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Phenomenological model for predicting stationary and non-stationary spectra of wave turbulence in vibrating plates

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Summary. Wave turbulence theory aims at describing the long time behaviour of weakly nonlinear and out-of-equilibrium systems. For thin vibrating plates, this framework allows predicting a Kolmogorov-Zakharov spectrum with an energy flux from the injection to the dissipative scales. Discrepancies between measured and theoretical velocity spectra have been mainly attributed to damping –acting at all scales and precluding the existence of a transparency window– as well as other effects such as the anisotropy brought by the forcing. As the complete derivation of a wave turbulence theory including the presence of a given damping law is still missing, an alternative approach is used in this contribution. A phenomenological equation for the energy spectrum is established from the general solutions (Rayleigh-Jeans and Kolmogorov-Zakharov) of the kinetic equation. This model equation is then used in order to establish the self-similar solutions of the system in the case of non-stationary turbulence. The phenomenological model is then enriched by adding a damping law close to experimental measurements. In that case of stationary wave turbulence, the model also predicts power spectra that are in close agreement with the measurements.

Introduction

Wave (or weak) turbulence theory (WTT) aims at describing the long time behaviour of weakly nonlinear systems with energy exchanges between scales. Under classical assumptions (such as dispersivity and existence of a transparency window in which the dynamics is assumed to be conservative), a broadband Kolmogorov-Zakharov spectrum is predicted, by analogy with hydrodynamic turbulence [1, 2]. Wave turbulence spectrum for elastic vibrating plates has been established theoretically in [3], by using the von Kármán model for geometrically nonlinear thin plates vibrating with large amplitudes. Two independent experiments, performed soon after on thin elastic plates [4, 5, 6], did not replicate these theoretical (and numerical) predicted spectra. Recently, an experimental and numerical study on the effect of damping on the turbulent properties has clearly established that: (i) in experiments, damping acts at all scales such that the assumption of the existence of a transparency window is questionable, (ii) modifying the damping alters the shape of the velocity power spectrum so that a direct comparison with the theory is out of reach in experimental conditions [7].

A modification of the wave turbulence theory allowing to derive the statistical properties of the out-of-equilibrium system in case of damping, although not impossible, remains an important theoretical effort. In order to bypass this difficulty while still searching for a model equation that could explain in a simple framework the observations, a phenomenological equation for predicting the spectra of wave turbulence in plates is proposed in this contribution. In fact, these phenomenological models have been already deduced in different situations from the WTT, providing a natural framework to investigate unsteady dynamics [8, 9]. The model equation is derived from the two known solutions, namely the Rayleigh-Jeans and the Kolmogorov-Zakharov spectra, so that they are naturally enclosed within the model, which can then be enriched to study the effects of *e.g.* the damping. Two results from the phenomenological model are shown in this paper. First, in a conservative case, its ability to retrieve self-similar spectra of forced and free turbulence which have been already observed in numerical simulations [10]. Secondly, the damping is added to the model equation.

Conservative Case

Model Equation

The complete theoretical derivation of the wave turbulence theory for the von Kármán thin plate equations has been obtained in [3]. The two solutions of the kinetic equation are found to be:

- The Rayleigh-Jeans (RJ) equilibrium solution, where energy E_ω is equally parted along all available modes, and such as $\partial_t E_\omega = 0$.
- The Kolmogorov-Zakharov (KZ) solution, for which an energy flux ε is transferred along the cascade until it is dissipated near ω_c, the cut-off frequency of the spectrum. Referring to [3], the energy spectrum in this case reads

$$E_\omega \propto \varepsilon^{\frac{1}{3}} \log^{\frac{1}{3}}\left(\frac{\omega_c}{\omega}\right). \tag{1}$$

By deriving Eq.(1) with respect to ω, one obtains that $\omega E_\omega^2 \partial_\omega E_\omega$ is equal to a constant. Hence deriving once again with respect to ω and using the fact that for the Rayleigh-Jeans solution, $\partial_t E_\omega = 0$, a model equation is found as

$$\partial_t E_\omega = \partial_\omega(\omega E_\omega^2 \partial_\omega E_\omega). \tag{2}$$

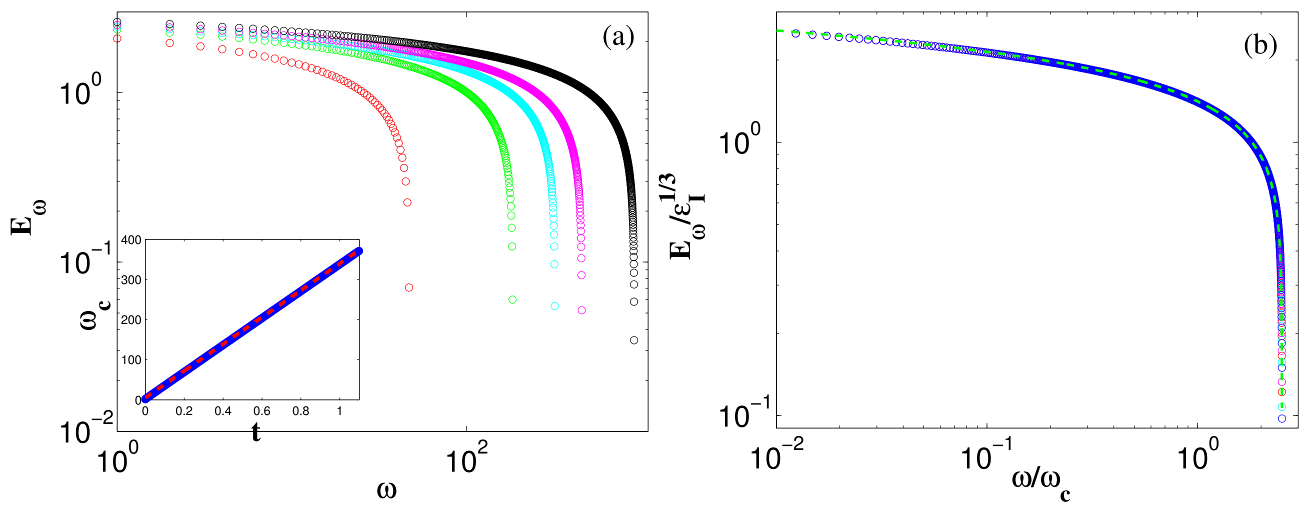

Figure 1: (a) Energy spectrum E_ω as a function of the frequency ω for times increasing from the left to the right, $\varepsilon_I = 1$. Insert : characteristic frequency ω_c as a function of time, red line : $\omega_c \propto t$. (b) Energy spectrum E_ω divided by $\varepsilon_I^{1/3}$ as a function of the rescaled frequency ω/ω_c for several energy fluxes $\varepsilon_I = 0.5, 1, 2, 5$. Green line : Kolmogorov-Zakharov spectrum $E_\omega \propto \log(\frac{\omega_c}{\omega})^{1/3}$.

The energy flux ε associated to this equation is

$$\varepsilon = -\omega E_\omega^2 \partial_\omega E_\omega. \tag{3}$$

Numerical simulations of the model equation Eq.(2) will now be conducted in various cases so as to investigate all the solutions. We begin with a classical case where an energy flux is imposed at low frequency.

Forced Turbulence

In order to simulate numerically Eq.(2), a finite volume method is used. The flux is computed at each space increment and the value of E_ω is defined at the centre of the mesh element. With this method, we apply a constant value ε_I over time for the flux ε at $\omega = 0$. A typical run consists in 2048 points in the ω direction and a time step equal to 10^{-7} seconds. The total duration of the simulation is 1.5 seconds.

Fig.1(a) displays in that framework the energy spectrum at different times and suggests that the cascade grows toward high frequencies following a self-similar behaviour. Moreover, the characteristic frequency, defined here by

$$\omega_c = \frac{\int_0^\infty E_\omega \omega d\omega}{\int_0^\infty E_\omega d\omega}, \tag{4}$$

shows clearly a linear relation with time. Strong dissipation has been included upon $\omega = 10^3$ so that when the cascade front reaches this scale, the system becomes quickly stationary and the characteristic frequency remains constant. This strong dissipation allows to stand in the theoretical framework of the wave turbulence theory. As it can be seen in Fig.1(b), a perfect match with the Kolmogorov-Zakharov spectrum is then observed. This figure also displays the classical amplitude scaling of the spectrum by $\varepsilon_I^{1/3}$ for fluxes $\varepsilon_I = 0.5, 1, 2, 5$, showing the validity of this scaling law with our data. The typical behaviour of energy spectra of vibrating plates in the stationary regime is so correctly described by the phenomenological equation Eq.(2).

In order to recover the numerical behaviour observed in Fig.1(a), the self-similar solutions of Eq.(2) are investigated. The solutions are thus written under the form

$$E_\omega = t^\alpha g(\frac{\omega}{t^\beta}), \tag{5}$$

with α and β two real unknowns and g a function to be determined. Inserting Eq.(5) into Eq.(2), one finds that α and β must fulfill the relationship

$$2\alpha = \beta - 1. \tag{6}$$

If we assume further that when injecting with a constant flux over time, the total energy of the plate is growing linearly with time t, the equality

$$\int_0^{+\infty} E_\omega d\omega = At \tag{7}$$

where A is a constant leads to a second relationship $\alpha+\beta=1$. This yields $\alpha=0$ and $\beta=1$ so that finally the self-similar solutions are necessarily under the form

$$E_\omega = g(\frac{\omega}{t}) \tag{8}$$

which shows that the self-similar solutions of Eq.(2) have a characteristic frequency growing linearly with time.

Non-stationary wave turbulence in plates has been studied numerically in [10] with a finite-difference, energy-conserving scheme, showing a cascade front propagating to higher frequencies and leaving in its wake a self-similar spectrum. When the plate is continuously excited with an harmonic forcing, the front cascade evolves linearly with time. These behaviours are perfectly recovered in the present study, showing the ability of the phenomenological model to determine thanks to a simple equation complex features of the turbulent regime. The case of free turbulence has also been considered in [10], where the cascade front then evolves to higher frequencies as $t^{1/3}$. This framework is here investigated through the use of the phenomenological equation.

Free Turbulence

Free turbulence considers the study of the propagation of an initial condition in the absence of forcing and damping. The system being conservative, the amount of initial energy K should be conserved, so that Eq.(7) is replaced by

$$\int_0^{+\infty} E_\omega d\omega = K. \tag{9}$$

In that case, the second relationship on the unknowns α and β turns to be now $\alpha=-\beta$, leading to $\alpha=-1/3$ and $\beta=1/3$. The self-similar solutions for the energy spectrum E_ω reads now

$$E_\omega = t^{-1/3} h(\frac{\omega}{t^{1/3}}). \tag{10}$$

For the free turbulence simulations, the dissipation introduced earlier at high frequencies in order to mimic an energy sink, is now removed. An energy flux ε_I is imposed for a few time steps and then stopped, thus fixing the origin of time. Then the simulation is run by imposing a vanishing energy flux at $\omega=0$, ensuring free turbulence.

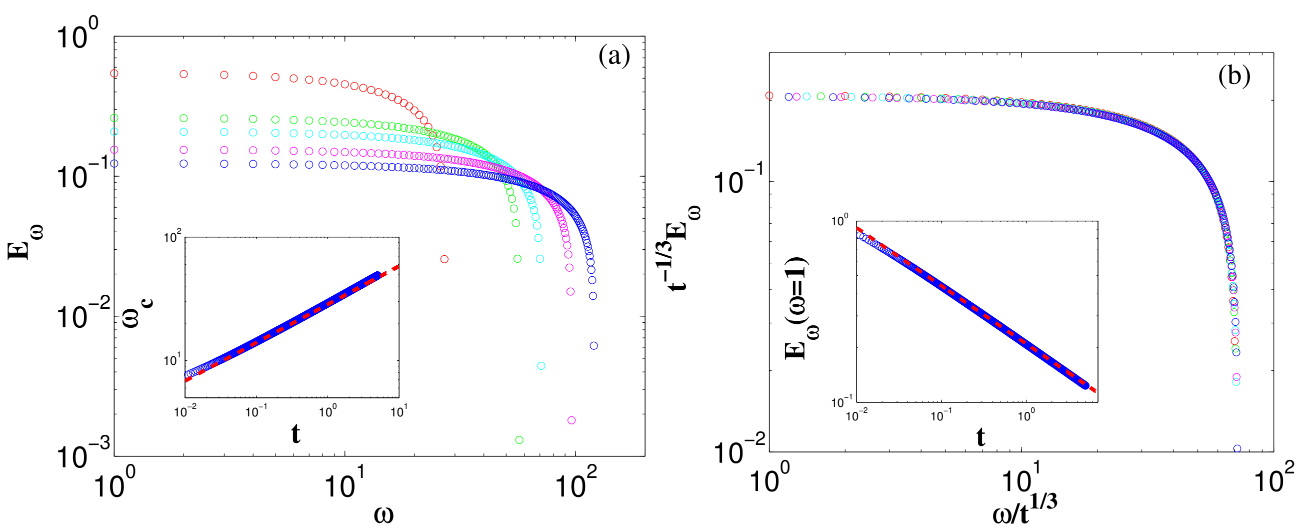

Figure 2: (a) Energy spectrum E_ω as a function of the frequency ω for (from top to bottom) $t=0,1,2,3,4$ s. Insert : characteristic frequency ω_c as a function of time, red line : $\omega_c \propto t^{1/3}$. (b) Self-similar function $h(\frac{\omega}{t^{1/3}})$ for the same data. Insert : $E_\omega(\omega=1)$ as a function of time, red line : $E_\omega(1) \propto t^{-1/3}$.

Fig.2(a) shows the evolution of the initial amount of energy K (in red) as a function of time. The two inserts describe the evolution of the characteristic frequency ω_c and the amplitude of the spectrum in $\omega=1$ as a function of time, displaying a behaviour respectively in $t^{1/3}$ and in $t^{-1/3}$. Those two observations and the final scaling displayed in Fig.2(b) are in agreement with the self-similar solution given by Eq.(10).

Non Conservative case : the effect of damping

Eq. (2) describes the evolution of the energy spectrum as a function of the frequency in a conservative framework. In [7], an experimental and numerical study on the effect of damping on the turbulent properties has clearly established that: (i) in experiments, damping acts at all scales such that the assumption of the existence of a transparency window is questionable, (ii) modifying the damping alters the shape of the velocity power spectrum so that a direct comparison with the theory is out of reach in experimental conditions. Those effects are investigated through the use of our phenomenological model.

Model equation

Adding a dissipation term in Eq.(2) is simply done by writing

$$\partial_t E_\omega = \partial_\omega(\omega E_\omega^2 \partial_\omega E_\omega) - \hat{\gamma} E_\omega, \tag{11}$$

where $\hat{\gamma}$ is a function of ω. Measurements in [7] have shown that, under some experimental conditions, the damping follows a power-law as $\hat{\gamma} = \alpha f^{0.6}$, where α is a parameter that can be tuned. The experimentally measured α are here used to deduce appropriate damping functions of the form $\hat{\gamma} = \hat{\alpha}\omega^{0.6}$ in Eq.(11). This equation is solved numerically for a constant flux of energy ε in $\omega = 0$. After some time, a stationary regime arises.

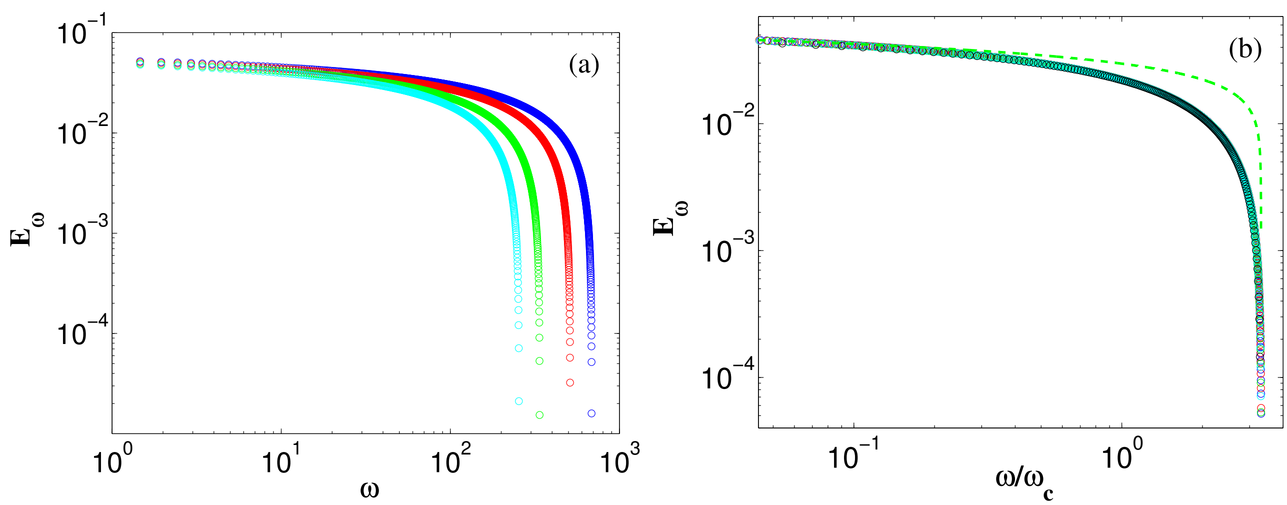

Figure 3: (a) Energy spectrum E_ω as functions of the frequency ω, $\varepsilon_I = 10^{-5}$. Blue: $\hat{\alpha} = 1.908 \times 10^{-5}$, red: $\hat{\alpha} = 3.0528 \times 10^{-5}$, green: $\hat{\alpha} = 5.9359 \times 10^{-5}$, cyan: $\hat{\alpha} = 9.3279 \times 10^{-5}$. (Right) Energy spectra displayed in (a) as functions of the rescaled frequency ω/ω_c. Green line : Kolmogorov-Zakharov spectrum $E_\omega \propto \log(\frac{\omega_c}{\omega})^{1/3}$

Fig.3(a) exhibits energy spectra obtained for each damping case of [7] for the same energy flux in $\omega = 0$. It appears that the more the system is damped, the smallest is the characteristic frequency of the system. Moreover, it looks like the dissipation process affects the energy transfers between scales such that the stationary spectra do not look like power law anymore.

Fig.3(b) displays the previous spectra as functions of the rescaled frequency ω/ω_c. They all converge in a same function that is steeper than the Kolmogorov-Zakharov spectrum (displayed in green in Fig.3(b)). This recalls what we have already reported in [7]: damping plays an important role in the discrepancies between theoretical and experimental spectra. However, since experiments allows studying only a very small range of scales which is not disturbed by the injection process or the fast decay of the dissipative scales, the fact that all curves can be rescaled in one function was not clear for us.

Now, thanks to the phenomenological model, a simple explanation of the behaviour of the cascade in presence of damping is provided. Actually, the amplitude of the energy spectrum depends on the balance between the conservative term $\partial_\omega(\omega E_\omega^2 \partial_\omega E_\omega)$ and the damping term $\hat{\gamma}_\omega E_\omega$. There is obviously no inertial range so that the stationary solution depends on the form of the dissipation function and is steeper than the Kolmogorov-Zakharov spectrum.

Investigating the behaviour of the characteristic frequency

In order to conclude this analysis, the physical interpretation of the characteristic frequency is clarified assuming that the cascade stops because the injected energy has been completely dissipated by all the excited modes, yielding for the dissipated power ε_D

$$\varepsilon_D = \int_0^{\omega_c} \hat{\gamma} E_\omega d\omega. \tag{12}$$

Using the general form of the energy spectrum $E_\omega \propto \varepsilon_I^{1/3}(\omega/\omega_c)^{-\beta}$ and a dissipation term $\hat{\gamma} \propto \hat{\alpha}\omega^\lambda$ with λ unknown, the energy budget of the cascade becomes

$$\varepsilon_D \propto \gamma_* \varepsilon_I^{1/3} \omega_c^{1+\lambda}. \tag{13}$$

Note that this relation does not depend on the slope of the spectra β. Finally, since ε_I corresponds to the injected power, it should be equal or at least proportional to ε_D. We obtain the following expression for the characteristic frequency

$$\omega_c \propto \hat{\alpha}^{-\frac{1}{1+\lambda}} \varepsilon_I^\theta \quad \text{with} \quad \theta = \frac{2}{3(1+\lambda)}. \tag{14}$$

Fixing λ and ε_I

It has been reported previously that for $\lambda = 0.6$ and $\varepsilon_I = 10^{-5}$, increasing the damping coefficient $\hat{\alpha}$ leads to decrease the characteristic frequency ω_c. This behaviour is compared to the theoretical predictions provided by Eq.(14) for chosen values of λ and ε. Fig.4(a) displays the characteristic frequency as a function of $\hat{\alpha}$ obtained by solving Eq.(11) with $\hat{\gamma} = \hat{\alpha}\omega^\lambda$, $\lambda = 0.6, 1, 2$, and $\varepsilon_I = 10^{-5}$. A perfect agreement is found with the theoretical predictions provided by Eq.(14).

Fixing λ and γ_*

Eq.(14) predicts also, for a given dissipation law $\hat{\gamma} = \hat{\alpha}\omega^\lambda$, the following relation between the injected flux and the characteristic frequency,

$$\omega_c \propto \varepsilon_I^\theta \quad \text{with} \quad \theta = \frac{2}{3(1+\lambda)}. \tag{15}$$

Fig.4(b) shows these predicted functions compared to results coming from numerical simulations of Eq.(11) for $\hat{\alpha} = 1$ and $\lambda = 0.1, 0.5, 1, 2$. In these cases, the theoretical predictions given by Eq.(14) are respectively $\theta = 0.60, 0.44, 0.33, 0.22$. The matching is very good, acting that the behaviour of the characteristic frequency can be fully explained theoretically by looking at the energy budget of the cascade.

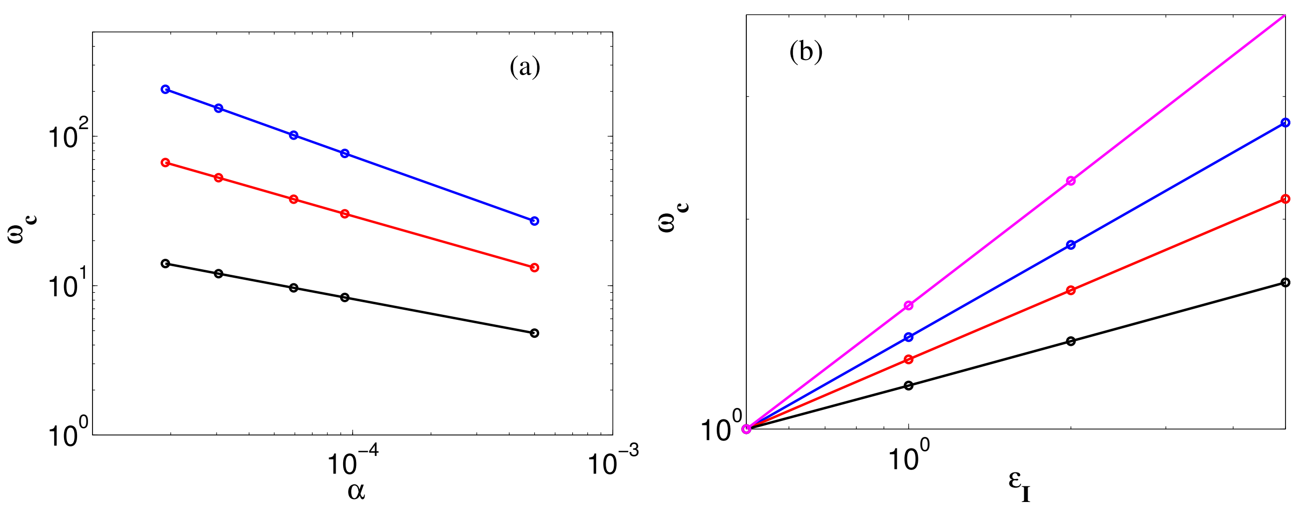

Figure 4: (a) Characteristic frequency ω_c as a function of the damping coefficient $\hat{\alpha}$. Black : $\lambda = 2$, red $\lambda = 1$, blue $\lambda = 0.6$. Lines: evolution laws predicted by Eq.(14). (b) Caracteristic frequency ω_c as a function of the injected flux ε_I. Black: $\lambda = 2$, red: $\lambda = 1$, blue: $\lambda = 0.5$, magenta: $\lambda = 0.1$. Lines: evolution laws predicted by Eq.(15).

Conclusions and outlooks

A phenomenological model built from the two solutions of the kinetic equation of wave turbulence in thin plates has been presented in this contribution and applied to the framework of non-stationary turbulence in two cases : the free turbulence and a turbulence excited by a constant flux of energy over time. By solving the phenomenological equation, a self-similar behaviour has been observed in both cases, and justified in a second time by an analysis in self-similar variables. Those two examples show the ability of our model to capture the most significant features of the dynamic of thin elastic plates and a comparison with direct non-stationary simulations of the Foppl Von Kármán equations presented in [10] will validate definitely the work presented here.

By including in the phenomenological model dissipation rates coming from the experiments, the conclusions drawn by [7] have been recalled : in vibrating plates, damping acts at all scales and breaks the transparency window required by the wave turbulence theory. Then, no more power law behaviour can be observed, and the slope of energy spectra doesn't represent the most important parameter to look at any more. On the contrary, studying the behaviour of the characteristic frequency can be done by writing the energy budget of the cascade. This process predicts evolution laws that are in perfect agreements with the results provided by the phenomenological model.

References

[1] V. E. Zakharov, V. S. Lvov, G. E. Falkovich (1992) "Kolmogorov Spectra of Turbulence I: Wave Turbulence", *S*pringer.

[2] S. Nazarenko (2011) "Wave turbulence", Vol. 825. *S*pringer.

[3] G. Düring, C. Josserand and S. Rica (2006) "Weak Turbulence for a Vibrating Plate: Can One Hear a Kolmogorov Spectrum?", *Physical Review Letters* **97**, 025503.

[4] A. Boudaoud, O. Cadot, B. Odille and C. Touzé (2008) "Observation of Wave Turbulence in Vibrating Plates", *Physical Review Letters* **100**, 234504.

[5] N. Mordant (2008) "Are there Waves in Elastic Wave Turbulence ?", *Physical Review Letters* **100**, 234505.

[6] B. Miquel and N. Mordant (2011) "Nonlinear dynamics of flexural wave turbulence", *Physical Review E*, **84**.

[7] T. Humbert, O. Cadot, G. Düring, C. Josserand, S. Rica and C. Touzé (2013) "Wave Turbulence in Vibrating Plates: the Effect of Damping", *Europhysics Letters* **102**(3), 30002.

[8] G. E. Falkovich, A. V. Shafarenko (1991) "Nonstationary Wave Turbulence", *Journal of Nonlinear Science* **1**, 457.

[9] C. Connaughton, A. Newell, Y. Pomeau (2003) "Non-stationary Spectra of Local Wave Turbulence", *Physica D* **184**, 64.

[10] M. Ducceschi, O. Cadot, C. Touzé and S. Bilbao (2013) "Dynamics of the wave turbulence spectrum in vibrating plates: a numerical investigation using a conservative finite difference scheme", submitted to *Physica D*.