, x_3, t)$. Taking into account the definition of the operator $\mathbf{A}_e(x_3)$, we deduce from (61) that $\varphi_r(\cdot, x_3, t)$ satisfies

$$\mathrm{div}\,\Big[\,\Big(\varepsilon(\cdot, x_3)\frac{\partial}{\partial t} + \sigma_e(\cdot, x_3)\Big)\,\mathbf{grad}\,\varphi_r(\cdot, x_3, t)\,\Big] = 0, \quad \text{in } S_{x_3} \times \mathbb{R}^+.$$

We deduce that the time Fourier transform of $\varphi_r(\cdot, x_3, t)$, denoted $\widetilde{\varphi}_r(\cdot, x_3, \omega)$ satisfies, using $\varphi_r(\cdot, x_3, 0) = \varphi_{r,0}(\cdot, x_3)$ and (60)

$$\mathrm{div}\,\Big[\,\Big(i\omega\,\varepsilon + \sigma_e\Big)\,\mathbf{grad}\,\widetilde{\varphi}_r\,\Big] = \mathrm{div}\,\Big(\,\varepsilon\,\mathbf{grad}\,\varphi_{r,0}\,\Big) \equiv -\mathrm{div}\,\Big(\,\sigma_e\,\mathbf{grad}\,\varphi_s^\infty\,\Big), \tag{65}$$

as $\widetilde{\varphi}_r(\cdot, x_3, \omega) = 0$ on ∂S_{x_3}, comparing (65) with (50), we conclude by a uniqueness argument that $\widetilde{\varphi}_r(\cdot, x_3, \omega) = \widehat{\varphi}_r(\cdot, x_3, \omega)$. Then, formulas (63) are a consequence of (55) and of the linearity of the Fourier transform, the causality and regularity of $k_e(x_3, t)$ and $k_m(x_3, t)$ follow from the similar properties of $\varphi_r(\cdot, x_3, t)$ and $\psi_r(\cdot, x_3, t)$.

The (easy) proof of (64) is left to the reader. □

5.2. The 1D effective model in time domain

Our goal is to identify the (transverse) limit fields E_T^0 and H_T^0. Denoting $V(x_3, t)$ and $I(x_3, t)$ the inverse Fourier transforms of the functions $\widehat{V}(x_3, \omega)$ and $\widehat{I}(x_3, \omega)$ (introduced in section 3.1), we deduce from (27) and (32), lemmas 4.1, 4.2 and 5.1 as well as (62) that

$$\left\{ \begin{array}{rcl} E_T^0(\mathbf{x}_T, x_3, t) & = & V(x_3, t)\, \mathbf{grad}\, \varphi_s^\infty(\mathbf{x}_T/\delta, x_3) \\ & & + \displaystyle\int_0^t V(x_3, s)\, \mathbf{grad}\, \varphi_r(\mathbf{x}_T/\delta, x_3, t-s)\, ds, \\ H_T^0(\mathbf{x}_T, x_3, t) & = & I(x_3, t)\, \widetilde{\mathbf{grad}}\, \psi_s^\infty(\mathbf{x}_T/\delta, x_3) \\ & & + \displaystyle\int_0^t I(x_3, s)\, \mathbf{grad}\, \psi_r(\mathbf{x}_T/\delta, x_3, t-s)\, ds, \end{array} \right. \tag{66}$$

where the functions φ_s^∞ and ψ_s^∞ are defined by (29, 33) and the functions φ_r and ψ_r by (60, 61). From the properties of the Fourier transform and lemma 5.1, we deduce from the frequency domain equations (59) that $V(x_3, t)$ and $I(x_3, t)$ satisfiy the integro-differential equations, for $x_3 \in \mathbb{R}, t > 0$,

$$\left\{ \begin{array}{l} C^\infty(x_3)\, \dfrac{\partial V}{\partial t} + G^\infty(x_3)\, V + \dfrac{\partial I}{\partial x_3}(x_3, t) + k_e(x_3, \cdot) * V(x_3, \cdot) = j, \\ L^\infty(x_3)\, \dfrac{\partial I}{\partial t} + R^\infty(x_3)\, I + \dfrac{\partial V}{\partial x_3}(x_3, t) + k_m(x_3, \cdot) * I(x_3, \cdot) = 0, \end{array} \right. \tag{67}$$

with zero initial conditions (as a trivial consequence of (6))

$$V(x_3, 0) = 0, \quad I(x_3, 0) = 0. \tag{68}$$

In (67), $*$ holds for time convolution, the convolution kernels $k_e(x_3, t)$ and $k_m(x_3, t)$ are characterized in lemma 5.1 and the "high frequency" homogenized coefficients $C^\infty(x_3), L^\infty(x_3)$ and so on are given by (54). Equation (67)

is a general Telegraphist's equation due to the presence of additional convolution terms whose existence require both losses and heterogeneity of the cross section (see remark 4.2). To guarantee that the equations (66) characterize the limit (rescaled) electric and magnetic fields, it suffices to prove that the evolution problem (67, 68) is well-posed. This is possible under hypothesis 4.1.

Theorem 5.1. *Assume that hypothesis 4.1 holds and* $j \in L^1_{loc}(\mathbb{R}^+, L^2(\mathbb{R}))$. *Then, the evolution problem (67, 68) admits a unique (weak) solution*

$$\big(V, I\big) \in C^0\big(\mathbb{R}^+; L^2(\mathbb{R})\big)$$

that also satisfies the a priori estimate, for any $T > 0$

$$\forall\, t \leq T, \quad \int_{\mathbb{R}} \Big(\,\big|\, V(x_3, T) \,\big|^2 + \big|\, I(x_3, T) \,\big|^2 \,\Big)\, dx_3 \leq C\, \|j\|_{L^1(0,T,L^2(\mathbb{R}))}. \quad (69)$$

We shall not derive the technical details of the proof that can be achieved in different ways, for instance using a Galerkin method as in [4]. The key point is the (formal) derivation of the a priori estimate (69) that relies on a coercivity property of the convolution operators appearing in (67).

Lemma 5.2. *For any* $x_3 \in \mathbb{R}$, *any* $T > 0$ *and any* $\xi(t) \in L^2(\mathbb{R}^+)$, *one has*

$$G^\infty(x_3) \int_0^T |\, \xi(t)\,|^2\, dt \;+ \int_0^T \int_0^t k_e(x_3, t-s)\, \xi(s)\, \xi(t)\, ds\, dt \;\geq\; 0,$$

$$R^\infty(x_3) \int_0^T |\, \xi(t)\,|^2\, dt \;+ \int_0^T \int_0^t k_m(x_3, t-s)\, \xi(s)\, \xi(t)\, ds\, dt \;\geq\; 0.$$

Proof The proof is based on causality arguments and Plancherel's theorem. Lemma 4.4 and Plancherel's theorem. We omit the details.

□

With the help of Lemma 5.2, the formal derivation of the estimate (69) follows a standard energy approach. The model (67) is a lower order perturbation of a 1D wave equation with variable (in x_3) velocity given by

$$c^\infty(x_3) = \Big(C^\infty(x_3)\, L^\infty(x_3) \Big)^{-\frac{1}{2}}.$$

It is easy to determine an upper bound for $c^\infty(x_3)$, from the variable velocity for 3D Maxwell's equation in the domain Ω, namely the bounded function:

$$c(\mathbf{x}_T, x_3) = \Big(\varepsilon(\mathbf{x}_T, x_3)\, \mu(\mathbf{x}_T, x_3) \Big)^{-\frac{1}{2}}.$$

Theorem 5.2.

$$\forall\, x_3 \in \mathbb{R}, \quad c^\infty(x_3) \;\leq\; c^+(x_3) \equiv \sup_{\mathbf{x}_T \in S_{x_3}} c(\mathbf{x}_T, x_3)\,. \tag{70}$$

Proof In the same way that we proved (46)(i), one easily shows that (this can also be obtained by taking the limit in (46)(i) when $\omega \longrightarrow +\infty$):

$$\int_{S_{x_3}} \mathbf{grad}\, \varphi_s^\infty(\mathbf{x}_T, x_3) \cdot \widetilde{\mathbf{rot}}\; \psi_s^\infty(\mathbf{x}_T, x_3)\, d\mathbf{x}_T = 1. \tag{71}$$

Since $\big|\, \mathbf{rot}\; \psi_s^\infty(\mathbf{x}_T, x_3) \,\big| = \big|\, \mathbf{grad}\;\; \psi_s^\infty(\mathbf{x}_T, x_3) \,\big|$, by writing

$$d\mathbf{x}_T = c(\mathbf{x}_T, x_3)\; \varepsilon(\mathbf{x}_T, x_3)^{\frac{1}{2}}\; \mu(\mathbf{x}_T, x_3)^{\frac{1}{2}}\; d\mathbf{x}_T$$

and using Cauchy-Schwartz inequality in (71), we find

$$1 \;\leq\; c^+(x_3)\;\; C^\infty(x_3)^{\frac{1}{2}}\; L^\infty(x_3)^{\frac{1}{2}}$$

that is to say (70). □

This theorem says that waves in the 1D limit model cannot propagate more rapidly that in the original 3D domain. It appears much less obvious to obtain a lower bound for $c^\infty(x_3)$, for instance $c^\infty(x_3) \geq c^-(x_3)$, the infimum of $c(\mathbf{x}_T, x_3)$ in the cross-section S_{x_3}, which we expect to be true.

Remark 5.1. *In the case of homogeneous cross sections (Example 2.3), using remark 3.3), it is easy to show that*

$$c^\infty(x_3) = \Big(\, \varepsilon(x_3)\, \mu(x_3) \,\Big)^{-\frac{1}{2}}.$$

5.3. A "simple" case of application.

We conclude this section by a simple case of application for which everything can be made much more explicit. It is the case where each cross section S_{x_3} is made of two concentric homogeneous annuli (see Figure 2 right picture). In what follows everything may depend on x_3 but, for simplicity of notation, we shall omit to mention the x_3 dependency. More precisely, let $\alpha_1 > 1$ and $\alpha_2 > 1$, $S(\equiv S_{x_3})$ is defined by $\overline{S} = \overline{S}_1 \cup \overline{S}_2$,

$$S_1 = \big\{\, \mathbf{x}_T \in \mathbb{R}^2 \;\; / \;\; 1 < r < \alpha_1 \,\big\}\,, \qquad S_2 = \big\{\, \mathbf{x}_T \in \mathbb{R}^2 \;\; / \;\; \alpha_1 < r < \alpha_2\, \alpha_1 \,\big\}\,,$$

where we have used the polar coordinates (r, θ) in the (x_1, x_2) plane. We assume that the physical coefficients satisfy

$$\big(\varepsilon, \mu, \sigma_e, \sigma_m\big)(x) = (\varepsilon_1, \mu_1, \sigma_{e,1}, \sigma_{m,1}) \text{ in } S_1, \quad (\varepsilon_2, \mu_2, \sigma_{e,2}, \sigma_{m,2}) \text{ in } S_2.$$

In such situation it is natural to look for solutions $(\varphi_s^\infty, \psi_s^\infty)$ of problem (29) and (33) that are independent of the variable θ which leads to solving simple ODE's in the r variable. After some computations, we find

$$\mathbf{grad}\, \varphi_s^\infty(r, \theta) = -\frac{\varepsilon_j}{\varepsilon_m}\, \frac{\mathbf{e}_r}{r} \quad \text{in } S_j \ , \ j = 1, 2, \ \ \mathbf{grad}\, \psi_s^\infty(r, \theta) = -\frac{\mathbf{e}_\theta}{2\pi\, r} \text{ in } S.$$

where $(\mathbf{e}_r, \mathbf{e}_\theta)$ is the usual orthonormal basis associated to polar coordinates and ε_m is given by: $\varepsilon_m = \varepsilon_2 \ln\, \alpha_1 + \varepsilon_1 \ln\, \alpha_2$. Let us emphasize that $\mathbf{grad}\, \psi_s^\infty$ is constant in all of S and independent of μ: this is specific to the circular symmetry. Using the above formulas in equation (54), we get the expression of the homogenized coefficients. The "magnetic" ones, L^∞ and R^∞, are obtained, up to the multiplicative constant $2\pi^{-1}$ by a weighted average on μ and σ_m

$$L^\infty = \frac{1}{2\,\pi}\Big(\, \mu_1 \ln\, \alpha_1 + \mu_2 \ln\, \alpha_2 \,\Big), \qquad R^\infty = \frac{1}{2\,\pi}\Big(\, \sigma_{m,1} \ln\, \alpha_1 + \sigma_{m,2} \ln\, \alpha_2 \,\Big).$$

Concerning the "electric" coefficients (C^∞, G^∞), this is slightly more complicated since the quantities that are averaged in a similar way are non linear functions of ε and σ_e

$$(C^\infty)^{-1} = \frac{1}{2\,\pi}\Big(\, \varepsilon_1^{-1} \ln\, \alpha_1 \, + \varepsilon_2^{-1} \ln\, \alpha_2 \,\Big),$$
$$G^\infty\, (C^\infty)^{-2} = \frac{1}{2\,\pi}\Big(\, \sigma_{e,1}\, \varepsilon_1^{-2} \, \ln\, \alpha_1 + \sigma_{e,2}\, \varepsilon_2^{-2} \, \ln\, \alpha_2 \,\Big).$$

The reader will observe that the squared velocity for the homogenized model, c^∞, appears as a weighted arithmetic average of the squared velocities of each homogeneous media

$$(c^\infty)^2 \ = \ \frac{\mu_1\, c_1^2 \ln\, \alpha_1 + \mu_2\, c_2^2 \ln\, \alpha_2}{\mu_1 \ln\, \alpha_1 + \mu_2 \ln\, \alpha_2}, \quad c_1^2 = (\varepsilon_1\, \mu_1)^{-1}, \quad c_2^2 = (\varepsilon_2\, \mu_2)^{-1}.$$

In particular, as expected: $\min\, (c_1, c_2) \ \leq \ c^\infty \ \leq \ \max\, (c_1, c_2)$. Finally, one can compute explicitly the electric and magnetic kernels $k_e(t)$ and $k_m(t)$ from

(63) after having determined analytically (φ_r, ψ_r), the solutions of (61). This is easily done, again, by looking (φ_r, ψ_r) as functions of r and θ. It can be shown that $k_e(t) = k_{e,0}\, e^{-t/\tau_e}, \quad k_m(t) = 0$, where

$$k_{e,0} = 2\pi \ln \alpha_1 \, \ln \alpha_2 \, \frac{\big(\, \sigma_{e,2}\, \varepsilon_1 - \sigma_{e,1}\, \varepsilon_2 \,\big)^2}{\varepsilon_m^3}, \quad \tau_e = \frac{\big(\, \varepsilon_1 / \ln \alpha_1 \,\big) + \big(\, \varepsilon_2 / \ln \alpha_e \,\big)}{\big(\, \sigma_{e,1} / \ln \alpha_1 \,\big) + \big(\, \sigma_{e,2} / \ln \alpha_2 \,\big)}.$$

Here again, the fact that the magnetic kernel $k_m(t)$ vanishes is specific to circular symmetry. Moreover, note that if $\sigma_{e,2}/\varepsilon_2 = \sigma_{e,1}/\varepsilon_1$, $k_e(t)$ also vanishes: this is a particular case of the remark 3.1.

Note also that $k_e(t)$ decays exponentially, which corresponds to a "short" memory effect, as soon as $\sigma_{e,1} > 0$ or $\sigma_{e,2} > 0$. Moreover, the fact that $k_e(t)$ is reduced to one simple exponential allows us to rewrite (67) as the coupling between the usual telegraphist's equation coupled, via the introduction of an auxiliary variable W (that will represent the nonlocal effects) to a simple linear ordinary differential equation (we reintroduce here the dependency of the coefficients with respect to x_3): the equations (72) can be written under the convenient form, $x_3 \in \mathbb{R}, t > 0$,

$$\left\{ \begin{array}{l} C^\infty(x_3)\ \dfrac{\partial V}{\partial t}(x_3,t) \,+\, G^\infty(x_3)\ V(x_3,t) \\ \qquad\qquad\qquad\qquad + \dfrac{\partial I}{\partial x_3}(x_3,t) \,+\, k_{e,0}(x_3)\, W(x_3,t) \,=\, j(x_3,t), \\ L^\infty(x_3)\ \dfrac{\partial I}{\partial t}(x_3,t) \,+\, R^\infty(x_3)\ I(x_3,t) \,+\, \dfrac{\partial V}{\partial x_3}(x_3,t) = 0, \\ \tau_e(x_3)\ \dfrac{\partial W}{\partial t}(x_3,t) \,+\, W(x_3,t) \,=\, \tau_e(x_3)\ V(x_3,t). \end{array} \right. \tag{72}$$

We conclude this section by presenting some numerical simulations realized with the above model. For simplicity we consider the problem with constant coefficients in x_3 (which corresponds to the pure cylindrical case), posed in a finite segment $[0, \ell\,]$ (with $\ell = 10$) with periodic boundary conditions to simulate infinity. We consider a gaussian source with a so called "Ricker" pulse in time: $j(x_3,t) = g(x_3)\ r(t - t_0)$ with

$$g(x_3) \equiv e^{-\gamma\,(x_3-5)^2}, \quad r(t) \equiv 2\,\pi^2 \big((t-1)^2 - 1\big)\, e^{-\pi^2 \left((t-1)^2 - 1\right)},$$

where we have chosen, arbitrarily $\gamma = -4\,\ln(10^{-20})$ and $t_0 = 1/10$. We consider the particular case where the only source of heterogeneity of the

cross section is the presence of electric conductivity in the outer layer:

$$\varepsilon_1 = \varepsilon_2 = \mu_1 = \mu_2 = 1, \quad \sigma_{m,2} = \sigma_{m,2} = \sigma_{e,1} = 0, \quad \sigma_{e,2} = 0.5$$

and

$$\alpha_1 = 8/5, \quad \alpha_2 = 5/4.$$

The dissipation phenomena are thus governed by the two following time constants, setting $\tau_\infty \equiv \left(C^\infty / G^\infty \right)$,

$$\tau_\infty = 10/3 = 3.3333, \quad \tau_e \simeq 2.9495.$$

To evaluate in this case the influence on the nonlocal effects induced by the heterogeneity of the cross section we compare a numerical simulation without these effects, by taking $k_{e,0} \equiv 0$ with a numerical simulation where $k_{e,0}$ is given its actual value namely:

$$k_{e,0} \simeq -0.4946.$$

The computations are made up to the final time $T_f = 40$, which permits to the electromagnetic waves, that propagate at speed 1, to travel along 4 times the distance ℓ. On Figure 3, we represent snapshots of the solutions (more precisely the electric potential, at a given time, as a function of $x_3 \in [0, \ell\,]$) at four instants near the final times T_f. The red curves correspond to the simulation without the non local term and the blue curves correspond to the full model. In the absence of the non local terms, one simply observes a decay of the amplitude of the wave without any distortion. With the full model, in addition to the pure dissipation, we observe a change of the shape of the pulse during its propagation: this is what could be called a "dispersive dissipation" phenomenon.

6. Conclusion

In this paper, we have derived 1D limit models for the propagation of electromagnetic waves in thin co-axial cables by means of asymptotic analysis of the original 3D model with respect to the small transverse dimensions of the cable. The resulting model is a generalized telegraphist's model whose coefficients are determined from the solutions of 2D scalar problems in normalized cross sections. In the case of heterogeneous cross sections involving lossy

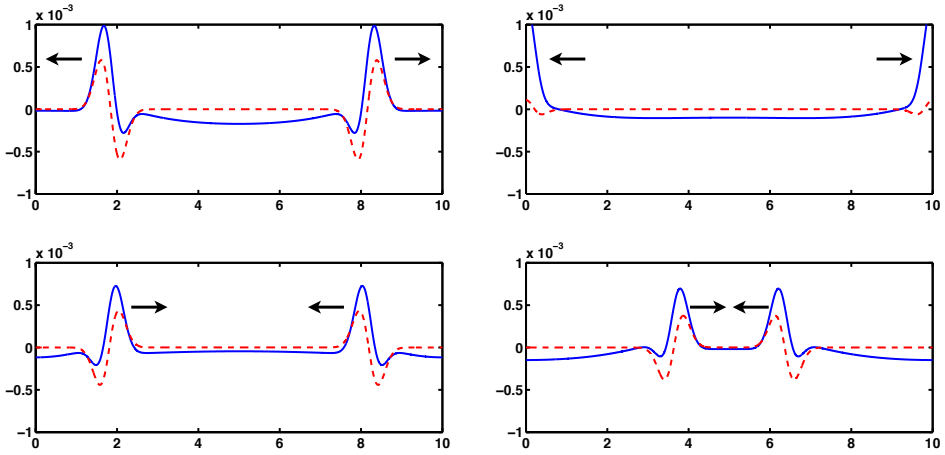


Figure 3: Snapshots of the electric potential $V(x_3, t)$ for $k_{e,0} \equiv 0$ (dashed line) or $k_{e,0} \simeq -0.4946$ at time $t \simeq 34.33$ (upper left), $t \simeq 36.16$ (upper right), $t \simeq 37.98$ (lower left) and $t \simeq 39.81$ (lower right). To help the reader, the arrows indicate the propagation direction of the pulses.

media, additional nonlocal time convolution terms appear in the equations of the model. The next step towards the complete mathematical justification of this approximate model is the derivation of error estimates, which is the object of a forthcoming work. Among the desirable enrichments of this model let us quote:

- the derivation of more accurate (higher order) models: such models should include weak dispersive effects.
- the modeling of the junctions between several cables: the objective is to justify or generalize Kirchhoff conditions (in the spirit of [9] and [10]).
- the introduction of small defects in both the geometry and the material properties of the cable.

From a more applied point of view, another objective is the development of robust numerical methods for the solution of the limit model in the presence of convolutions term, in a more general situation than the one considered in section 5.3.

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