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▶ **To cite this version:**

Maurizio Falcone, Dante Kalise, Axel Kröner. A semi-Lagrangian scheme for Lp-penalized minimum time problems. 21st International Symposium on Mathematical Theory of Networks and Systems, Jul 2014, Groningen, Netherlands. hal-01089877

HAL Id: hal-01089877
https://hal.science/hal-01089877

Submitted on 2 Apr 2015

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A semi-Lagrangian scheme for L^p-penalized minimum time problems

Maurizio Falcone[1] and Dante Kalise[2] and Axel Kröner[3]

***Abstract*— In this paper we consider a semi-Lagrangian scheme for minimum time problems with L^p-penalization. The minimum time function of the penalized control problem can be characterized as the solution of a Hamilton-Jacobi Bellman (HJB) equation. Furthermore, the minimum time converges with respect to the penalization parameter to the minimum time of the non-penalized problem. To solve the control problem we formulate the discrete dynamic programming principle and set up a semi-Lagrangian scheme. Various numerical examples are presented studying the effects of different choices of the penalization parameters.**

I. Introduction

In this paper we consider L^p-penalized minimum time problems for dynamical systems with $1 \leq p < \infty$. In the literature there exist several publications considering these type of problems. In [18] the authors analyzed optimal control problems for dynamical systems with different cost functionals involving L^p-norms of the control. They present several approaches how to handle the nonsmooth case $p = 1$. Further, in a recent preprint [3] L^1-regularization for open-loop optimal control for finite horizon problems of a dynamical system is considered and regularization and discretization error estimates are derived. In [16] a minimum time problem is solved by regularization. In contrast to these open-loop approaches we present a closed-loop approach leading to robust controls and only minimal changes in the implementation are necessary when considering different p-norms in the penalization. Results for closed-loop optimal control problems with exit times can be found in Masiloff [17] and Bardi, Capuzzo-Dolcetta [4, Chapter IV]. Error estimates for a semi-Lagrangian scheme for the classical minimum time problem are presented in [5]. In the context of differential games error estimates are derived in [19]. Numerical examples for semi-Lagrangian schemes for optimal control problems with exit times are shown in [9], where also an error estimate for the fully discretized problem is presented under an assumption on the approximation error of the value function.

In this paper we consider a minimum time problem with different L^p-penalizations. We show that for given p the corresponding minimum time converges to the minimum time of the non-penalized problem if the penalization parameter tends to zero. Moreover, the minimum time function can be characterized as the solution of a Hamilton-Jacobi Bellman (HJB) equation. To solve the problem numerically we formulate the discrete dynamic programming principle and set up a semi-Lagrangian scheme. Various numerical examples are presented illustrating the effect of the choice of the penalization parameters on the value function and the trajectory. For $p = 1$ we observe sparse controls.

Semi-Lagrangian schemes were first introduced to solve linear and nonlinear hyperbolic problems in [8], since the 60's they have been very popular among the meteorological community since they have good stability properties which allow to use large time steps in the integration of evolutionary problems. In the framework of control problems it has been shown that they can be obtained in a very natural way by applying a discrete dynamic programming principle, see e.g. [10], [11]. For a general overview about semi-Lagrangian schemes we refer to the monograph [13]. Convergence and a priori error estimates have been proved for many classical control problems such as the minimum time problem, the infinite horizon, see [5], [14], [12]. In view of control applications, it is also important to note that an efficient algorithm for Hamilton-Jacobi Bellman equations in high dimensions has been presented in [6].

The presented results are interesting for optimal feedback control of partial differential equations, in particular, L^1-control costs have given increasing attention in the last years leading to sparse controls which is interesting, e.g., in actuator placement problems, see [20]. The approximation of sparse controls for semilinear elliptic equations was analyzed in [7]. Directional sparsity for optimal control of partial differential equations was considered in [15]. Convergence and regularization results for optimal control problems with sparsity functional are derived in [21].

The paper is organized as follows. In Section II we introduce the control problem and characterize the value function as the solution of an HJB equation. Furthermore, we show a convergence result for the solution of the regularized problem with respect to the penalization parameter γ. In Section III we discretize the problem and formulate the discrete dynamic programming principle. In Section IV we present several numerical examples studying the behaviour for different penalization parameters γ and p on the value function and the trajectory.

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II. The control problem

In this section the minimum time problem with penalization is introduced and the corresponding value function is characterized as the solution of a stationary HJB equation. We recall some regularity results and consider the convergence of the optimal time, state, and control of the penalized problems for penalization parameter γ tending to zero.

Throughout the paper we consider the Banach space $\mathbb{R}^n$, $n \in \mathbb{N}$, equipped with the norm

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \quad x \in \mathbb{R}^n$$

for $1 \leq p < \infty$ and for time horizon $T > 0$ we use the usual notation for Lebesgue spaces

$$L^p((0,T),\mathbb{R}^n) = \big\{ f\colon (0,T) \to \mathbb{R}^n \text{ measurable} \mid \int_0^T \|f(t)\|_p^p \, dt < \infty \big\}. \quad \text{(II.1)}$$

Further, we define the set of admissible controls by

$$\mathcal{A} = L^p((0,T), A) \quad \text{(II.2)}$$

with $A \subset \mathbb{R}^m$, $m \in \mathbb{N}$, compact. We define a target

$$\mathcal{T} = \{ x \in \mathbb{R}^n \mid \|x\| \leq \delta \} \quad \text{(II.3)}$$

for $\delta > 0$ small. $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Further, let $C > 0$ be a generic constant. Throughout the paper we make the following assumptions:

Assumption 2.1: Let the dynamics $f\colon \mathbb{R}^n \times A \to \mathbb{R}^n$ be continuous and locally Lipschitz uniformly in $\alpha \in A$.

Further we make the following controllability assumption:

Assumption 2.2: There holds

$$\inf_{\alpha \in A} f(x,\alpha) \cdot n(x) < 0 \quad \forall x \in \partial\mathcal{T},$$

where $n(x)$ is the outer normal to $\mathcal{T}$ at x.

Under these assumptions on the dynamics we introduce the controlled dynamical system

$$\begin{cases} y'(s) = f(y(s), \alpha(s)), \\ y(0) = x \end{cases} \quad \text{(II.4)}$$

for control $\alpha \in \mathcal{A}$ and initial point $x \in \mathbb{R}^n$. Sometimes we write y_x^α to indicate that we mean the state corresponding to control α with initial point x.

The discounted minimum time problem for (II.4) is given by

$$\begin{cases} v(x) = \min\limits_{\alpha \in \mathcal{A}} \int_0^{t_x(\alpha)} e^{-\lambda s} ds, \\ \qquad \text{subject to the constraint} \\ \quad \frac{d}{ds} y_x^\alpha(s) = f(y_x^\alpha(s), \alpha(s)), \quad s \in (0, t_x(\alpha)), \\ \quad y_x^\alpha(0) = x \\ \quad \text{and } y_x^\alpha(t_x(\alpha)) \in \mathcal{T} \end{cases} \quad (P_T^0)$$

with

$$t_x(\alpha) = \begin{cases} +\infty, & \text{if } \{ t > 0 \mid y_x^\alpha(t) \in \mathcal{T} \} = \emptyset, \\ \min \{ t > 0 \mid y_x^\alpha(t) \in \mathcal{T} \}, & \text{else} \end{cases}$$

and $\lambda \geq 0$. Here we use the notation y_x^α for the state to express its dependence on the initial state x and the control α.

The regularized minimum time problem for (II.4) is given by

$$\begin{cases} v(x) = \min\limits_{\alpha \in \mathcal{A}} \int_0^{t_x(\alpha)} e^{-\lambda s} ds + \gamma \int_0^{t_x(\alpha)} \|\alpha(s)\|_p^p \, e^{-\lambda s} ds, \\ \qquad \text{subject to the constraint} \\ \quad \frac{d}{ds} y_x^\alpha(s) = f(y_x^\alpha(s), \alpha(s)), \quad s \in (0, t_x(\alpha)), \\ \quad y_x^\alpha(0) = x, \\ \quad \text{and } y_x^\alpha(t_x(\alpha)) \in \mathcal{T} \end{cases} \quad (P_T^\gamma)$$

for $1 \leq p < \infty$, $\lambda \geq 0$, $\gamma > 0$.

We assume that both problems have a solution, which is guaranteed under local controllability assumptions on the target (see [4] for more precise definitions and general conditions for existence). For a discussion of necessary optimality conditions we refer to [18]. To shorten the notation we introduce

$$l^\gamma\colon \mathbb{R}^N \times A \to \mathbb{R}, \quad l^\gamma(x,\alpha) = 1 + \gamma \|\alpha\|_p^p. \quad \text{(II.5)}$$

Furthermore, we define

$$\mathcal{R} = \{ x \in \mathbb{R}^n \mid v(x) < \infty \},$$

which depends in particular on γ. The dynamic programming principle for these problems is given by

$$v(x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^{t_x(\alpha) \wedge t} l^\gamma(y_x^\alpha(s), \alpha(s)) e^{-\lambda s} ds + \chi_{\{ t < t_x(\alpha) \}} v(y_x^\alpha(t)) e^{-\lambda t} \right\} \quad \text{(II.6)}$$

for all $x \in \mathbb{R}^N$ if $\lambda > 0$, for all $x \in \mathcal{R}$ if $\lambda = 0$, all $t \geq 0$, and given $\gamma \geq 0$, see [4, pp. 254]. Here, χ denotes the characteristic function. The corresponding Hamiltonian is given by

$$H^\gamma(x,p) = \sup_{\alpha \in A} (-f(x,\alpha)p - l^\gamma(x,\alpha)) \quad \text{(II.7)}$$

for $x, p \in \mathbb{R}^n$.

Under this assumption we obtain from [4, Chapter IV, Proposition 3.13] the following characterization of the value function.

Proposition 2.3: Let Assumption 2.1 and 2.2 be satisfied. Then,

(i) if $\lambda > 0$ the value function v is the complete solution in the space of bounded and continuous functions $BC(\overline{\mathcal{T}^c})$, $\mathcal{T}^c = \mathbb{R}^n \setminus \mathcal{T}$, of

$$\begin{cases} \lambda u + H^\gamma(x, \nabla u(x)) = 0 & \text{in } \mathcal{T}^c, \\ u(x) = 0 & \text{on } \partial\mathcal{T}. \end{cases} \quad \text{(II.8)}$$

(ii) if $\lambda = 0$ the value function v is the unique solution of

$$\begin{cases} H^\gamma(x, \nabla u(x)) = 0 & \text{in } \mathcal{R} \setminus \mathcal{T}, \\ u(x) = 0 & \text{on } \partial\mathcal{T}, \\ \lim\limits_{x \to x_0} u(x) = +\infty & \text{for } x_0 \in \partial\mathcal{R} \end{cases} \tag{II.9}$$

continuous in $\mathcal{R} \setminus \operatorname{int} \mathcal{T}$,

(iii) if $\lambda = 0$, the function u defined by Kruzkov transform

$$u(x) = \begin{cases} 1 - e^{-v(x)}, & x \in \mathcal{R}, \\ 1, & \text{else} \end{cases} \tag{II.10}$$

for given value function v is the complete solution in $BC(\overline{\mathcal{T}^c})$ of

$$\begin{aligned} u(x) + \sup \Big\{ & - f(x,\alpha)^T \nabla u(x) - l^\gamma(x,\alpha) \\ & + (l^\gamma(x,\alpha) - 1)u \Big\} = 0 \quad \text{in } \mathcal{T}^c, \\ u(x) = 0 \quad & \text{in } \partial\mathcal{T}. \end{aligned}$$

A formal definition of "complete solution" can be found in [4, Chapter IV, p. 256]. For $\gamma = 0$ we obtain the HJB equation for the minimum time problem (P_T^0). In the following we will prove convergence of the minimum time of the penalized problem with respect to the regularization parameter γ and weak convergence of the optimal state and control to the corresponding state and control of the non-penalized problem. Thereby we restrict the consideration to linear systems of the type

$$\begin{cases} y'(s) = Fy(s) + B\alpha(s), \\ y(0) = x \end{cases} \tag{II.11}$$

with matrices $F \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$. In case of a nonlinear dynamics the specific character of the nonlinearity has to be taken into account when analyzing the weak convergence.

Theorem 2.4: Let α^0 be a minimizing control of the non-regularized problem (P_T^0) and α^γ of the regularized problem (P_T^γ). We denote the corresponding state and time by (y^0, t^0) and (y^γ, t^γ), respectively. Then there holds

$$t^\gamma \to t^0 \in \mathbb{R}$$

for $\gamma \to 0$. Further, after reparametrization of both solutions such that they satisfy

$$\begin{cases} y'(s) = t(Fy(s) + B\alpha(s)) & \text{in } (0,1), \\ y(0) = x, \end{cases} \tag{II.12}$$

there exists subsequences with

$$(\alpha^\gamma, y^\gamma) \rightharpoonup (\alpha^0, y^0) \in L^p((0,1), \mathbb{R}^m) \times H^1((0,1), \mathbb{R}^n) \tag{II.13}$$

for $\gamma \to 0$ and $1 < p < \infty$.

Proof: The minimum time t^0 and corresponding control α^0 are feasible for the regularized problem. Therefore for given $\gamma \geq 0$ the minimum time t^γ and corresponding optimal control α^γ of (P_T^γ) satisfy

$$t^\gamma + \gamma \int_0^{t^\gamma} \|\alpha^\gamma(s)\|_p^p \, ds \;\leq\; t^0 + \gamma \int_0^{t^0} \left\|\alpha^0(s)\right\|_p^p ds. \tag{II.14}$$

Hence, we have

$$\limsup_{\gamma \to 0} t^\gamma \leq t^0. \tag{II.15}$$

After reparameterization the triple $(\alpha^\gamma, y^\gamma, t^\gamma)$ satisfies

$$\begin{cases} y'(s) = t^\gamma(Fy(s) + B\alpha(s)) & \text{in } (0,1), \\ y(0) = x \end{cases} \tag{II.16}$$

and there holds

$$\|y^\gamma\|_{H^1((0,1),\mathbb{R}^n)} \leq C \left(\|\alpha^\gamma\|_{L^\infty((0,1),\mathbb{R}^m)} + \|x\| \right) \tag{II.17}$$

using (II.15), Gronwall's inequality, and the boundedness of A. Since the controls are bounded we obtain that the state y^γ is uniformly bounded in $H^1((0,1), \mathbb{R}^n)$. Thus there exist

$$(\alpha^*, y^*, t^*) \in L^p((0,1), A) \times H^1((0,1), \mathbb{R}^n) \times \mathbb{R}_0^+,$$

$1 < p < \infty$, such that for a subsequence

$$\begin{aligned} \alpha^\gamma &\rightharpoonup \alpha^* \quad \text{in } L^p((0,1), A), \\ y^\gamma &\rightharpoonup y^* \quad \text{in } H^1((0,1), \mathbb{R}^n), \\ t^\gamma &\to t^* \quad \text{in } \mathbb{R}_0^+. \end{aligned}$$

Thereby we used the fact that A is closed with respect to weak convergence. We further obtain

$$y^\gamma \to y^* \quad \text{in } C^0((0,1), \mathbb{R}^n). \tag{II.18}$$

Consequently, the limit satisfies the state equation

$$\begin{cases} y_t^*(s) = t^*(Fy^*(s) + B\alpha^*(s)) \\ y^*(0) = x. \end{cases} \tag{II.19}$$

Furthermore using the continuity of y^* and the convergence of y^γ in $C^0((0,1), \mathbb{R}^n)$ we have

$$\begin{aligned} &\|y^*(t^*) - y^\gamma(t^\gamma)\| \leq \|y^*(t^*) - y^\gamma(t^*)\| + \|y^\gamma(t^\gamma) - y^\gamma(t^*)\| \\ &\leq \|y^*(t^*) - y^\gamma(t^*)\| + \|y^\gamma(t^\gamma) - y^*(t^\gamma)\| \\ &\quad + \|y^*(t^\gamma) - y^*(t^*)\| + \|y^*(t^*) - y^\gamma(t^*)\| \longrightarrow 0 \end{aligned}$$

for $\gamma \to 0$ implying that $y^*(t^*) = 0$. Hence, (α^*, y^*, t^*) is feasible for the non-regularized problem.

With the condition $t^* \leq t^0$ from (II.15) and since t^0 is minimal we conclude $t^0 = t^*$. Since t^0 is unique, the whole family t^γ converges to t^0. ∎

III. Discretization of the semi-Lagrangian scheme

In this section we apply the discrete dynamic programming principle to obtain a characterization of the time discrete value function. Furthermore we formulate a semi-Lagrangian scheme.

For the discretization of the dynamical system we choose an explicit Euler scheme. For fixed time step $h > 0$ we define the temporal mesh

$$0 = t_0 < t_1 < \cdots < t_N$$

with $t_{i+1} - t_i = h$, $i = 0, \ldots, N-1$. For solving the dynamical system we apply an explicit Euler scheme

$$\begin{cases} y_{m+1} = y_m + hf(y_m, \alpha_m), \\ y(0) = x \end{cases} \tag{III.1}$$

for $\alpha_m \in A$ and $x \in \mathbb{R}^n$. The control is given by

$$\alpha^h(s) = \alpha_m, \quad s \in [t_m, t_{m+1}), \quad m = 0, \ldots, N-1.$$

We consider the discrete minimum time problem with penalization and characterize the value function using the dynamic programming principle.

Theorem 3.1: For $\lambda = 0$ the value function of the minimum time problem is characterized by

$$v_h(x) = \min_{\alpha^h \in A} \left\{ \nu v_h(x + hf(x, \alpha^h)) - \nu \right\} + 1 \tag{III.2}$$

with

$$\nu = e^{-h(1+\gamma \left\| \alpha^h \right\|_p^p)}$$

and $x \in \mathbb{R}^n$.

Proof: The dynamic programming principle for the time discrete minimum time problem is given as follows

$$v_h(x) = \inf_{\alpha^h \in A} \left\{ h \left(1 + \gamma \left\| \alpha_i \right\|_p^p \right) + v_h(x + hf(x, \alpha^h)) \right\},$$

for $x \in \mathbb{R}^n$, see [4, Proposition 4.1, p. 389]. We have

$$e^{-v_h(x)} = \sup_{\alpha^h \in A} \left\{ e^{\left(-h - \gamma h \left\| \alpha^h \right\|_p^p \right) - v_h(x + hf(x, \alpha^h))} \right\}$$

and applying Kruzkov transform

$$V_h(x) := \begin{cases} 1 - e^{-v_h(x)}, & \text{if } x \in \mathcal{R}, \\ 1, & \text{if } x \notin \mathcal{R} \end{cases}$$

we obtain

$$\begin{aligned} V_h(x) &= 1 - \sup_{\alpha^h \in A} e^{\left(-h - \gamma h \left\| \alpha^h \right\|_p^p \right) - V_h(x + hf(x, \alpha^h))} \\ &= \inf_{\alpha^h \in A} \left\{ \frac{1 - e^{-V_h(x + hf(x, \alpha^h))} - 1}{e^{h\left(1 + \gamma \left\| \alpha^h \right\|_p^p\right)}} + 1 \right\}. \end{aligned}$$

Again applying Kruzkov transform we obtain

$$v_h(x) = \inf_{\alpha^h \in A} \left\{ \frac{v(x + hf(x, \alpha^h)) - 1}{e^{h\left(1 + \gamma \left\| \alpha^h \right\|_p^p\right)}} + 1 \right\}$$

and hence,

$$v_h(x) = \min_{\alpha^h \in A} \left\{ \nu v(x + hf(x, \alpha^h)) - \nu \right\} + 1,$$

with

$$\nu = e^{-h\left(1 + \gamma \left\| \alpha^h \right\|_p^p\right)}$$

which gives the assertion. ∎

Equation (III.2) is the starting point for the numerical scheme applied to the minimum time problem. We discretize equation (III.2) in space. The HJB equation is defined on the full space. To solve the equation numerically we consider a bounded computational domain $\Omega \subset \mathbb{R}^n$. Let

$$\overline{\Omega} = \bigcup_j S_j$$

be a regular triangulation of the computational domain for a family of simplices S_j and let x_i denote the nodes of the triangulation with $i = 1, \ldots, L$, $L \in \mathbb{N}$. We set

$$k = \max_j \operatorname{diam}(S_j)$$

and define

$$\begin{aligned} I_{\mathcal{T}} &= \{\, i \in I \mid x_i \in \mathcal{T} \cap \Omega \,\}, \\ I_{\text{out}} &= \{\, i \in I \mid x_i + hf(x_i, \alpha) \notin \Omega \quad \forall \alpha \,\}, \\ I_{in} &= I \setminus (I_{out} \cup I_{\mathcal{T}}), \\ I &= \{\, 1, \ldots, L \,\}. \end{aligned} \tag{III.3}$$

Then we consider

$$\begin{aligned} &v_h(x_i) = \min \left\{ \nu v_h(x_i + hf(x_i, \alpha^h)) - \nu \right\} + 1 && i \in I_{in}, \\ &v_h(x_i) = 0 && i \in I_{\mathcal{T}}, \\ &v_h(x_i) = 1 && i \in I_{\text{out}}. \end{aligned} \tag{P_T^{kh}}$$

For the computation of the numerical examples we solve (P_T^{kh}).

IV. Numerical examples

In this section we present two numerical examples illustrating the properties of our scheme, in terms of the effects produced by including a penalization term of the control in the minimum time problem formulation. The numerical implementation follows the general guidelines for the construction of low-order semi-Lagrangian schemes for HJB equations, and in particular is based on the solvers described in, for instance, [4, Appendix A] and [1]. For the scheme (P_T^{kh}), the computation of the arrival point $v_h(x_i + hf(x_i, \alpha^h))$ is replaced by a linear interpolation $I_1[V]$ defined upon the grid point values $V = \{v_h(x_i)\}_{i=1}^L$ (we refer to [13, pp. 46-47] for specific details), and the resulting system is solved via the fixed point iteration

$$V^{n+1} = S(V^n)$$

$$[S(V)]_i = \begin{cases} \min \left\{ \nu I_1[V](x_i + hf(x_i, \alpha^h)) - \nu \right\} + 1 & \\ & i \in I_{in}, \\ 0 & i \in I_{\mathcal{T}}, \\ 1 & i \in I_{\text{out}} \end{cases}$$

which in our case is stopped when two consecutive iterations hold $\|V^n - V^{n+1}\| \leq k^2$. We set the discount factor $\lambda = 0.01$ for both examples.

A. 2D Eikonal equation

The first test case that we consider is a minimum time problem for a two-dimensional Eikonal equation. The domain is $\Omega = [-1, 1]^2$, and the system dynamics are given by

$$f(x, y, a) = \begin{pmatrix} a_x \\ a_y \end{pmatrix}, \quad \|a\|_2 = \|(a_x, a_y)^T\|_2 \leq 1 .$$

The control set $A = \{\|a\|_2 \leq 1,\ a \in \mathbb{R}^2\}$ is discretized into 32 different directions, whereas the grid contains 81^2 points with $k = 0.025$. The target set $\mathcal{T}$ is specified as $\mathcal{T} = \{ x \in \mathbb{R}^2 \mid \|x\|_2 \leq 0.2 \}$, and the discretization parameters hold $h = 0.9k$. It is well-known that for the unpenalized minimum time problem, the exact solution corresponds to the distance function to the target, and the optimal trajectories point to the origin.

For the penalized optimal control problem, such symmetry is preserved for any value of γ, as long as $p = 2$, as shown in Figure 1. Note that with this choice of parameters, all the controls lying in the boundary of the control set (the unitary ball with $p = 2$) introduce the same control cost; although the value function is different from the unpenalized control problem, the optimal control field remains the same. A radically different situation is observed when $p = 1$, as the introduction of such a norm leads to optimal sparse solutions. In Figure 2, it can be observed that the optimal trajectories computed with $p = 1$ differ from those with $p = 2$ around a vicinity of the coordinate axis, i.e. there is a non-trivial section of the state space where the optimal trajectories are related to the directions $(1, 0), (0, 1), (-1, 0)$, and $(0, -1)$ instead of the directions pointing to the origin.

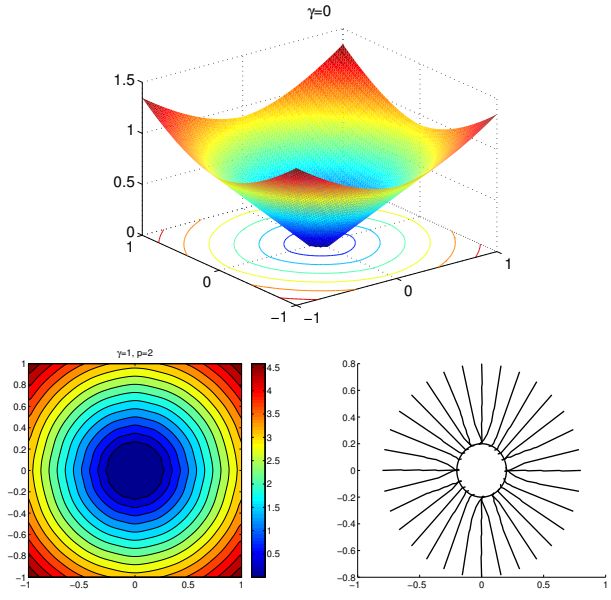

Fig. 1: Eikonal equation. Results with $p = 2$: value function (left) and optimal trajectories (right). Note that this setting yields similar trajectories as in the unpenalized minimum time problem (top).

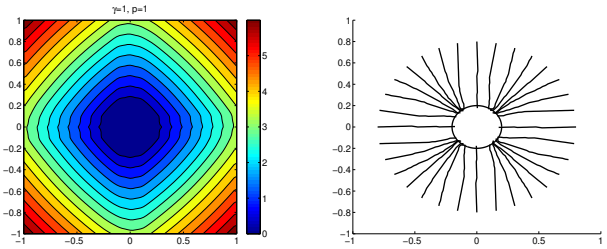

Fig. 2: Eikonal equation. Results with $p = 1$: value function (left) and optimal trajectories (right). The application of the $||\cdot||_1$ norm induces a change in the shape of the value function and a sparsity pattern in the control field which is reflected in the optimal trajectories, creating a sector in the state space where sparse controls are preferred instead of directions pointing to the origin.

B. Van der Pol oscillator

In a second example, we consider a two-dimensional, nonlinear system dynamics given by the Van der Pol oscillator

$$f(x, y, a) = \begin{pmatrix} y \\ (1 - x^2)y - x + a \end{pmatrix},$$

with test parameters given by

$$\Omega = [-2\,, 2]^2\,, \quad A = [-1\,, 1]\,, \quad h = 0.3k\ .$$

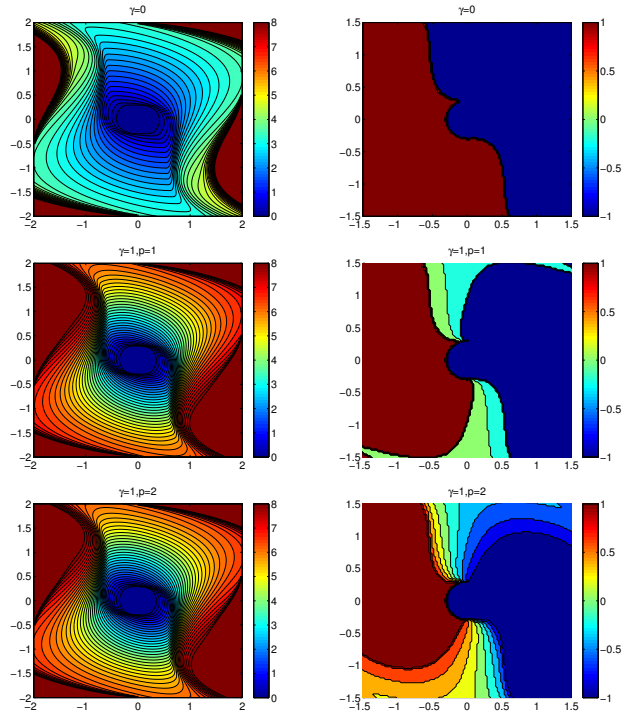

Fig. 3: Van der Pol oscillator. Value functions (left) and associated control fields (right), with different values of γ and p. In the top, with $\gamma = 0$, the minimum time problem yields a bang-bang control structure. The inclusion of a penalization with $p = 1$ (middle), preserves the shape of the value function, but replaces the bang-bang control field with an (approximate) bang-zero-bang structure. Finally, for $p = 2$ (bottom), the control field transits along all the available discrete values, whereas there is no significant change in the value function with respect to the case $p = 1$.

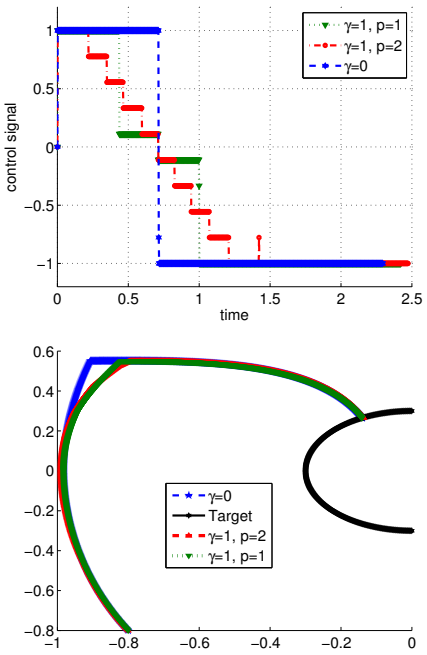

Fig. 4: Van der Pol oscillator: computations along a trajectory. Left: different control signals for different values of p. Right: Optimal trajectories associated to the control signals.

The control space is discretized into 10 equidistant points, and the target set is specified as $\mathcal{T} = \{ x \in \mathbb{R}^2 \mid ||x||_2 \leq 0.2 \}$. The grid contains 101^2 grid points with $k = 0.04$. An interesting result can be observed in Figures 3 and 4, in the sense that the inclusion of penalization costs in the control changes the bang-bang structure of the control field (see for instance [2]), and replaces it by an approximated bang-zero-bang control for the case $p = 1$, whereas for $p = 2$ it translates in the uses of all the discrete control values. The phenomenon of a zero-arc was already mentioned in [18] and was later also analyzed in [3]. In Figure 3, it can be seen that there is a change in the control field when the control penalization is considered, whereas no significant change in the shape of the value function is observed, which differs from the previous test.

The effect in the computation of a trajectory can be observed in Figure 4; variations on the optimal trajectories are not significant, while the associated control signals exhibit a different use of the control capabilities of the system. This result is relevant from a point of view of applications where it is desirable to have a setting avoiding the use of maximum control energy over long periods of time.

V. Concluding remarks

In this paper we have introduced a semi-Lagrangian scheme for the solution of a Hamilton-Jacobi-Bellman equation related to penalized minimum time problems. Although the minimum time problem with penalization in the control has been studied in an open loop context, the present work aims at developing a discrete dynamic programming framework yielding to feedback controllers. In this context, the extension of the existing results for minimum time problems is not trivial, and further developments in this direction are currently undertaken. The numerical experiments presented are consistent with what is reported in the literature of both semi-Lagrangian schemes and penalized minimum time optimal control problems, and suggest that the application of dynamic programming is a correct approach to follow.

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