

Reconstruction of Independent sub-domains in a Hamilton-Jacobi Equation and its Use for Parallel Computation

Adriano Festa

▶ **To cite this version:**

Adriano Festa. Reconstruction of Independent sub-domains in a Hamilton-Jacobi Equation and its Use for Parallel Computation. NETCO 2014 - New Trends in Optimal Control, Jun 2014, Tours, France. hal-01025112

HAL Id: hal-01025112
https://inria.hal.science/hal-01025112

Submitted on 17 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

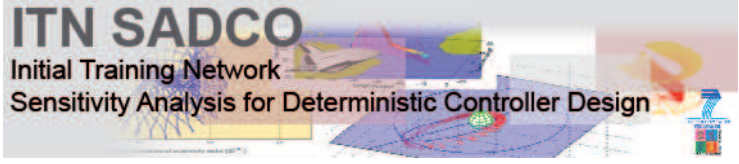
Reconstruction of Independent sub-domains in a Hamilton-Jacobi Equation and its Use for Parallel Computation

Adriano Festa

UMA, ENSTA-ParisTech.

25th June 2014

NetCo 2014, Tours, France.



Introduction

Patchy Decomposition

- Cacace, Cristiani, Falcone and Picarelli, *A patchy dynamic programming scheme for a class of Hamilton-Jacobi-Bellman equation*, SIAM J. Sc. Comp. (2012)

- *Reconstruction of some "Sub-Domains of Invariance" through the resolution of the problem on a course grid passing by the synthesis of the controls*

- **Goal:** solve the problem in parallel on a fine grid
- Good point: Some cases of interest where this idea works well
- **Open questions:**
  - Convergence, error introduced,
  - Extension of this idea to a wider class of problems

Patchy Decomposition: example

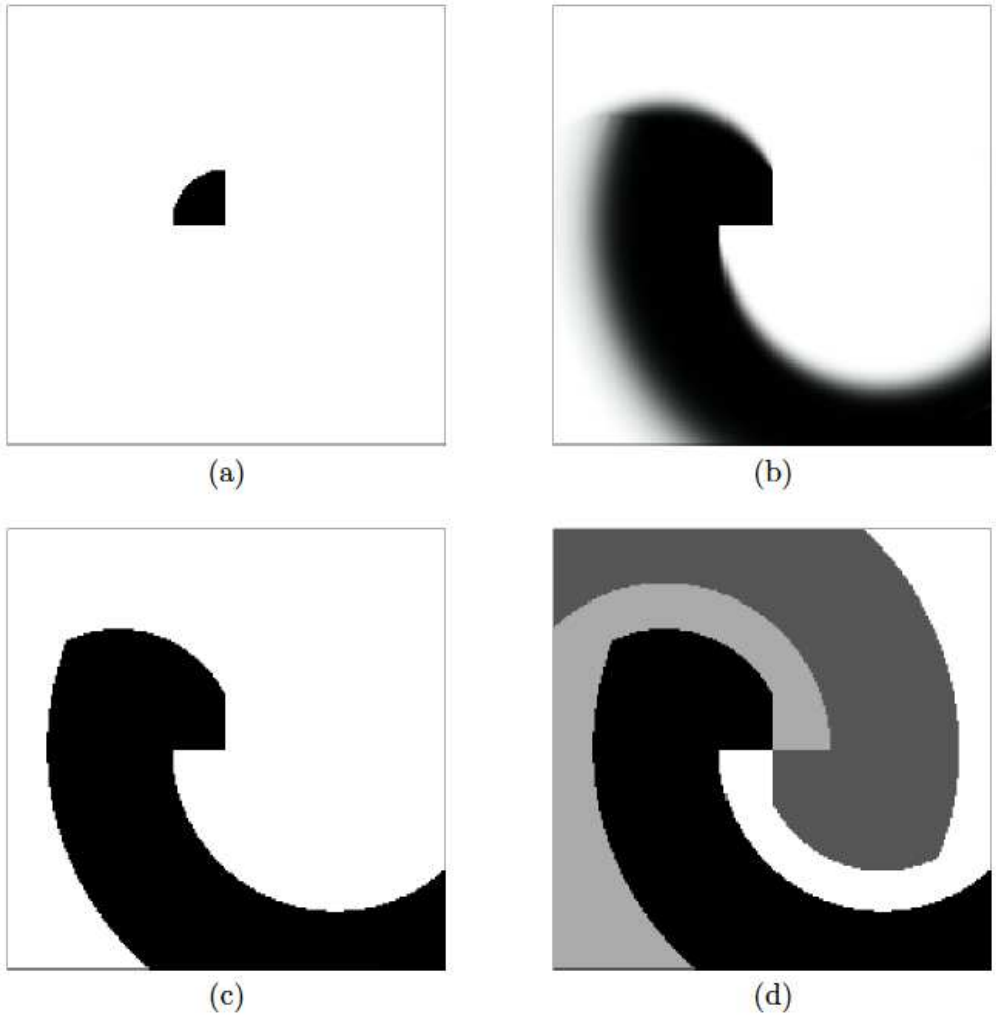


Figure: Some steps of the *Patchy Algorithm* (thanks to the authors)

Decomposition for Differential Games

(Deduced from F. Vinter, preprint 2014)

Let us consider, for an H not necessarily convex

$$\begin{cases} \lambda v(x) + H(x, Dv(x)) = 0 & x \in \Omega \\ v(x) = g(x) & x \in \Gamma \end{cases}$$

Considered a **decomposition of the boundary** $\Gamma := \bigcup_{i \in \mathbb{I}} \Gamma_i$, with $\mathcal{I} := \{1, ...m\} \subset \mathbb{N}$, we call $v_i : \bar{\Omega} \to \mathbb{R}$ a Lipschitz continuous viscosity solution of the problem

$$\begin{cases} \lambda v_i(x) + H(x, Dv_i(x)) = 0 & x \in \Omega \\ v_i(x) = g_i(x) & x \in \Gamma \end{cases}$$

where the functions $g_i : \Gamma \to \mathbb{R}$ is a regular function such that

$$g_i(x) = g(x), \text{ if } x \in \Gamma_i,$$
$$g_i(x) > g(x), \text{ otherwise.}$$

Define

$$\begin{aligned} I(x) &:= \{i \in \{1,\dots,m\} | v_i(x) = \min_j v_j(x)\}, \\ \Sigma &:= \{x \in \mathbb{R}^N | card(I(x)) > 1\}. \end{aligned}$$

Theorem

Assume the following condition satisfied: for arbitrary $x \in \Sigma$, any convex combination $\{\alpha_i \,|\, i \in I(x)\}$ and any collection of vectors $\{p_i \in \partial^L v_i(x) \,|\, i \in I(x)\}$ we have

$$\lambda \bar{v} + H\left(x, \sum_i \alpha_i p_i\right) \leq 0\,. \tag{E}$$

Then, for all $x \in \mathbb{R}^N \setminus \mathcal{T}$,

$$v(x) \;=\; \bar{v}(x) \;:=\; \min_i \{v_1(x),\dots,v_m(x)\}.$$

Example (I):

The equation considered is, with $\Omega := (-1,1) \times (-1,1)$,

$$\begin{cases} \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1 & x \in \Omega \\ v(x) = 0 & x_1 \in \partial\Omega; \end{cases}$$

(evidently the solution is the **distance function** from the boundary), we associate, considered to the easiest division $\partial\Omega = \Gamma := \cup_i \Gamma_i = \cup[(\pm 1, \pm 1), (\pm 1, \pm 1)]$, to the $\Gamma_1 := [(-1,-1),(-1,1)]$ the function $g_1 : \partial\Omega \to \mathbb{R}$ defined as

$$\begin{cases} g_1(x) := 0 & x \in \Gamma_1 \\ g_1(x) := \gamma(1 + x_2) & x \in \Gamma \setminus \Gamma_1; \end{cases}$$

with $\gamma > 0$.

Example (II):

The unique viscosity solution of such a problem is

$$v_1(x) = (1+\gamma) - \max(|x_1 - \gamma|, |x_2|)$$

and finally, the original value function $v(x) = 1 - \max(|x_1|, |x_2|)$ is recovered as $v(x) = \min_{i=1,\ldots,4} v_i(x)$.

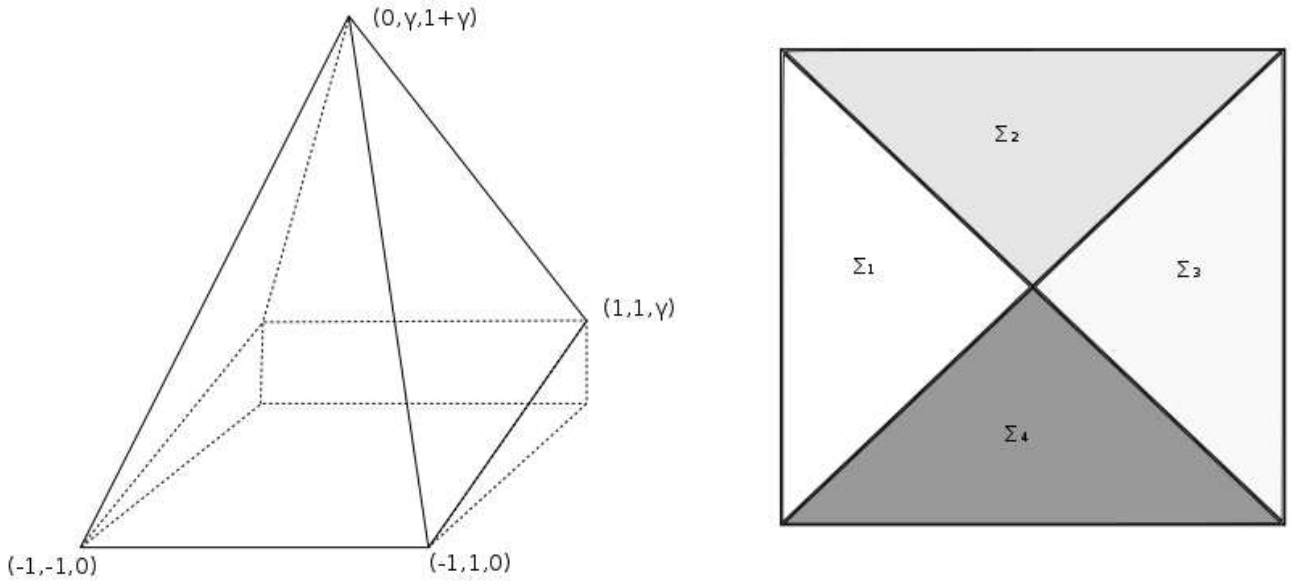


Figure: the auxiliary solution v_1 and the "regions of coincidence".

Independent subdomains reconstruction

Differential Games Problem

Let the dynamics be given by

$$\begin{cases} \dot{y}(t) = f(y(t), a(t), b(t)), & a.e. \\ y(0) = x, \end{cases}$$

$x \in \Omega \subseteq \mathbb{R}^n$ open, $a, b \in \mathcal{A}, \mathcal{B} = \{\mathbb{R}^+ \to A,\ measureable\}$, A, B compact sets. A solution is a trajectory $y_x(t, a(t), b(t))$.

The goal is to find the $sup - inf$ **optimum** over $\mathcal{A}, \mathcal{B}$ of

$$J_x(a, b) := \int_0^{\tau_x(a,b)} l(y_x(s, a(s), b(s)), a(s), b(s)) e^{-\lambda s} ds + e^{-\lambda \tau_x(a,b)} g(y_x(\tau_x(a, b))), \quad \lambda \geq 0,$$

where $\tau_x(a, b) := \min \{t \in [0, +\infty) \mid y_x(t, a(t), b(t)) \notin \Omega\}$.

the value function of this problem is

$$v(x) := \sup_{\phi \in \Phi} \inf_{a \in \mathcal{A}} J_x(a, \phi(a)),$$

$$\Phi := \{\phi : \mathcal{A} \to \mathcal{B} : t > 0, a(s) = \tilde{a}(s) \textbf{ for all } s \leq t \textbf{ implies } \phi[a](s) = \phi[\tilde{a}](s) \textbf{ for all } s \leq t\}.$$

we will assume the *Isaacs' conditions* verified.

Theorem

The value function of the problem is a viscosity solution of the HJ equation associated with

$$H(x, p) := \min_{b \in \mathcal{B}} \max_{a \in \mathcal{A}} \{-f(x, a, b) \cdot p - l(x, a, b)\}.$$

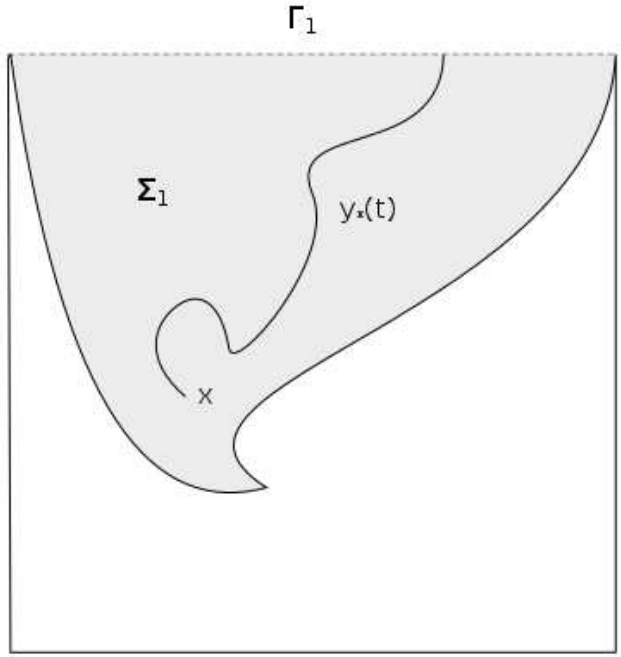
Independent Sub-Domains

Definition

A closed subset $\Sigma \subseteq \overline{\Omega}$ is an independent sub-domain of the problem (11) if, given a point $x \in \Sigma$ and an **optimal control** $(\bar{a}(t), \bar{\phi}(\bar{a}(t))$

(i.e. $J_x(\bar{a}, \bar{\phi}(\bar{a})) \leq J_x(a, \bar{\phi}(a))$ for every choice of $a \in \mathcal{A}$, and $J_x(\bar{a}, \bar{\phi}(\bar{a})) \geq J_x(\bar{a}, \phi(\bar{a}))$ for every choice of $\phi \in \Phi$),

the trajectory $y_x(\bar{a}(t), \bar{\phi}(\bar{a}(t))) \in \Sigma$ for $t \in [0, \tau_x(\bar{a}, \bar{\phi}(\bar{a}))]$.



Independent Domains Decomposition

Proposition

Given a collection of $n-1$ dimensional subsets $\{\Gamma_i\}_{i=\mathcal{I}}$ such that $\Gamma = \cup_{i=1}^{m}\Gamma_i$, the sets defined as

$$\Sigma_i := \left\{x \in \overline{\Omega} | v_i(x) = v(x)\right\}, \quad i = 1, ..., m$$

where v_i, v are defined accordingly to Theorem (1)*, are independent sub-domains of the original problem.*

Proof.

By contradiction using the DPP. □

Numeric framework, SemiLagrangian scheme

Consider a **structured grid of simplices** S_j, such that $\overline{\Omega} \in \cup_j S_j$, denoting x_m, $m = 1, ..., N$, the nodes of the triangulation,

$$\Delta x := \max_j diam(S_j)$$

G is *internal nodes*, ∂G *boundary points*, Ψ *ghost nodes*.

Mapping the values at the nodes onto $V = (V_1, ..., V_N)$, it is possible to obtain the following scheme in **fixed point form**

$$V = T(V),$$

$$[T(V)]_i = \begin{cases} \max_{b \in B} \min_{a \in A} \left\{ \frac{1}{1+\lambda h} \mathbb{I}[V](x_i - hf(x_i, a, b)) - hl(x_i, a, b) \right\} & x_i \in G, \\ g(x_i) & x_i \in \partial G, \\ +\infty & x_i \in \Psi. \end{cases}$$

An estimate needed

Theorem

Let v and V be the solutions of the continuous and the discrete problem. Under some regularity hypotheses

$$||v - V||_\infty \leq C(\Delta x)^q,$$

where C is a positive constant independent from Δx, $q \in \mathbb{R}^+$ depending on the regularity of the problem.

An estimate needed

Theorem

Let v and V be the solutions of the continuous and the discrete problem. Under some regularity hypotheses

$$||v - V||_\infty \leq C(\Delta x)^q,$$

where C is a positive constant independent from Δx, $q \in \mathbb{R}^+$ depending on the regularity of the problem.

Examples in the SL case: differential games with a Lipschitz solution

$$\|v - V\|_\infty \leq Ch^{\frac{1}{2}} \left(1 + \left(\frac{\Delta x}{h}\right)^2\right),$$

An estimate needed

Theorem

Let v and V be the solutions of the continuous and the discrete problem. Under some regularity hypotheses

$$||v - V||_\infty \leq C(\Delta x)^q,$$

where C is a positive constant independent from Δx, $q \in \mathbb{R}^+$ depending on the regularity of the problem.

Examples in the SL case: differential games with a Lipschitz solution

$$\|v - V\|_\infty \leq Ch^{\frac{1}{2}} \left(1 + \left(\frac{\Delta x}{h}\right)^2\right),$$

Optimal Control problem with $\lambda > 0$:

$$\|v - V\|_\infty \leq 2(M_v + M_{v_h})h^{\frac{1}{2}} + \left(\frac{L_l}{\lambda(\lambda - L_f)}\frac{\Delta x}{h}\right).$$

A necessary condition

Proposition

Assumed $x \in \Omega$ such that $v_i(x) = v(x)$ for a certain $i \in \mathbb{I}$, an $\epsilon \in [0, \Delta x)$ and a direction $d \in B(0,1)$, such that, for a x_j node of the grid G, $x_j = x + \epsilon d$ and $v(x_j) < v_i(x_j)$; then

$$|V_i(x_j) - V(x_j)| \leq 2(C(\Delta x)^q + M\Delta x)$$

C as in the previous statement and $M := \max\{L_{v_i}, i \in \mathcal{I}\}$ where L_{v_i} is the Lipschitz constant of the function v_i.

Independent Subdomains Reconstruction

INDEPENDENT SUB-D. RECONSTRUCTION ALGORITHM (IRA).

- Given a collection of indices vectors such that $union(x_i, i = 1, ..., m) = y$

1) (Resolution of auxiliary problems)
for $i = 1...m$ solve iteratively the problem
$V_i = T_i(V_i)$ with $\partial G := \{x_j | j \in y_i\}$
end

2) **(Reconstruction of the value function)**
obtain V as $V = \min_{i=1...m}\{V_i\}$

3) (Reconstruction of the Sub-Domains)
for i=1...m
initialize $w_i = \emptyset$
for j=1...N
if $\| V_i(j) - V(j) \| \leq \varepsilon(\Delta x)$
then add j to vector w_i
end
the index set of the $i-$subset is w_i
end

Some Observations

(IRA) builds an approximation of the independent sub-domains. Moreover we have:

- the approximation **exceeds the desired** set, i.e. $\Sigma_i \subseteq \overline{\Sigma}_i$.
- The same relation holds for the **projection of the approximation** on a finer grid, i.e. for $\Delta x_1 \geq \Delta x_2$,

$$Proj_{\Delta x_2} \overline{\Sigma}_i^{\Delta x_1} \supseteq \Sigma_i$$

Example of Reconstruction (I)

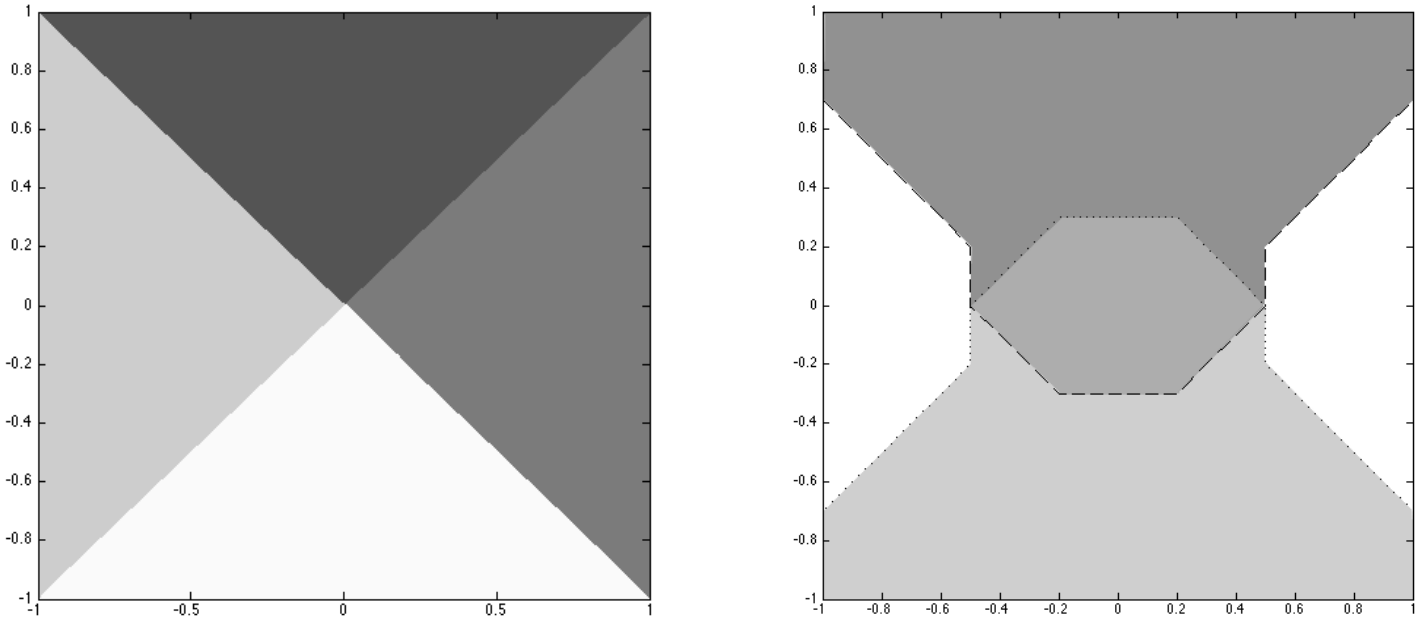


Figure: Distance function: Exact decomposition and two (of the four) approximated independent subsets found with a course grind of 15^2 points.

Example of Reconstruction (II)

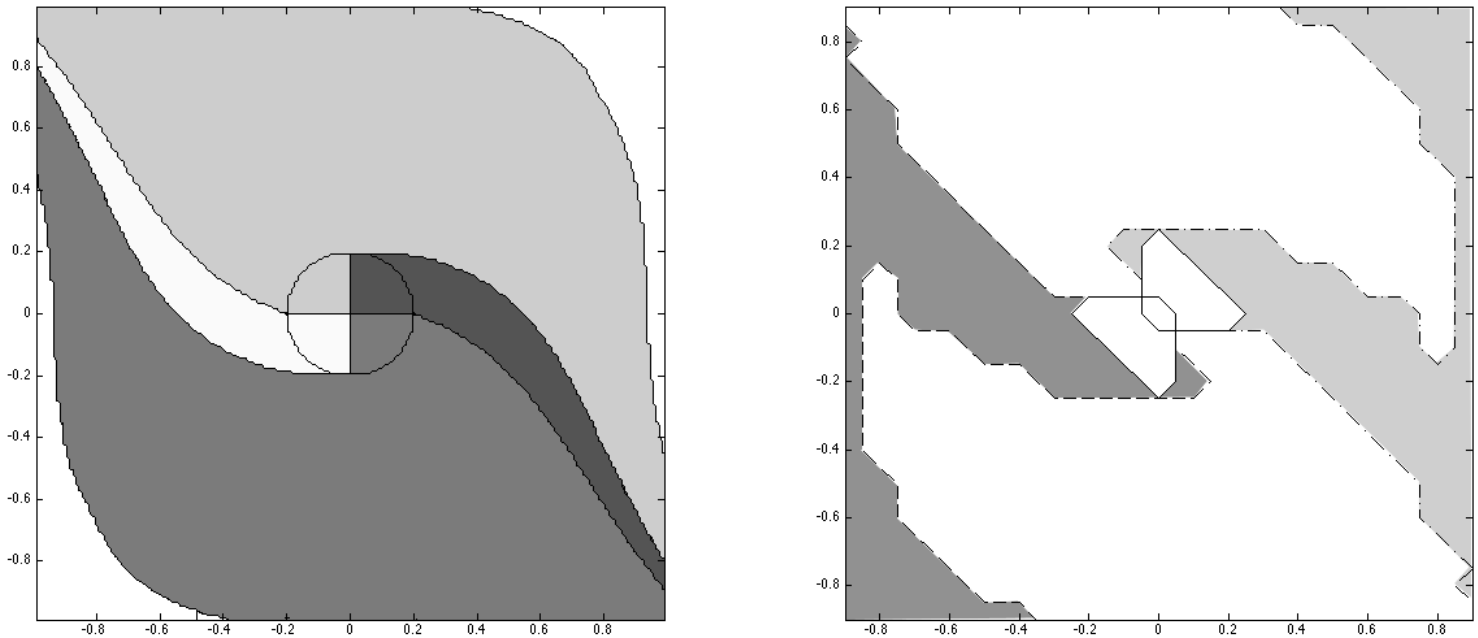


Figure: Van Der Pol: Exact decomposition and two (of the four) approximated independent subsets with a course grind of 15^2 points.

Use for Parallel Computing

Consider:

- a coarse grid K of discretization step Δx_k
- a **fine grid** G of step Δx_G

on the domain Ω.
The two triangulations are chosen such that $\Delta x_G \ll \Delta x_K$.

Independent Sub-Domain Parallelization

INDEPENDENT-SETS ALGORITHM (ISA).

- Given a collection of indices vectors such that $union(z_i, i = 1, ..., M) = y$

1) (Reconstruction of the approximated independent sub-domains)
Using IRA get a collection w_i, $i = 1, ..., M$ of indexes sets.

2) (Projection on the fine grid)
Extend indexes sets w_i on the fine grid G_j getting W_i

3) (Resolution on the fine grid)
for $i = 1, ..., M$ solve iteratively the problem

$$V_i = T(V_i)$$

end

4) (Assembly of the final Solution)
for $j = 1, ..., N_2$

$$\overline{V}(x_j) = min\{V_i \text{ s.t. } j \in w_i\}$$

end

Convergence Result:

Proposition

*Called $\overline{V}$ the **exact discrete solution** of the (ISA) algorithm (i.e. all $V_i = T_i(V_i)$ are verified exactly) and v the exact solution of the **original continuous problem**, then, for a $C > 0$, independent from Δx, the following estimate holds*

$$\|\overline{V} - v\|_\infty \leq C(\Delta x_G)^q.$$

Convergence Result:

Proposition

*Called $\overline{V}$ the **exact discrete solution** of the (ISA) algorithm (i.e. all $V_i = T_i(V_i)$ are verified exactly) and v the exact solution of the **original continuous problem**, then, for a $C > 0$, independent from Δx, the following estimate holds*

$$\|\overline{V} - v\|_\infty \leq C(\Delta x_G)^q.$$

Proof.

We know that the inclusion property holds. Then for $x_j \in G$, it is assured the existence of $i \in \{1, ..., M\}$ s.t. $v(x_j) = v_i(x_j)$, and $|\overline{V}(x_j) - V_i(x_j)| = 0$. Then

$$|\overline{V}(x_j) - v(x_j)| \leq |\overline{V}(x_j) - V_i(x_j)| + |V_i(x_j) - v(x_j)| \leq \|V_i - v_i\|_\infty \leq C(\Delta x_G)^q$$

for the arbitrariness of the choice of x_j we have the thesis. □

A convergence result for a domain partition:

Corollary

Let us consider a partition of the grid K in subsets B_i $i = 1, ..., M$ such that for every $i = 1, ..., M$

- $Z_i \subseteq B_i$
- $\forall z_j \in B_i$, $|V_i(j) - V(j)| \leq 2(C(\Delta x_K)^q + M\Delta x_K)$
- *each B_i is connected (for each $z_j \in B_i$, there exists a direction d s.t. for $r \in [0, \Delta x_k]$, $z_j + dr = z_{\bar{j}} \in B_i$ with $j \neq \bar{j}$)*

then the following estimate for $\hat{V} := \min\{V_i(j) | x_j \in B_i\}$, where the V_i is calculated on the projection on G of the set B_i:

$$\|\hat{V} - v\|_\infty \leq C\Delta x_G + 2(C(\Delta x_K)^q + M\Delta x_K)$$

Tests: Distance function

Eikonal equation on the set $\Omega := (-1,1)^2$ with the boundary value fixed to zero on $\Gamma := \partial\Omega$.
$\lambda = 1$ (non linear monotone scaling of the solution);
the correct viscosity solution is

$$v(x) + \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1, \qquad v(x) = 1 - \frac{\min\{e^{|x_1|}, e^{|x_2|}\}}{e}.$$

N. of variables	Δx	Time elapsed	$\max_i \lvert\overline{\Sigma}_i\rvert/\lvert\Omega\rvert$	$\max_i \lvert\Sigma_i\rvert/\lvert\Omega\rvert$
5^2	0.4	$1 \cdot 10^{-3}$	50%	
7^2	0.28	$2 \cdot 10^{-3}$	43%	
10^2	0.2	$4 \cdot 10^{-3}$	38%	
15^2	0.133	$2 \cdot 10^{-2}$	35%	25%
20^2	0.1	$5 \cdot 10^{-2}$	33%	
40^2	0.05	3	29%	
50^2	0.04	11	28.3%	

Tests: Distance function

Table: Time necessary for the resolution on the original domain and the reduced one with various discretization steps (15^2 (IRA))

N. of variables	Δx	Time on Ω	Time on $\max_i \lvert\bar{\Sigma}_i\rvert$
25^2	0.08	0.13	0.035
50^2	0.04	7.02	0.68
75^2	0.026	57.5	4.8
100^2	0.02	$1.5 \cdot 10^3$	16.6
200^2	0.01	–	$3 \cdot 10^3$

Table: Approximation error Error in norm $\Delta\infty$ (and Δ_1)

	50^2	100^2	200^2
original	$1.2 \cdot 10^{-2}(1.1 \cdot 10^{-2})$	$6.5 \cdot 10^{-3}(3.6 \cdot 10^{-3})$	$2.5 \cdot 10^{-3}(1.6 \cdot 10^{-3})$
2-subs.	$1.2 \cdot 10^{-2}(7.2 \cdot 10^{-3})$	$6.5 \cdot 10^{-3}(3.7 \cdot 10^{-3})$	$2.5 \cdot 10^{-3}(1.4 \cdot 10^{-3})$
4-subs.	$9 \cdot 10^{-3}(7.2 \cdot 10^{-3})$	$4.6 \cdot 10^{-3}(3.6 \cdot 10^{-3})$	$1.4 \cdot 10^{-3}(1.3 \cdot 10^{-3})$
8-subs.	$9 \cdot 10^{-3}(7.2 \cdot 10^{-3})$	$4.6 \cdot 10^{-3}(3.6 \cdot 10^{-3})$	$1.4 \cdot 10^{-3}(1.3 \cdot 10^{-3})$

Tests: Van Der Pool Oscillator

We consider $\Gamma := \partial B(0, \rho)$ (in this case $\rho = 0.2$) and $\Omega := (-1, 1)^2 \setminus \bar{B}(0, \rho)$. The nonlinear system will be:

$$f(x, a) = \begin{pmatrix} x_2 \\ (1 - x_1^2)x_2 - x_1 + a \end{pmatrix}.$$

The others parameters of the system are:

$$A = [-1, 1], \quad \lambda = 1, \quad l(x, y, a) = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad g(x) \equiv 0.$$

Table: Van Der Pol: Comparison of the accuracy of the decomposition with various discretization steps

N. of variables	Δx	Time elapsed	$\max_i \lvert\overline{\Sigma}_i\rvert/\lvert\Omega\rvert$	$\max_i \lvert\Sigma_i\rvert/\lvert\Omega\rvert$
5^2	0.4	$1.4 \cdot 10^{-3}$	62%	
10^2	0.2	0.011	55%	
20^2	0.1	0.103	47%	42.2%
30^2	0.06	1.47	45%	
40^2	0.05	5.6	44.6%	
50^2	0.04	16.3	44.1%	

Tests: Van Der Pool Oscillator

Table: Van Der Pol: Approximation error Error in norm $\Delta\infty$ (and Δ_1)

	50^2	100^2	200^2
original	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
2-subsets	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
4-subsets	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
8-subsets	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$

Tests: A Pursuit-Evasion game

We will consider a Pursuit evasion game, where two agents have the opposite goal to reduce/postpone the time of capture.

The dynamics considered are the following:

$$f(x,a,b) := \begin{pmatrix} f_1(x)(a_1 - b_1) \\ f_2(x)(a_2 - b_2) \end{pmatrix}$$

where the functions f_1, f_2 are $f_1(x) := x_2 + 1$ and $f_2(x) := 1$.
The running cost $l(x,a,b) := x_1 + 0.1$

Controls are $A = B(0,1)$ for the pursuer and $B = B(0,1/2)$ for the evader.
Capture happens trajectory is driven to touch $B(0,0.2)$, then $\Gamma := \partial B(0,0.2)$

Tests: A Pursuit-Evasion game

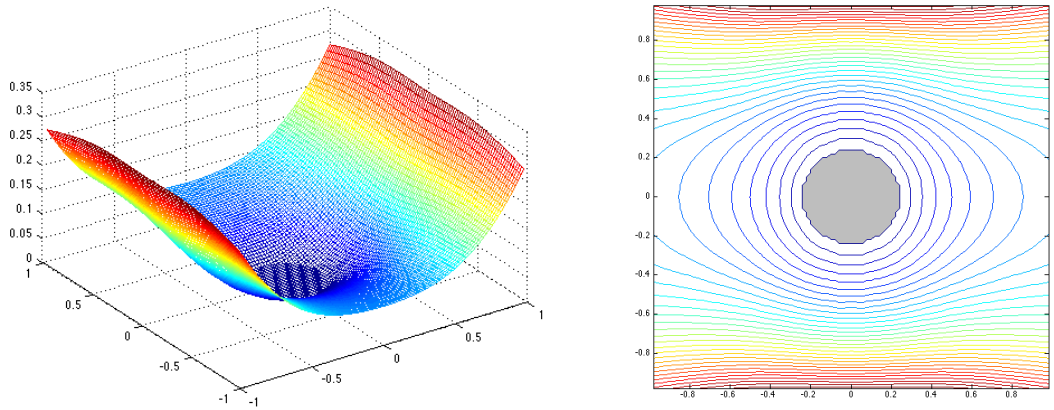


Figure: Approximated value function

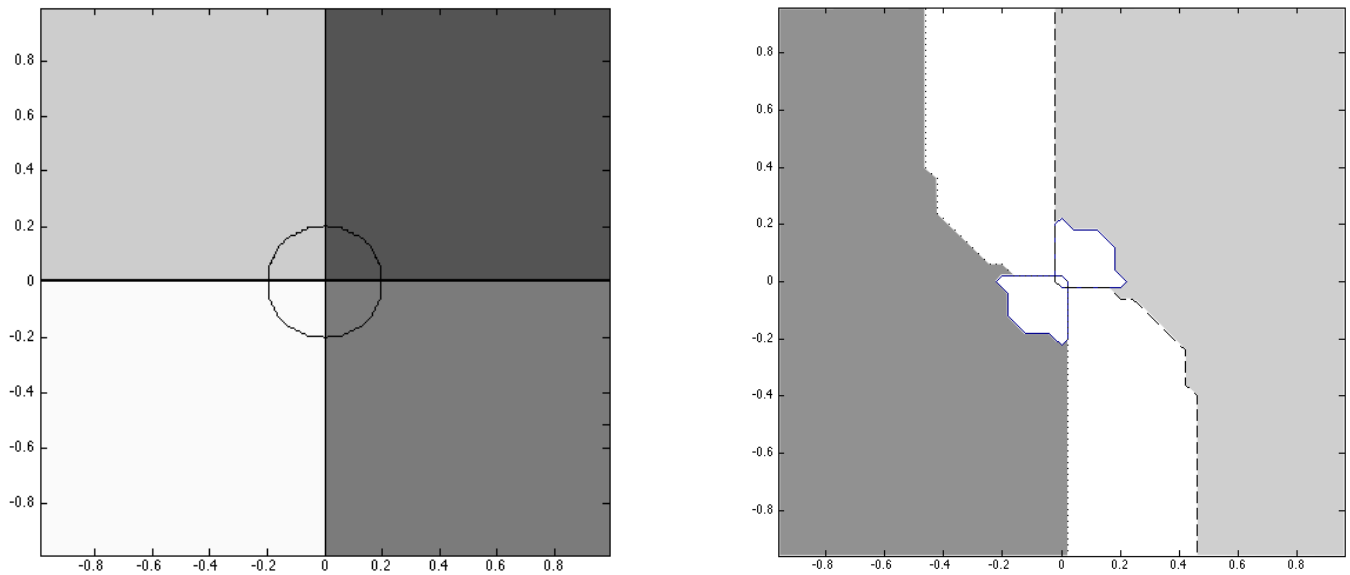


Figure: Exact decomposition and two (of the four) approximated independent subsets found with a course grind of 40^2 points.

Tests: A Pursuit-Evasion game

Table: Comparison of the accuracy of the decomposition with various discretization steps

N. of variables	Δx	Time elapsed	$\max_i \lvert\overline{\Sigma}_i\rvert/\lvert\Omega\rvert$	$\max_i \lvert\Sigma_i\rvert/\lvert\Omega\rvert$
5^2	0.4	10^{-3}	60%	
10^2	0.2	0.008	46%	
30^2	0.06	1.38	38%	25%
50^2	0.04	15.9	36.1%	

The HJ equation associated verifies the decomposability condition (E); because considering the norm (it is possible because $|f_i(x)| > 0$ for $i = 1, 2$)

$$\|p\|_* := \max_{a \in B(0,1)} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} a^T \cdot p$$

the Hamiltonian associated is equivalent to

$$H(x, p) := \|p\|_* - \frac{\|p\|_*}{2} - (x_1^2 + 0.1)$$

evidently convex everywhere with respect to p.

Independent Domain Parallelization: Summary

- The gain is variable, but can be considerable high (no communication between the sub-domains)
- There is a direct dependence of the technique on the **initial decomposition** of the boundary
- Convergence for $\Delta_G \to 0^+$ is guaranteed and the error in the approximation is bounded by an estimate
- It is proved also **convergence for a partition decomposition** (as the *Patchy Dec.*) but it is guaranteed a convergence only for $\Delta_K \to 0^+$, and relative estimation of the error.

Conclusions and Perspectives

- We have shown a constructive manner to obtain a decomposition of the domain of a HJ eq. verifying (E), in independent subdomains, subsets which have the properties of being computed, independently from each others.
- A critical point is **the estimation of the function** $\varepsilon(\cdot)$.
- The critical occurrence about the balance of the dimension of the subsets can be solved with a recursive refinement of the division of Γ.
- A more serious limit appears in **presence of flat regions**: in that case it is, for the moment, impossible to obtain a satisfactory reduction of the dimension of the decomposed domains without solving the problem on a sufficiently fine grid.

Bibliography

- S. CACACE, E. CRISTIANI, M. FALCONE AND A. PICARELLI, *A patchy dynamic programming scheme for a class of Hamilton-Jacobi-Bellman equation*, SIAM J. Scientific Computing, vol. 34 (2012) no. 5, pp. 2625–2649.
- F. CAMILLI, M. FALCONE, P. LANUCARA AND A. SEGHINI, A domain decomposition Method for Bellman Equations, Cont. Math. 180, (1994) pp. 477–483.
- A. FESTA AND R. VINTER preprint: ”Decomposition of Differential Games” 2014.
- A. FESTA preprint: ”Reconstruction of independent sub-domains in a Dynamic Programming Equation and its Application to Parallel Computation” , 2014, `http://arxiv.org/abs/1405.3521`

Bibliography

- S. CACACE, E. CRISTIANI, M. FALCONE AND A. PICARELLI, *A patchy dynamic programming scheme for a class of Hamilton-Jacobi-Bellman equation*, SIAM J. Scientific Computing, vol. 34 (2012) no. 5, pp. 2625–2649.
- F. CAMILLI, M. FALCONE, P. LANUCARA AND A. SEGHINI, A domain decomposition Method for Bellman Equations, Cont. Math. 180, (1994) pp. 477–483.
- A. FESTA AND R. VINTER preprint: ”Decomposition of Differential Games” 2014.
- A. FESTA preprint: ”Reconstruction of independent sub-domains in a Dynamic Programming Equation and its Application to Parallel Computation” , 2014, `http://arxiv.org/abs/1405.3521`

Thank you.