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NUMERICAL APPROXIMATION OF LEVEL SET POWER MEAN CURVATURE FLOW

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Abstract. In this paper we investigate the numerical approximation of a variant of the mean curvature flow. We consider the evolution of hypersurfaces with normal speed given by H^k, $k \geq 1$, where H denotes the mean curvature. We use a level set formulation of this flow and discretize the regularized level set equation with finite elements. In a previous paper we proved an a priori estimate for the approximation error between the finite element solution and the solution of the original level set equation. We obtained an upper bound for this error which is polynomial in the discretization parameter and the reciprocal regularization parameter. The aim of the present paper is the numerical study of the behavior of the evolution and the numerical verification of certain convergence rates. We restrict the consideration to the case that the level set function depends on two variables, i.e. the moving hypersurfaces are curves. Furthermore, we confirm for specific initial curves and different values of k that the flow improves the isoperimetrical deficit.

Key words. geometric evolution equations, level set formulation, viscosity solution, finite elements

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1. Introduction.

Geometric evolution equations, especially curvature-dependent interface motion has been studied for many years in both pure and applied mathematics. One application example is the evolution of soap films and the behavior of the boundaries of oil drops on a surface of water which evolve into disks. In material science, for example, the evolving surfaces might be grain boundaries in alloys which separate differing orientations of the same crystalline phase. In image processing, for example, one wants to identify a dark shape in a light background in a two-dimensional image. Therefore a so-called snake contour is evolved so that it wraps around the shape. We refer to [23] for a more detailed exposition of these applications and, e.g., [23, 15, 16, 41] and references therein for further applications.

The most famous case for such an interface motion is the mean curvature flow of closed n-dimensional hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$ or as special case the curve shortening flow of closed curves in the plane. Under this flow the hypersurface moves in normal direction so that the normal speed equals the (mean) curvature and a convex initial hypersurface shrinks to a round point in finite time, where in case of closed, embedded plane curves this even holds if the convexity assumption is left out, cf. [37] and [29, 31, 36].

Mean curvature flow can be formulated in parametric form, where the moving hypersurface is given by a parametrization over a fixed hypersurface which depends on the evolution time as variable, cf. [37]; a special case is graphical mean curvature flow, where the hypersurface is given as the graph of a height function, cf. [26]. A third possibility is a phase field approach to mean curvature flow, cf. [44] and references therein, and the fourth which will be considered in the following in more detail is to consider the PDE resulting from the level set formulation. The level set formulation is

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powerful because it can handle topological changes of the moving hypersurface. Level set methods were introduced by Sethian and Osher, see [46, 52, 30], and have been applied to a wide range of problems.

In the present paper we are concerned with the level set formulation of a modified version of the mean curvature flow. Instead of the mean curvature we prescribe the normal speed of the evolution to be a power $k \geq 1$ of the mean curvature and assume that the initial hypersurface has positive mean curvature. In our previous paper [40] we proved a priori error estimates for a finite element approximation of this flow. The aim of the present paper is the numerical study of the behavior of the evolution and the numerical verification of certain convergence rates. We restrict the consideration to the case that the level set function depends on two variables, i.e. the moving hypersurfaces are curves. Furthermore, we confirm for specific initial curves and different values of k that the flow improves the 'isoperimetrical deficit'.

To specify how our flow looks like we give the following parametric formulation of this flow. Let M be a smooth n-dimensional compact manifold without boundary (at the moment it is sufficient to assume only $k > 0$) and $x_0 : M \to \mathbb{R}^{n+1}$ a smooth embedding such that $M_0 = x_0(M)$ has positive mean curvature, then we consider a solution of the following fully nonlinear parabolic initial value problem. Find $T > 0$ and a smooth mapping

$$x : [0, T) \times M \to \mathbb{R}^n \tag{1.1}$$

with

$$\begin{aligned} x(0, \cdot) &= x_0, \\ \frac{d}{dt} x(t, \xi) &= -H^k \nu. \end{aligned} \tag{1.2}$$

Here, H and ν denote the mean curvature and the outer normal of $x(t, M)$ at $x(t, \xi)$, respectively. We call this a power mean curvature flow (PMCF).

Why is this flow interesting and how does this flow behave? This flow has been considered in a series of papers under different aspects. In [50] it is shown that the flow (1.2) exists on a maximal, finite time interval and that, approaching the final time, the surfaces contract to a point. In [51] the flow is considered in the case $k \geq 1$. It is shown that if initially the ratio of the biggest and smallest principal curvature at every point is close enough to 1, depending only on k and the dimension n of the hypersurfaces, then this is maintained under the flow. As a consequence the authors of [51] obtain that, when rescaling appropriately as the flow contracts to a point, the evolving surfaces converge to the unit sphere. The paper [49] shows that for $k \geq n-1$ the flow improves a certain 'isoperimetrical difference'. As singularities may develop before the volume goes to zero, a weak level-set formulation for such flows is developed and it is shown that the monotonicity of the isoperimetrical difference is still valid. This proves the isoperimetrical inequality for $n \leq 7$. A further reason which makes this flow interesting is that mean curvature flow is used in denoising images, cf. [1] for such applications, and we expect that PMCF with the possibility to choose different values for $k \geq 1$ is an interesting alternative for this purpose. We mention the remarks in [43] which state that in the case $0 < k \leq 1$ equation (1.2) plays a key role in the context of image processing.

As announced above we will perform our calculation for curves, so it is of interest to give references for this case, both of theoretical and practical nature. In [6, 7] a finite element approximation of the parametric formulation of the flow (1.2) in the

case of curves is formulated and stability bounds are derived, see also [8]. In [42] the evolution of plane curves driven by a nonlinear function of curvature and anisotropy is considered with a focus on the analysis of the parametric formulation of such a flow, see also [3, 4]. The analysis of boundaries of shapes in the context of morphological and shape image processing leads to an equation of the form (1.2) in the case of curves. This has been introduced in [1, 2, 48] and we mention especially the case $k = \frac{1}{3}$ which is the so-called affine curvature equation, cf. [5] and see also (1.8) and the following text lines. In summary one can say that our flow (1.2) (in the case $k \geq 1$) plays an important role for applications and has received a lot attention so far but the numerical approximation of its level set formulation has apart from our own previous work [40] not been analyzed yet.

We introduce our notation. The Euclidean norm of $\mathbb{R}^n$ is denoted by $|\cdot|$. For an open subset Ω of $\mathbb{R}^n$ and $m \in \mathbb{N}^*$, $p \geq 1$ we denote the corresponding Sobolev spaces by $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$. The dual spaces are denoted by $W^{-m,p}(\Omega) = W_0^{m,p}(\Omega)^*$ and the dual pairing by

$$W^{-m,p}(\Omega) \times W_0^{m,p}(\Omega) \ni (F, \varphi) \mapsto \langle F, \varphi \rangle = F\varphi \in \mathbb{R}. \tag{1.3}$$

In the following we will introduce a (stationary) level set formulation for (1.2) and start for this purpose by recalling the (time-dependent) level set formulation for the mean curvature flow, i.e. the case $k = 1$ in (1.2). Let $M_0 \subset \mathbb{R}^{n+1}$ be a given initial hypersurface and choose a continuous function $u_0 : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$M_0 = \{x \in \mathbb{R}^{n+1} : u_0(x) = 0\}. \tag{1.4}$$

If $u : [0, \infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is the unique viscosity solution of

$$\frac{d}{dt}u = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = \mathcal{H}(Du, D^2u) \tag{1.5}$$

in $\mathbb{R}^{n+1} \times (0, \infty)$ with $u(0, \cdot) = u_0$ in $\mathbb{R}^{n+1}$, where

$$\mathcal{H}(p, Y) = \sum_{i,j} \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) Y_{ij} \tag{1.6}$$

for $p = (p_i) \in \mathbb{R}^n \setminus \{0\}$ and $Y = (Y_{ij}) \in \mathbb{R}^{n \times n}$. We call the family of the

$$M(t) = \{x \in \mathbb{R}^{n+1} : u(t, x) = 0\}, \quad t > 0, \tag{1.7}$$

a (time dependent) level set mean curvature flow. Equation (1.5) is a quasilinear, degenerate and possibly singular (if $Du = 0$) parabolic equation. Existence and uniqueness of a solution for this equation is proved in [17, 18, 27].

If we include our nonlinearity H^k of the mean curvature in this formulation we get instead of (1.5) the fully nonlinear, degenerate and possibly singular parabolic equation

$$\frac{d}{dt}u = |Du| \left(\operatorname{div}\left(\frac{Du}{|Du|}\right)\right)^k = \left(\sum_{i,j} \left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2}\right) D_i D_j u\right)^k |Du|^{1-k} \tag{1.8}$$

To our knowledge an existence proof for (1.8) is only known for the case $0 < k \leq 1$, cf. [43] and the references therein. But the case under consideration in the present

paper is $k > 1$. In case $k > 1$ the proof presented in [43] does not work any more because the linear growth of the elliptic part of the operator needed to apply classical arguments is not available.

The time dependent formulation 1.8 in the $k = \frac{1}{3}$ case, i.e. the affine curvature equation, is used for image processing, cf. [1, 30]. In [43] equation (1.8) in case $0 < k \leq 1$ is approximated by a family of regularized equations and rates of convergence of the corresponding solutions are obtained.

In the case $k = 1$, of course, we have mean curvature flow, and the corresponding equation (1.5) has been studied intensively analytically and numerically, cf., e.g., [14, 20, 23, 39]. We want to point out the paper [21] by Deckelnick, where the solution u^ε of a regularized version of (1.5) is approximated by a finite difference scheme which was originally proposed by Crandall and Lions [20]. In Deckelnick's paper rates for the convergence of the discrete solution to the solution u of the (not regularized) level set equation are proved. The total error consists of a regularization error of the form

$$\|u - u^\varepsilon\|_{L^\infty(\Omega)} \leq c_\alpha \varepsilon^\alpha \tag{1.9}$$

with $\alpha \in (0, \frac{1}{2})$ arbitrary and c_α a positive constant, see [21, Theorem 1.2] for details, and a discretization error which is a *polynomial* expression in the numerical parameter and the reciprocal regularization parameter. Furthermore, the concrete value for the convergence order of the discretization error (and hence for the total approximation error) is very low; the main point here is that this rate is of polynomial order.

This is not self-evident as can be seen in the paper [24]. There the viscosity solution u of (1.5) is approximated by a solution u_ε of the regularized equation and then the regularized equation is approximated by a solution $u_{\varepsilon,h}$ of a semi discrete problem. The regularization error is again of the form (1.9) but the error $u_\varepsilon - u_{\varepsilon,h}$ measured in a certain energy norm, cf. [23, Theorem 6.4], is only of order $c_\varepsilon h$, where, and this is the important point, the constant c_ε depends *exponentially* on $\frac{1}{\varepsilon}$. Numerical tests as written there, however, suggest that the resulting bound overestimates the error. In the special case of two dimensions, i.e. the moving hypersurfaces are curves, Deckelnick and Dziuk [24] prove L^∞-convergence (without rates) of the discrete solution provided $h = h(\varepsilon)$ sufficiently small, where 'sufficiently small' is not given by an explicit formula or polynomial dependence.

Let us now consider the case $k \geq 1$ which is the relevant one in the present paper. To circumvent the above mentioned problem with the growth in the elliptic part of the operator if $k > 1$ Schulze [50] uses a stationary level set formulation at which the nonlinearity due to the exponent k affects only lower order terms. We present Schulze's stationary level set formulation of the PMCF.

Let $\Omega \subset \mathbb{R}^{n+1}$ be open, connected and bounded having smooth boundary $\partial\Omega$ with positive mean curvature. Here, $\partial\Omega$ plays the role of the initial hypersurface. We call the level sets $\Gamma_t = \partial\{x \in \Omega : u(x) > t\}$, $t \geq 0$, of the continuous function $0 \leq u \in C^0(\bar{\Omega})$ a (stationary) level set PMCF, if u is a viscosity solution of

$$\begin{aligned} \operatorname{div}\left(\frac{Du}{|Du|}\right) &= -\frac{1}{|Du|^{\frac{1}{k}}} && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.10}$$

For a definition of a viscosity solution for this equation we refer to [40, Section 2]. If u is smooth in a neighborhood of $x \in \Omega$ with non vanishing gradient and satisfies in this neighborhood (1.10), then the level set $\{u = u(x) | x \in \Omega\}$ moves locally at

x according to (1.2). Using elliptic regularization of level set PMCF we obtain the equation

$$\begin{aligned} \operatorname{div}\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}\right) &= -(\varepsilon^2+|Du^\varepsilon|^2)^{-\frac{1}{2k}} && \text{in } \Omega, \\ u^\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.11}$$

which has a unique smooth solution u^ε for sufficiently small $\varepsilon > 0$, cf. [50, Section 4]; moreover, there is $c_0 > 0$ such that

$$\|u^\varepsilon\|_{C^1(\bar\Omega)} \leq c_0 \tag{1.12}$$

uniformly in ε and (for a subsequence)

$$u^\varepsilon \to u \in C^{0,1}(\bar\Omega) \tag{1.13}$$

in $C^0(\bar\Omega)$. We call u a weak solution of (1.10), which is unique for $n \leq 6$. All the above facts are proved in [50, Section 4] under the assumption that $k \geq 1$. A weak solution of (1.10) satisfies (1.10) in the viscosity sense, cf. Section [40, Section 2]. Furthermore Schulze's existence result is restricted to the case $k \geq 1$.

What looks as a disadvantage at first glance, namely the fact that our level set function does not depend on the time explicitly and hence no explicit Euler method is applicable (as for example in Deckelnick's paper [21]), has the advantage that we have a divergence structure for the elliptic part of the operator which would not be the case if we would use the time dependent level set formulation (1.8) (in addition we would lack a proof of existence of a solution).

To our knowledge our previous paper [40] is the only numerical analysis result for Schulze's level set formulation so far. In [40] we proved an explicit rate for the convergence of the solution u^ε of (1.11) to the solution u of equation (1.10) which depends on k, see Section 3, where we recall the result. Using the divergence structure of the elliptic part of (1.11) we proved existence of a finite element approximation u_h^ε of u^ε and an approximation rate. Summarized we have a total approximation error

$$u - u_h^\varepsilon = (u - u^\varepsilon) + (u^\varepsilon - u_h^\varepsilon) \tag{1.14}$$

which consists of the regularization error (first bracket in equation (1.14)) and the discretization error (second bracket in equation (1.14)). We obtained a *polynomial* rate in $\frac{1}{\varepsilon}$ and h for the total approximation error provided the discretization parameter is sufficiently small compared with the regularization parameter (the coupling between the discretization and the regularization parameter is also of polynomial order). The order of convergence is polynomial (in contrast, e.g., to [23, Theorem 6.4], where the authors obtain exponential order of convergence) and comparable to the one obtained in [21]. In both cases despite from being of polynomial order the precise order is rather of theoretical value. As in the remarks following [23, Theorem 6.4] stating that experiments indicate that the proven approximation error overestimates the real error, we have a similar behavior for our situation, cf. Sections 2, 3, and 4.

Furthermore, we validate for some examples that the flow improves (i.e. decreases) the isoperimetrical deficit

$$A(t)^{\frac{n+1}{n}} - c_{n+1}V(t), \tag{1.15}$$

where $A(t)$ denotes the n-dimensional volume of the evolving hypersurface, $V(t)$ the $(n+1)$-dimensional volume of the enclosed subset of $\mathbb{R}^{n+1}$ and c_{n+1} the Euclidean isoperimetrical constant, cf. [50] for a proof of this property. The isoperimetrical deficit (1.15) is nonnegative and zero if and only if the hypersurface is a sphere. Since we restrict ourselves to the case $n = 1$ the evolving hypersurfaces are curves and the variables in the isoperimetrical deficit become arc length and enclosed area. And there holds $c_2 = 4\pi$. Geometrically more interesting is the case $n \geq 2$ because then one can have non convex initial hypersurfaces with positive mean curvature which develop topological changes under the flow. To validate our level set ansatz with respect to its convergence behavior and numerical properties restricting ourselves to the curve case seems to be a reasonable and legitimate technical simplification.

The paper is organized as follows. In Section 2 we prove an error estimate for the discretization error and validate it numerically. In Section 3 we recall and calculate in detail an error estimate for the regularization error from our previous paper [40] and provide numerical examples. In Section 4 we provide numerical examples for the total approximation error. In Section 5 we illustrate with an example the influence of the exponent k on the behavior of the flow.

2. Discretization error. In this section we assume that the space dimension $n+1$ is 2 or 3 and that Ω is convex. The latter is only a restriction if $n+1 = 3$ since $\partial\Omega$ has positive mean curvature by our assumptions in the introduction. We fix $\varepsilon > 0$ at a small value and approximate the solution u^ε of equation (1.11) by a finite element solution u_h^ε which seems to be an appropriate method in view of the divergence structure of the operator. The goal is to analyze the discretization error $u^\varepsilon - u_h^\varepsilon$.

Let (T_h, Ω_h) be a quasi-uniform triangulation of Ω with mesh size $0 < h < h_0$, h_0 sufficiently small, and $V_h \subset H^1(\Omega_h)$ the finite element space given by

$$V_h = \left\{ v \in C^0(\bar{\Omega}_h) : v_{|\partial\Omega_h} = 0,\ v_{|T} \text{ linear } \forall T \in T_h \right\}. \tag{2.1}$$

In view of the convexity of Ω there holds $\Omega_h \subset \Omega$. A function $u_h \in V_h$ will be also considered as a function on Ω by extending it by zero in $\Omega \setminus \Omega_h$. Then $v_h \in H^1(\Omega)$. Our variational formulation is given as in our previous paper [40] by

$$\int_{\Omega_h} \frac{\langle Du_h^\varepsilon, Dv_h \rangle}{\sqrt{\varepsilon^2 + |Du_h^\varepsilon|^2}} dx = \int_{\Omega_h} (\varepsilon^2 + |Du_h^\varepsilon|^2)^{-\frac{1}{2k}} v_h dx \quad \forall\ v_h \in V_h. \tag{2.2}$$

For formal reason we might consider boundary tetrahedrons (boundary triangles in case $d = 2$) to be extended to a boundary tetrahedron with one 'curved face'. Therefore we will replace a boundary element $T \in T_h$ (i.e. $n+1$ vertexes of T lie on $\partial\Omega$) by $\tilde{T} = T \cup B$ with

$$B = \{tp + (1-t)Pp \mid 0 \leq t \leq 1, p \in F\}, \tag{2.3}$$

where F is the boundary face of T, i.e. $n+1$ vertexes of F lie on $\partial\Omega$, and Pp is the unique minimizer of $\operatorname{dist}(p, \cdot)_{|\partial\Omega}$. We denote the resulting triangulation by $\tilde{T}_h$. This leaves the space of finite element functions we use (namely V_h) unchanged. Note, that the boundary strip $\Omega \setminus \Omega_h$ has measure $O(h^2)$.

A similar equation as (2.2) is the stationary level set formulation for the inverse mean curvature flow which is used in [28]. There also a total approximation error, a discretization error and a regularization error appears. Furthermore, a rate for the

discretization error ($O(h)$ for the H^1-error and $O(h^2)$ for the L^2-error) is proved. But the dependence of the constants on the regularization parameter which appear in these error estimates is not analyzed theoretically. In contrast to [40] in [28] no theoretical estimate for the regularization error (and hence for the total approximation error) is given, and such a rate seems to be an open problem so far, cf. [28, Remark 4]. But this issue is addressed numerically in [28] and calculations suggest that the regularization error is $O(\varepsilon)$, where ε here also denotes the regularization parameter.

We remark that when we considered the discretization in our previous paper, see [40, Section 6], the space V_h consisted of continuous functions on Ω_h which are piecewise polynomials of degree ≤ 2 (and not linear as here) and we assumed that $\Omega \subset \mathbb{R}^2$ and existence of a solution of (2.2) in this case was shown. The reason for this is that in [40, Section 6] error bounds for the discretization error are proved which contain the dependence of ε explicitly. Therefore an estimate for the norm of the inverse of L_ε and its dual L_ε^*, where L_ε is the derivative of the regularized differential operator, see (2.8) for a definition, is calculated via the intermediate step of some rather technical sup-norm estimates and inverse inequalities which make it necessary to consider higher order elements. Since these ε-dependencies do not play a role for the discretization error under consideration in this section we present the proof for the present case in easier form and also for $W^{1,p}$-norms in Theorem 2.2 with general $p \geq n+1$ (contrary to [40, Section 6], where $p < 4$ is assumed) which is necessary to prove an optimal L^2-error estimate, cf. Theorem 2.3.

We start with a definition and properties of the linear operator L_ϵ and its dual.

Let $p > 1$. We define for $\varepsilon > 0$ and $z \in \mathbb{R}^n$

$$|z|_\varepsilon := f_\varepsilon(z) := \sqrt{|z|^2 + \varepsilon^2} \tag{2.4}$$

and denote derivatives of f_ε with respect to z^i by $D_{z^i} f_\varepsilon$. There holds

$$D_{z^i} f_\varepsilon(z) = \frac{z_i}{|z|_\varepsilon}, \quad D_{z^i} D_{z^j} f_\varepsilon(z) = \frac{\delta_{ij}}{|z|_\varepsilon} - \frac{z_i z_j}{|z|_\varepsilon^3}. \tag{2.5}$$

We define the operator Φ_ε by

$$\Phi_\varepsilon : W_0^{1,p}(\Omega) \to W^{-1,p^*}(\Omega), \quad \Phi_\varepsilon(v) = -D_i \left(\frac{D_i v}{|Dv|_\varepsilon} \right) - \frac{1}{|Dv|_\varepsilon^{\frac{1}{k}}}, \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$, so that (1.11) can be written as

$$\Phi_\varepsilon(u^\varepsilon) = 0. \tag{2.7}$$

We denote the derivative of Φ_ε in u^ε by

$$L_\varepsilon := D\Phi_\varepsilon(u^\varepsilon) \tag{2.8}$$

and have for all $\varphi \in W_0^{1,p}(\Omega)$ that

$$\begin{aligned} L_\varepsilon \varphi &= -D_i \left(D_{z^i} D_{z^j} f_\varepsilon(Du^\varepsilon) D_j \varphi \right) + \frac{1}{k} f_\varepsilon(Du^\varepsilon)^{-1-\frac{1}{k}} D_{z^j} f_\varepsilon(Du^\varepsilon) D_j \varphi \\ &=: -D_i(a^{ij} D_j \varphi) + b^i D_i \varphi, \end{aligned} \tag{2.9}$$

where we use the convention to sum over repeated indices. The coefficients a^{ij} and b^i are in $C^\infty(\bar{\Omega})$. Note, that the estimate (1.12) is not available for higher order

derivatives of u^ε but since we fix ε in the present section, this does not have an effect on the following considerations.

The linear operator

$$L_\epsilon : W_0^{1,p}(\Omega) \to W^{-1,p^*}(\Omega) \tag{2.10}$$

and its adjoint operator L_ϵ^* are topological isomorphism, cf. Corollary 7.2 in the Appendix.

We get from [13, Theorem 8.5.3] for $L = L_\epsilon$ or $L = L_\epsilon^*$ and $F \in W^{-1,p^*}(\Omega)$ that there is a unique solution $u_h \in V_h$ of

$$\langle Lu_h, \varphi_h\rangle = F\varphi_h \quad \forall \varphi_h \in V_h, \tag{2.11}$$

where $u \in H^1(\Omega)$ is the unique solution of $Lu = F$ and there holds the estimate

$$\|u_h\|_{W^{1,p}(\Omega)} + \|u - u_h\|_{W^{1,p}(\Omega)} \le c\|u\|_{W^{1,p}(\Omega)}. \tag{2.12}$$

Furthermore, if $F \in L^p(\Omega)$ we have

$$\|u - u_h\|_{W^{1,p}(\Omega)} + h\|u - u_h\|_{L^p(\Omega)} \le ch^2\|F\|_{L^p(\Omega)}. \tag{2.13}$$

Remark 2.1. Note, that we actually used the assertion of [13, Theorem 8.5.3] with slightly different assumptions. The difference from the assumptions we need to the one assumed in [13, Theorem 8.5.3] are as follows.

(i) We assume a right-hand side $F \in W^{-1,p^*}(\Omega)$ (instead $F \in L^p(\Omega)$).
(ii) We consider the equation on Ω (instead of a polygonal domain) and use as discretization the triple $(\tilde{T}_h, \Omega, V_h)$.

There holds the following theorem.

Theorem 2.2. *For every $p > n + 1$ and small $h > 0$ there exists a constant $0 < c = c(\|u^\varepsilon\|_{W^{2,2}(\Omega)}, p)$ such that (2.2) has a solution $u_h^\varepsilon \in V_h$ satisfying*

$$\|u^\varepsilon - u_h^\varepsilon\|_{W^{1,p}(\Omega)} \le ch. \tag{2.14}$$

This solution is unique in a small $W^{1,p}$-neighborhood of u^ε in V_h.

Proof. The proof of this lemma is adapted from [40, Section 6], where the result is proved for $p < 4$ and quadratic finite elements.

We set

$$\bar{B}_\rho^h = \{v_h \in V_h : \|u^\varepsilon - v_h\|_{W^{1,p}(\Omega)} \le \rho\}, \tag{2.15}$$

where we choose

$$\rho = h^\lambda \tag{2.16}$$

for an arbitrary and now fixed $\frac{n+1}{p} < \lambda < 1$.

We will obtain u_h as the unique fixed point in $\bar{B}_\rho^h$ of the operator $T : V_h \to V_h$ with

$$L_\varepsilon(w_h - Tw_h) = \Phi_\varepsilon(w_h), \quad w_h \in V_h. \tag{2.17}$$

We show that $\bar{B}_\rho^h \neq \emptyset$, that T is a contraction and that $T(\bar{B}_\rho^h) \subset \bar{B}_\rho^h$.

(i) Let $I_h u^\varepsilon$ be the interpolation of u^ε, i.e. the continuous piecewise linear function on Ω_h which is equal to u^ε at all nodes of Ω_h. We extend $I_h u^\varepsilon$ by zero to a function on Ω. In view of

$$\|I_h u^\epsilon - u^\epsilon\|_{W^{1,p}(\Omega)} \leq ch \tag{2.18}$$

we have $I_h u^\varepsilon \in \bar{B}_\rho^h$ for small h.

(ii) Let $v_h, w_h \in \bar{B}_\rho^h$, $\xi_h = v_h - w_h$ then using (2.17) we conclude

$$\begin{aligned} L_\varepsilon(Tv_h - Tw_h) &= L_\varepsilon \xi_h + \Phi_\varepsilon(w_h) - \Phi_\varepsilon(v_h) \\ &= (L_\varepsilon - D\Phi_\varepsilon(v_h + \Theta\xi_h))\xi_h \\ &=: F \end{aligned} \tag{2.19}$$

with a $\Theta \in (0,1)$. In order to estimate $\|F\|_{W^{-1,p^*}}(\Omega)$ which leads to an estimate of $\|Tv_h - Tw_h\|_{W^{1,p}(\Omega)}$ in view of Corollary 7.2 we choose $\psi \in W_0^{1,p^*}(\Omega)$ with $\|\psi\|_{W^{1,p^*}(\Omega)} \leq 1$ and estimate $\langle F, \psi \rangle$. To do so we use a mean value theorem for which we need the following auxiliary estimate

$$\begin{aligned} &\|Du^\varepsilon - (Dv_h + \Theta D\xi_h)\|_{L^\infty(\Omega)} \\ &\qquad \leq \|Du^\varepsilon - DI_h u^\varepsilon\|_{L^\infty(\Omega)} + \|DI_h u^\varepsilon - D\tilde{v}_h\|_{L^\infty(\Omega)} \\ &\qquad \leq ch + c\rho h^{-\frac{n+1}{p}}, \end{aligned} \tag{2.20}$$

where $\tilde{v}_h = v_h + \Theta\xi_h \in \bar{B}_\rho^h$ and where we used an inverse estimate. The resulting estimate implies

$$\begin{aligned} \|Tv_h - Tw_h\|_{W^{1,p}(\Omega)} &\leq c(h + \rho h^{-\frac{n+1}{p}})\|\xi_h\|_{W^{1,p}(\Omega)} \\ &\leq \frac{1}{4}\|\xi_h\|_{W^{1,p}(\Omega)} \end{aligned} \tag{2.21}$$

for small h.

(iii) Let $w_h \in \bar{B}_\rho^h$. There holds

$$\begin{aligned} &\|Tw_h - u^\varepsilon\|_{W^{1,p}(\Omega)} \\ &\quad \leq \|Tw_h - TI_h u^\varepsilon\|_{W^{1,p}(\Omega)} + \|TI_h u^\varepsilon - I_h u^\varepsilon\|_{W^{1,p}(\Omega)} \\ &\qquad + \|I_h u^\varepsilon - u^\varepsilon\|_{W^{1,p}(\Omega)} \\ &\quad \leq \frac{\rho}{2} + \|TI_h u^\varepsilon - I_h u^\varepsilon\|_{W^{1,p}(\Omega)} + ch \end{aligned} \tag{2.22}$$

It remains to estimate the norm on the right-hand side. There holds

$$\begin{aligned} \|TI_h u^\varepsilon - I_h u^\varepsilon\|_{W^{1,p}(\Omega)} &\leq c\|\Phi_\varepsilon(I_h u^\varepsilon)\|_{W^{-1,p^*}(\Omega)} \\ &= c\|\Phi_\varepsilon(I_h u^\varepsilon) - \Phi_\varepsilon(u^\varepsilon)\|_{W^{-1,p^*}(\Omega)} \\ &\leq ch \end{aligned} \tag{2.23}$$

again by a mean value theorem estimate. In view of (2.16) there holds

$$T(\bar{B}_\rho^h) \subset \bar{B}_\rho^h. \tag{2.24}$$

□

In the following theorem we improve the L^p-error estimate of Theorem 2.2, therefore we use a duality argument as in [28].

THEOREM 2.3. *For $p > n+1$ there holds*

$$\|u^\varepsilon - u_h^\varepsilon\|_{L^p(\Omega)} \leq ch^2 \tag{2.25}$$

with $c = c(\|u^\varepsilon\|_{W^{2,2}(\Omega)}, p) > 0$.

Proof. From the definitions of u^ε and u_h^ε we get for all $\varphi_h \in V_h$

$$\int_\Omega \left(\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|_\varepsilon} - \frac{\nabla u_h^\varepsilon}{|\nabla u_h^\varepsilon|_\varepsilon} \right) \cdot \nabla \varphi_h dx + \int_\Omega \left(|\nabla u^\varepsilon|_\varepsilon^{\frac{1}{k}} - |\nabla u_h^\varepsilon|_\varepsilon^{\frac{1}{k}} \right) \varphi_h dx = 0 \tag{2.26}$$

This equation can be written as

$$\int_\Omega (A_h^\varepsilon \nabla e_h^\varepsilon) \cdot \nabla \varphi_h dx + \int_\Omega (a_h^\varepsilon \cdot \nabla e_h^\varepsilon) \varphi_h dx = 0 \tag{2.27}$$

with

$$\begin{aligned} A_h^\varepsilon =& \int_0^1 D^2 f_\varepsilon(\nabla u^\varepsilon + t\nabla(u_h^\varepsilon - u^\varepsilon))dt \\ a_h^\varepsilon =& \frac{1}{k} \int_0^1 f_\varepsilon(\nabla u^\varepsilon + t\nabla(u_h^\varepsilon - u^\varepsilon))^{\frac{1}{k}-1} Df_\varepsilon(\nabla u^\varepsilon + t\nabla(u_h^\varepsilon - u^\varepsilon))dt \\ e_h^\varepsilon =& u_h^\varepsilon - u^\varepsilon \end{aligned} \tag{2.28}$$

and for later purposes we set

$$\begin{aligned} \bar{A}_h^\varepsilon =& D^2 f_\varepsilon(\nabla u^\varepsilon) \\ \bar{a}_h^\varepsilon =& \frac{1}{k} f_\varepsilon(\nabla u^\varepsilon)^{\frac{1}{k}-1} Df_\varepsilon(\nabla u^\varepsilon). \end{aligned} \tag{2.29}$$

We define $\varphi \in W_0^{1,p^*}(\Omega)$ by

$$L_\varepsilon^* \varphi = |e_h^\varepsilon|^{p-1} \operatorname{sgn}(e_h^\varepsilon) \tag{2.30}$$

and let $\varphi_h \in V_h$ be the finite element solution of this equation. We test (2.30) with e_h^ε and get in view of the symmetry of $\bar{A}_h^\varepsilon$ that

$$\begin{aligned} \int_\Omega |e_h^\varepsilon|^p dx =& \int_\Omega (A_h^\varepsilon \nabla e_h^\varepsilon) \cdot \nabla \varphi_h dx + \int_\Omega (a_h^\varepsilon \cdot \nabla e_h^\varepsilon) \varphi_h dx \\ \overset{(2.27)}{=}& \int_\Omega \left((A_h^\varepsilon - \bar{A}_h^\varepsilon) \nabla e_h^\varepsilon \right) \cdot \nabla \varphi_h dx + \int_\Omega \left((a_h^\varepsilon - \bar{a}_h^\varepsilon) \cdot \nabla e_h^\varepsilon \right) \varphi_h dx \\ \leq& c \int_\Omega |\nabla e_h^\varepsilon|^2 |\nabla \varphi_h| dx + c \int_\Omega |\nabla e_h^\varepsilon|^2 \varphi_h dx \\ \leq& c \|\varphi_h\|_{W^{1,p^*}(\Omega)} \|e_h^\varepsilon\|_{W^{1,2p}(\Omega)}^2. \end{aligned} \tag{2.31}$$

In view of Corollary 7.2 and (2.30) we get

$$\|\varphi_h\|_{W^{1,p^*}(\Omega)} \leq c \left(\int_\Omega |e_h^\varepsilon|^{(p-1)p^*} dx \right)^{\frac{1}{p^*}} = \quad c \|e_h^\varepsilon\|_{L^p(\Omega)}^{\frac{p}{p^*}}, \tag{2.32}$$

so that

$$\|e_h^\varepsilon\|_{L^p(\Omega)} \leq ch^2. \tag{2.33}$$

Here, we used (2.31) and Theorem 2.2 to estimate $\|e_h^\varepsilon\|_{W^{1,2p}(\Omega)}$. □

In the following we validate the convergence rates of Theorem 2.2 and Theorem 2.3 with a numerical example. Figure 1 shows the discretization error in the case of a unit circle as initial curve and $\varepsilon = 0.1$ fixed. The discrete solutions for different values of the discretization parameter h are compared with the discrete solution $\tilde{u}^\varepsilon$ on a fine grid with grid size $h = 0.005$. We compare on $\Omega_{0.005}$ and extend $u_h^\varepsilon = 0$, where it is not defined. The discretization errors $\|\tilde{u}^\varepsilon - u_h^\varepsilon\|_{\Omega_h}$ for $\|\cdot\|_{\Omega_h} = \|\cdot\|_{L^2(\Omega_h)}$, $\|\cdot\|_{\Omega_h} = \|\cdot\|_{H^1(\Omega_h)}$ and $\|\cdot\|_{\Omega_h} = \|\cdot\|_{L^\infty(\Omega_h)}$ are plotted and behave as shown in Theorem 2.3 and Theorem 2.2.

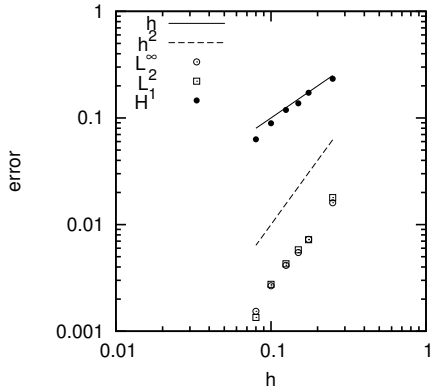

Fig. 1: Discretization error for the unit circle as initial curve where $k = 1$, $\varepsilon = 0.1$.

In Figure 2 we see that the discretization error close to the boundary has a pattern which results from the approximation of the smooth domain Ω by the polygonal domain Ω_h. The discrete solution u_h^ϵ is zero at the polygonal boundary $\partial\Omega_h$ while u^ϵ is zero on the curved boundary $\partial\Omega$.

3. Regularization error. We specify our a priori estimate for the regularization error presented in [40] which depends on the choice of certain constants. Let

$$\gamma > 1 + k \tag{3.1}$$

and $\alpha, s > 0$ so that

$$\beta_1(\alpha, s) > \beta_2(\alpha, s), \tag{3.2}$$

where

$$\beta_1(\alpha, s) := \frac{2 - s + \alpha(2 - \frac{1}{k})}{\gamma(2 - \frac{1}{k}) + \frac{1}{k} - 1}, \quad \beta_2(\alpha, s) := \frac{\alpha + ks}{\gamma - k - 1} \tag{3.3}$$

and choose

$$0 < r < \frac{\alpha}{\gamma}. \tag{3.4}$$

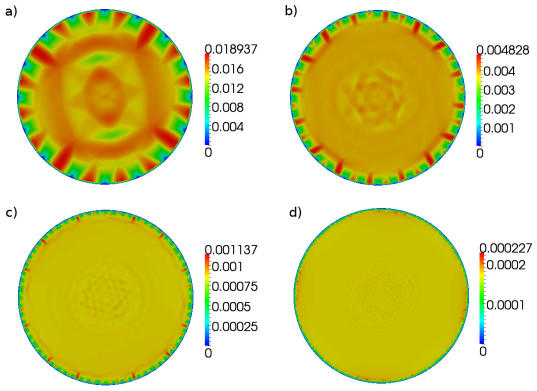

Fig. 2: Discretization error for the unit circle as initial curve (we compare with a discrete solution on a fine grid with size h_0). The pictures show the difference $u^\varepsilon_{h_0} - u^\varepsilon_h$ for $\varepsilon = 0.09$, $k = 1$, $h_0 = 0.025$ and a) $h = 0.4$, b) $h = 0.2$, c) $h = 0.1$, d) $h = 0.05$.

Note, that (3.2) obviously holds for sufficiently small α, s. There holds the following theorem, cf. [40, Theorem 3.1].

THEOREM 3.1. *There is $c = c(k, \Omega) > 0$ such that*

$$\|u^\varepsilon - u\|_{C^0(\bar{\Omega})} \leq c\varepsilon^{\min(r,s)} \tag{3.5}$$

for all $\varepsilon > 0$.

The order of convergence stated in the previous theorem can be written more explicitly which is content of the following lemma.

COROLLARY 3.2. *For the evolution with normal speed H^k, $k \geq 1$, the regularization error with respect to the C^0-norm is of order $O(\varepsilon^{\frac{1}{\lambda}})$ for all $\lambda > 2k$.*

Proof. We rewrite the right-hand side of the estimate in Theorem 3.1. In (3.2) we may assume that α, s are related by $s = \frac{\alpha}{\gamma}$. We multiply (3.2) by γ and get

$$\frac{2 - s + \alpha(2 - \frac{1}{k})}{2 - \frac{1}{k} + \frac{\frac{1}{k}-1}{\gamma}} > \frac{\alpha + sk}{1 - \frac{k+1}{\gamma}}. \tag{3.6}$$

Now, we multiply with the denominator of the left-hand side and sort by α and s on each side which leads to

$$2 - s + \alpha\left(2 - \frac{1}{k}\right) > \alpha\frac{2 - \frac{1}{k}}{1 - \frac{k+1}{\gamma}} + s\frac{\frac{1}{k} - 1 + k\left(2 - \frac{1}{k} + \frac{\frac{1}{k}-1}{\gamma}\right)}{1 - \frac{k+1}{\gamma}} \tag{3.7}$$

and after rearranging terms to

$$2 > s\frac{\frac{1}{k} - \frac{k+1}{\gamma} + k\left(2 - \frac{1}{k} + \frac{\frac{1}{k}-1}{\gamma}\right) + (2 - \frac{1}{k})(k+1)}{1 - \frac{k+1}{\gamma}}. \tag{3.8}$$

We may let γ tend to infinity without changing the value of s (by adapting α correspondingly). Hence the right-hand side of (3.8) converges to $4ks$ as $\gamma \to \infty$ and the claim follows. □

We recall an interpolation lemma, cf [33, PDE II, Lemma 1.4.13].

LEMMA 3.3. *For $0 < \beta < \alpha \leq 1$ and a function $v : \Omega \to \mathbb{R}$ holds*

$$[v]_\beta \leq 2^{1-\frac{\beta}{\alpha}} [v]_\alpha^{\frac{\beta}{\alpha}} \|v\|_{C^0(\Omega)}, \tag{3.9}$$

where these expressions might become ∞ and

$$[v]_\alpha = \sup_{x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\alpha}. \tag{3.10}$$

Since u^ε is uniformly bounded in the C^1-norm, cf. (1.12), we can use Lemma 3.3 to get

$$\|u - u^\varepsilon\|_{C^{0,\beta}(\Omega)} \leq c(\beta) \varepsilon^{\lambda(1-\beta)} \tag{3.11}$$

for every $0 < \beta < 1$ and $0 < \lambda < \frac{1}{2k}$.

In the case $k = 1$ which means mean curvature flow we can realize in Corollary 3.2 every power of ε which lies in $(0, \frac{1}{2})$. This is in accordance with the corresponding rate for the case of the time dependent level set regularization as considered in Deckelnick's paper [21, Theorem 1.2] and Mitake's paper [43, Theorem 1].

The following numerical examples indicate that the regularization error is even smaller than stated in Corollary 3.2. As a first example we calculate the regularization error in the case of the evolution of a unit circle as initial curve for which the exact solution of equation (1.10) is known. Let $\partial B_{r_0}(0) \subset \mathbb{R}^2$, i.e. a circle with radius $r_0 > 0$, be the initial curve then the exact solution u is given as

$$u(r) = \frac{r_0^{k+1} - r^{k+1}}{k+1}, \tag{3.12}$$

where r denotes the radius variable in polar coordinates in $\mathbb{R}^2$ with center in 0. As special case we choose $k = 1$, $r_0 = 1$, i.e.

$$u(r) = \frac{1}{2} - \frac{r^2}{2}. \tag{3.13}$$

In Figure 3 the error $\|u^\varepsilon - u\|$ is plotted in this special case (and for $k = 1.5$ and $k = 2$), where $\|\cdot\|$ stands for $\|\cdot\| = \|\cdot\|_{L^2(\Omega_h)}$, $\|\cdot\| = \|\cdot\|_{H^1(\Omega_h)}$ and $\|\cdot\| = \|\cdot\|_{L^\infty(\Omega_h)}$. We remark that our theoretical estimate in Corollary 3.2 does not provide information about an estimate with respect to $\|\cdot\|_{H^1(\Omega_h)}$. The functions u^ε are calculated by using linear finite elements on a fine grid with mesh size $h = 0.0125$. The L^2-error converges a little bit faster and the H^1-error a little bit slower than of quadratic order to zero.

In Figure 4 the scenario is the same as in Figure 3 apart from the fact that we now choose an ellipse (half axes lengths 1 and 2) as initial curve. Furthermore since we do not have an exact solution u for this case we use instead a solution u_h^ε with $\varepsilon = 0.1$ and small $h = 0.0125$. In Figure 5 we plot a section (along the long and short half axes of the initial curve) of the solution u^ε in the case of the circle and in Figure 6 in the case of an ellipse as initial curve for different values of ε.

In accordance with our a priori estimate in Corollary 3.2 the regularization error in Figure 3 seems to become larger for increasing k. This can be also seen from Figure 5. There we also observe that the approximation quality around the singularity of the flow deteriorates for $k = 2$ when changing ε from 0.5 to 0.17.

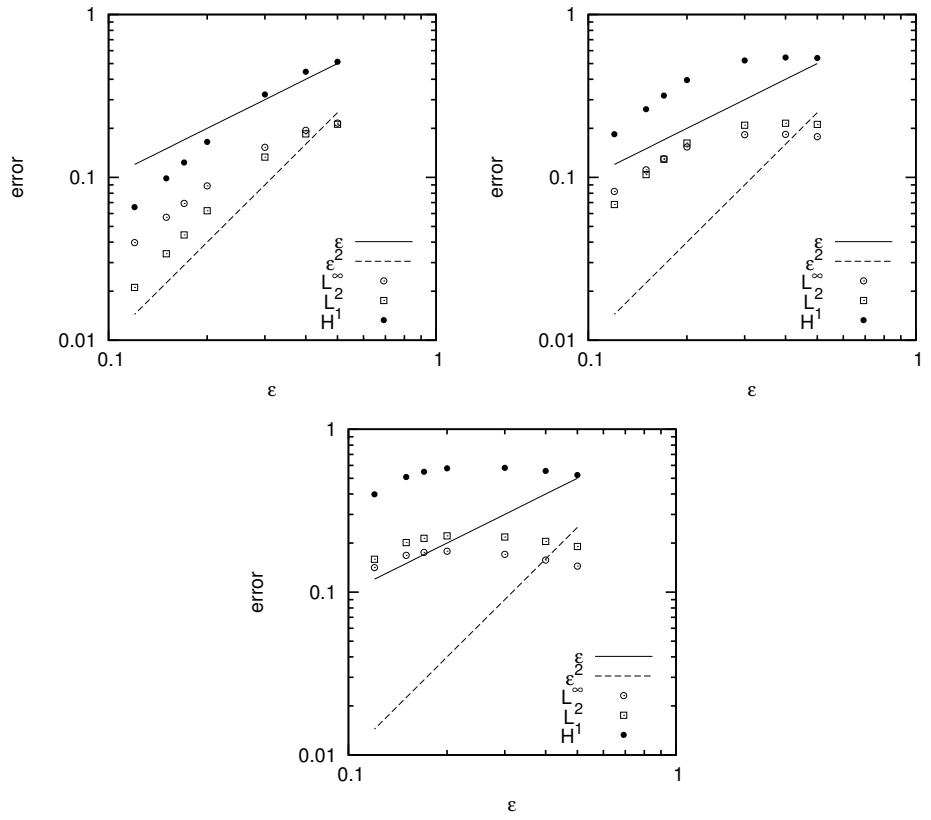

Fig. 3: Regularization error in case of a circle as initial curve for $k = 1, 1.5, 2$.

Figure 7 shows level sets of $u^{0.1}$ for the case of the ellipse as initial curve and different values of k. We remark that our theory covers only the case $k \geq 1$ but in the special case of convex curves we also have a level set solution for general $k > \frac{1}{3}$ which follows from the classification of the behavior of the evolution of curves by powers of the curvatures presented in Section 1 of [4]. Our observations are as follows. For $k = 0.5$ we see for $\varepsilon = 0.1$ a quite good approximation of the phenomenon of shrinking to a 'round point' and further lessening of ε does not show significant improvements. For all k the inner level line for $\varepsilon = 0.1$ seems to be already 'round' while for $k = 2$ this seems to be far from a 'point'.

4. Total approximation error. In the inequalities (2.14) and (2.25) appear constants c on the right-hand sides which depend on the solution u^ε of the regularized equation. To get an estimate for $u - u_h^\varepsilon$ in terms of ε and h one has to make this dependence explicit. In our paper [40] we showed that there is a $\gamma > 0$ such that if we couple h and ε by $h = \varepsilon^\gamma$ and use finite elements of order 2 (and quadratic boundary approximations), then there holds that the error $u - u_h^\varepsilon$ converges to zero with a polynomial rate in h with respect to the sup-norm. To estimate with respect to the sup-norm is natural since u is (only, in general,) a C^0-limit of u^ε and (as viscosity solution) Lipschitz continuous. A value for γ and the convergence rate can be obtained by adapting it at each stage of the proofs in [40] as described there which

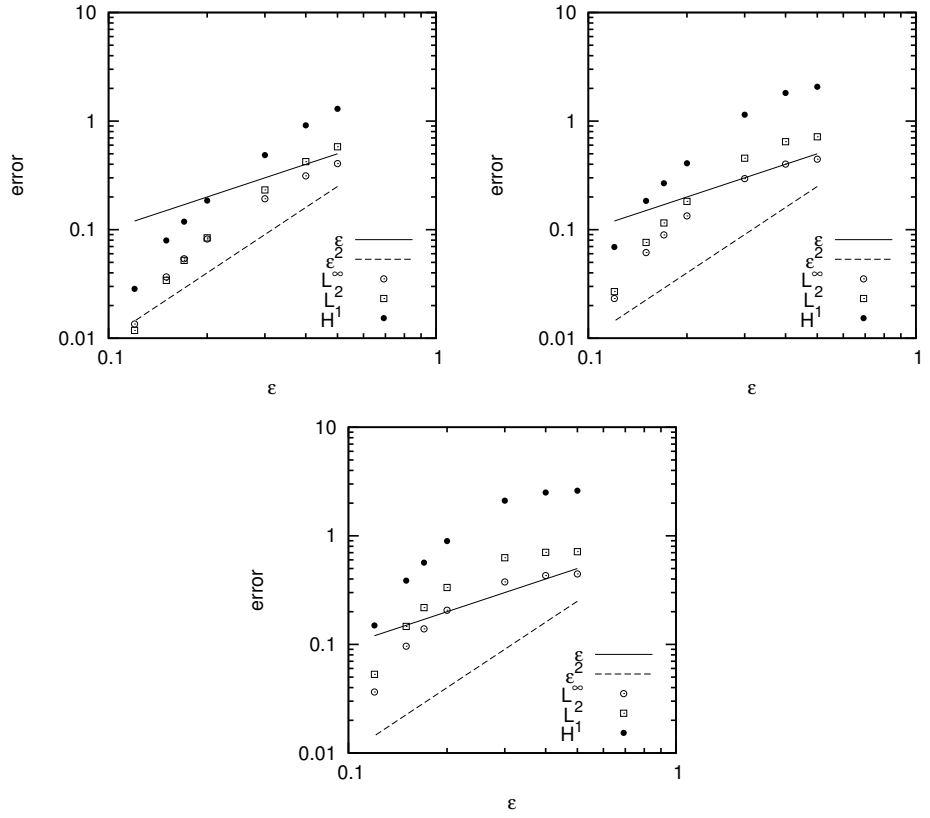

Fig. 4: Regularization error in case of an ellipse as initial curve for $k = 1, 1.5, 2$.

leads to a rather technical large value of no practical interest. The main point is that we have a polynomial rate and not an exponential rate. As said before and explained by comparing the situation with [23, Theorem 6.4], where the authors proved even only an exponential estimate, this is non-trivial. We let us furthermore inspire from the scenario of [23, Theorem 6.4] which overestimates the error rate as practical results indicate, cf. [23]. Therefore we start our calculations with the from practical point of view comfortable setting of continuous and piecewise linear finite elements, a polygonal boundary approximation and a coupling between ε and h by setting $\varepsilon = h$ which already lead to convergence.

Figure 8 shows the total approximation errors for the unit circle as initial curve in the cases $k = 1, 1.5, 2$. Although we have only for the sup-norm a theoretical estimate we also plot the H^1-error; we remark that in the situation of the circle the solution is of class $C^\infty(\bar{B}_{r_0}(0))$. Figure 9 shows the same scenario as Figure 8 apart from the fact that we now consider the ellipse (half axes with lengths 1 and 2) as initial curve. Furthermore, as reference solution we consider a solution with $h = 0.05$ and $\varepsilon = 0.05$.

5. Effect of k on behavior of the flow for an example case. The phenomenon of becoming round can be measured by the isoperimetrical deficit

$$l(t)^2 - 4\pi a(t), \tag{5.1}$$

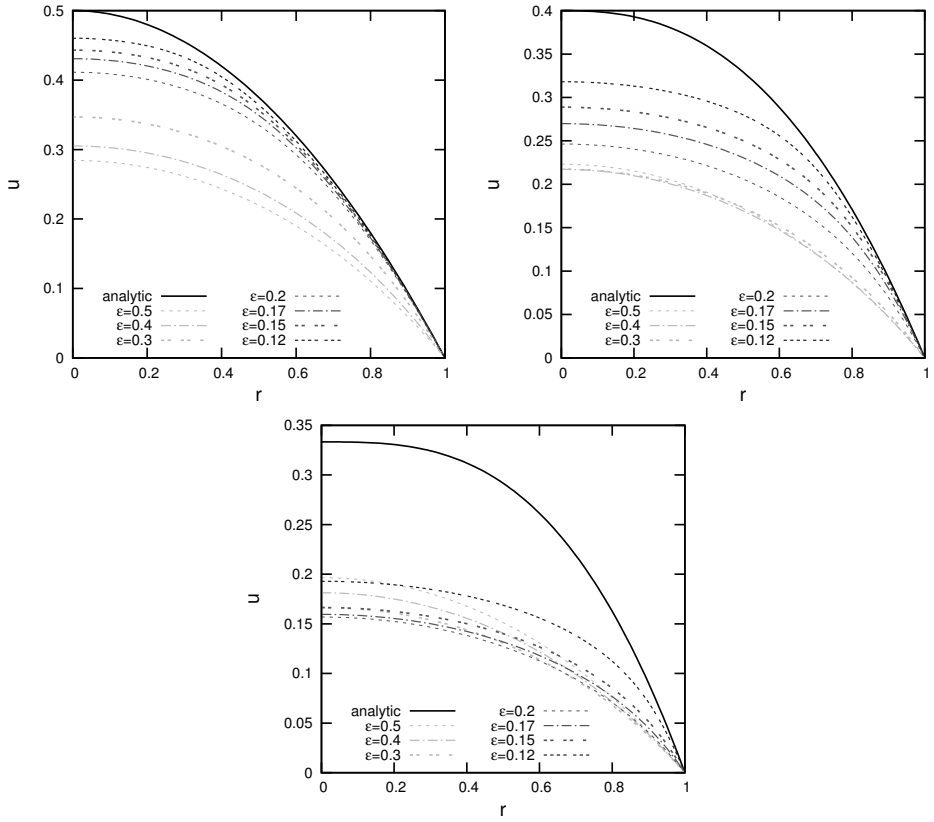

Fig. 5: Radial solution for a circle as initial curve for $k = 1, 1.5, 2$.

where $l(t)$ denotes the length of the curve and $a(t)$ the enclosed area at time t. According to theoretical results in [50] we confirm the monotonicity of this deficit during the evolution in the special case of the ellipse as initial curve, see Figure 10. Furthermore, we see that with increasing k (and $\varepsilon = 0.05$) the curves transform faster into a circle (they are not yet shrinked to a point except for $k = 0.5$, see Figure 7, where the 'point' is reached quite well). In Figure 5 and Figure 6 we see when comparing the exact solutions for the circle for different values of k and the approximate solutions for the ellipse with $\epsilon = 0.15$ for different values of k, respectively, that the flow reaches the singularity for larger k earlier.

6. Implementation. To calculate the finite element approximation u_h^ε of u^ε we used a discretization with unstructured grids, see Figure 11. These were generated by the mesh generator Gmsh, see [34]. We solved the non-linear equation (2.2) with a Newton method which uses a bi-conjugate gradient stabilized solver (BiCGSTAB) and SSOR preconditioning. For the implementation we used PDELab, a discretization module for solving PDEs which depends on the Distributed and Unified Numerics Environment (DUNE). As further references concerning PDELab we refer to [47, 11], information about DUNE can be found in [12, 9, 10, 25]. In order to get solutions for small ε we used a warm-start, i.e. we decreased ε stepwise to the desired small value and performed on each stage a calculation with the solution for the previous ε

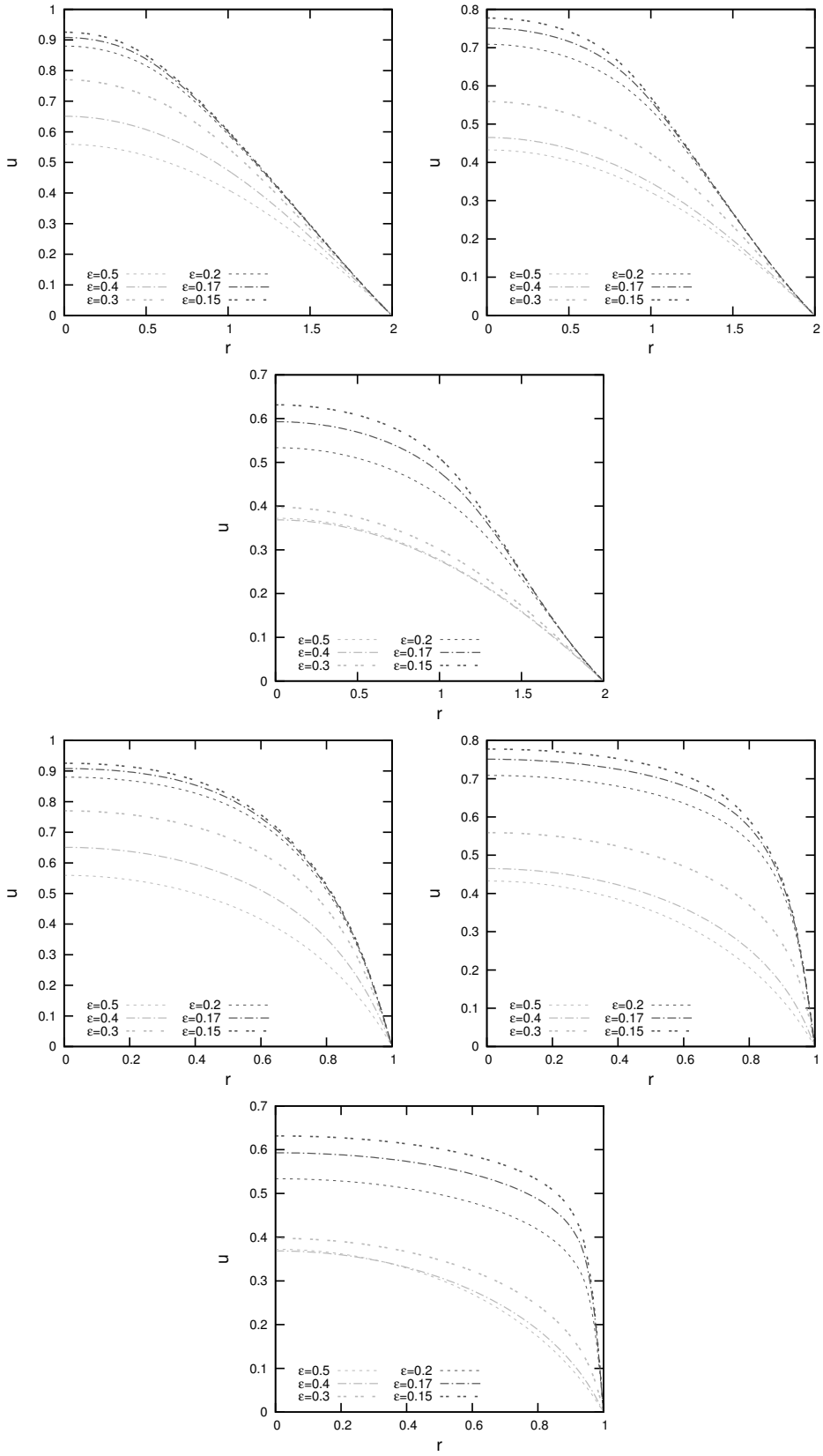

Fig. 6: Solution for ellipse as initial curve. Picture 1–3: Section in direction of the long half axis of the initial curve for $k = 1, 1.5, 2$; Picture 4–6: the same for the short half axis.

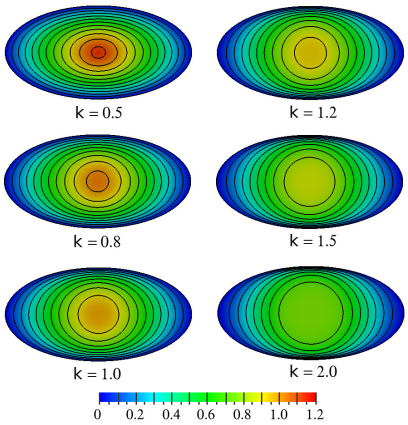

Fig. 7: Solution for ellipse for $\varepsilon = 0.1$.

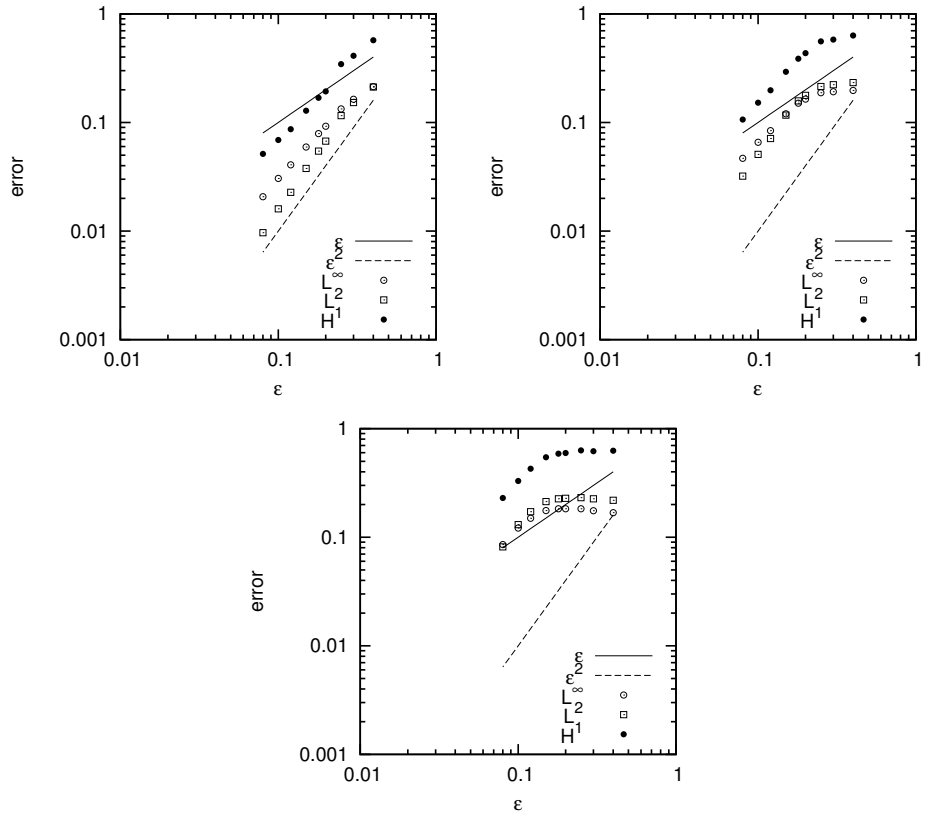

Fig. 8: Total approximation error, $\varepsilon = h$, for $k = 1, 1.5, 2$, in case of a circle as initial curve.

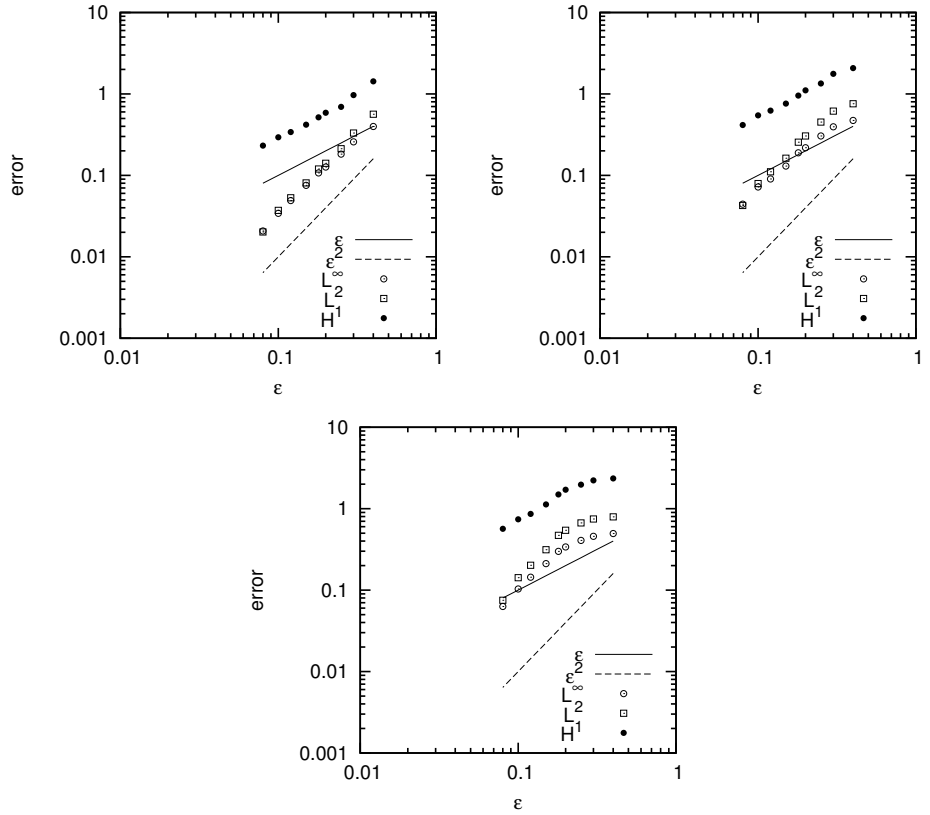

Fig. 9: Total approximation error, $\varepsilon = h$, for $k = 1, 1.5, 2$ in case of ellipse with half axes 1 and 2 as initial curve.

as initial value.

7. Appendix. Since $L_\epsilon : H_0^1(\Omega) \to H^{-1}(\Omega)$ is a topological isomorphism by classical L^2-theory this also holds for $L_\epsilon^* : H_0^1(\Omega) \to H^{-1}(\Omega)$. We define the to L_ϵ associated uniformly, elliptic regular Dirichlet bilinear form of order 1 by

$$B : W_0^{1,p}(\Omega) \times W_0^{1,p^*}(\Omega) \to \mathbb{R}, \quad B[u,v] = \int_\Omega a^{ij} D_i u D_j v + b^i D_i u v \; dx \tag{7.1}$$

and set

$$\begin{aligned} N_{p^*} =& \{v \in W_0^{1,p^*}(\Omega) : B[\psi, v] = 0 \text{ for every } \psi \in C_0^\infty(\Omega)\} \\ N_p =& \{v \in W_0^{1,p}(\Omega) : B[v, \phi] = 0 \text{ for every } \phi \in C_0^\infty(\Omega)\} \end{aligned} \tag{7.2}$$

From Fredholm's alternative, cf. [54, Theorem 10.7], we deduce that for every $F \in W^{-1,p^*}(\Omega)$ the equation

$$B[u, \varphi] = F\varphi \quad \forall \varphi \in W_0^{1,p^*}(\Omega) \tag{7.3}$$

has a solution $u \in W_0^{1,p}(\Omega)$ if and only if

$$v \in N_{p^*} \Rightarrow Fv = 0. \tag{7.4}$$

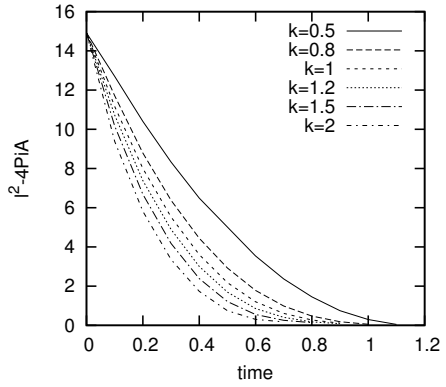

Fig. 10: Isoperimetrical deficit.

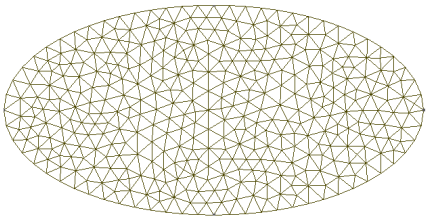

Fig. 11: Mesh for the discretization with size $h = 0.15$ for ellipse with half axes 1 and 2.

If $\dim N_{p^*} = \dim N_p = 0$ then for every $F \in W^{-1,p^*}(\Omega)$ equation (7.3) has a unique solution.

LEMMA 7.1. $\dim N_{p^*} = \dim N_p = 0$.

Proof. Let $v \in N_{p^*}$. From [54, Theorem 7.6] we get $v \in W_0^{1,p'}(\Omega)$ for all $1 < p' < \infty$, especially for $p' = 2$. Since we know from L^2-theory that (7.3) has a unique solution $u \in W_0^{1,2}(\Omega)$ if $p = 2$ and $F = 0$ we deduce that $v = 0$. Analogously we obtain the remaining claim. □

By bounded inverse theorem we conclude the following result.

COROLLARY 7.2. *L_ϵ, L_ϵ^* are topological isomorphisms.*

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