Finite element approximation of level set motion by powers of the mean curvature

Axel Kröner, Eva Kröner, Heiko Kröner

Finite element approximation of level set motion by powers of the mean curvature

AXEL KRÖNER*, EVA KRÖNER†, AND HEIKO KRÖNER‡

Abstract. In this paper we study the level set formulations of certain geometric evolution equations from a numerical point of view. Specifically, we consider the flow by powers greater than one of the mean curvature and the inverse mean curvature flow. Since the corresponding equations in level set form are quasilinear, degenerate and especially possibly singular a regularization method is used in the literature to approximate these equations to overcome the singularities of the equations. Motivated by the paper [29] which studies the finite element approximation of inverse mean curvature flow we prove error estimates for the finite element approximation of the regularized equations for the flow by powers of the mean curvature. We validate the rates with numerical examples. Additionally, the regularization error in the rotational symmetric case for both flows is analyzed numerically. All calculations are performed in the 2D case.

Key words. mean curvature flow, level set equation, regularization, viscosity solution, finite elements

AMS subject classifications. 35J60, 35J70, 35J75 , 65L60, 35D40

1. Introduction. Huisken and Ilmanen [39] used the inverse mean curvature flow to prove the Riemannian Penrose inequality in general relativity. Later its level set formulation was extended to the flow by powers $k > 1$ of the mean curvature by Schulze [51] who also proved a certain inequality using this flow. The paper [39] arouse the interest for a numerical analysis of this special level set approach to inverse mean curvature flow which lead to the paper [29] by Feng, Neil and Prohl who introduced a finite element discretization for the level set formulation of inverse mean curvature flow as it appears in [39]. They prove error estimates in the H^1-norm and the L^2-norm and confirm their rates by numerical examples. Furthermore, they focus on the aspect that their finite element method approximates the regularized equation (instead of the equation for level set inverse mean curvature flow) and present some numerical examples in which they study the corresponding regularization error.

The contribution of this paper is to study the finite element approximation of the regularized equation for level set inverse mean curvature flows as described in [29] for the different setting characterized by the fact that here we have flows by powers $k \geq 1$ of the mean curvature, as considered in [51]. We prove rates for the H^1- and L^2-error and confirm them by numerical examples. In the second part of the paper we study the regularization error in the rotational symmetric case for the flow by power $k \geq 1$ of the mean curvature numerically, similarly as in [29]. We obtain rates within the range of the corresponding theoretical estimate from [40]. Moreover, similar to this estimate we observe that this rate improves when $k \geq 1$ decreases. The third part of the paper deals with the regularization error for level set inverse mean curvature flow in a simplified rotational symmetric setting. Contrary to [29] we respect for an estimate of the regularization error the artificial boundary values. We confirm the obtained estimate in the simplified setting that the issue of artificial boundary values

*Department of Mathematics, Humboldt University of Berlin, Berlin, Germany; CMAP, Ecole Polytechnique, CNRS, Université Paris Saclay, and Inria, France. (axel.kroener@math.hu-berlin.de).

†Geophysics, Institute of Environmental Sciences, University of Koblenz-Landau, Forsttraße 7, 76829 Landau, Germany. (kroener@uni-landau.de).

‡Mathematisches Institut, Abteilung für Reine Mathematik, Albert-Ludwigs-Universität, Eckerstraße 1, 79104 Freiburg im Breisgau, Germany (heiko.kroener@math.uni-freiburg.de).

is omitted by a numerical example. The last section is devoted to some numerical examples for the non-symmetric case.

We give a short overview over some related publications. For the behaviour of the classical flows we refer to [3, 4, 27, 33, 38, 50, 52]. For level set formulations for mean curvature flow, see, e.g., [28, 42, 46, 53, 54] and its interpretation as the value function of a deterministic two-person game see [41]. For applications in image processing of geometric PDEs we refer to [1, 2, 16, 17]. For6 geometric flows describing the evolution of convex and non-convex curves see [5, 30, 32, 36, 43, 49]. The approximation of geometric evolution equations with finite elements is considered in [6, 7, 8, 29, 45, 57, 56], by finite difference schemes in [21, 47], and by semi-Lagrangian schemes in [15].

The paper is organized as follows. Section 2 introduces the setting of the level set flow by powers of the mean curvature (level set PMCF). Section 3 deals with the finite element approximation of regularized level set PMCF, proves error estimates and presents numerical examples. Section 4 presents numerically obtained rates for the regularization error of level set PMCF. Section 5 introduces the regularized level set inverse mean curvature flow formulation (level set IMCF), from [39]. Section 6 shows theoretically and numerically obtained rates for the regularization error of regularized level set IMCF. The final Section 7 contains some numerical examples in which we simulate level set PMCF in the non-rotational symmetric case and we give a short description of the implementation used for the numerical computations presented in this paper. Finally, we give some remarks on an alternative level set formulation sometimes used in the literature for the mean curvature flow case.

2. Level set PMCF. Let $\Omega \subset \mathbb{R}^{n+1}$ be open, connected and bounded having smooth boundary $\partial\Omega$ with positive mean curvature which we consider as initial hypersurface. We call the level sets $\Gamma_t := \partial\{x \in \Omega : u(x) > t\}$, $t \geq 0$, of the continuous function $0 \leq u \in C^0(\bar{\Omega})$ a level set PMCF, if u is a viscosity solution of

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = -\dfrac{1}{|Du|^{\frac{1}{k}}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

cf. [40, Section 2] for a definition viscosity solution in this case. If u is smooth in a neighborhood of $x \in \Omega$ with non vanishing gradient and satisfies in this neighborhood (2.1), then the level set $\{u = u(x)\}$, $x \in \Omega$, is locally at x a smooth hypersurface and moves at x in the direction of its outer normal with speed H^k where H is its mean curvature in x. Using elliptic regularization of level set PMCF we obtain the equation

$$\begin{cases} \operatorname{div}\left(\dfrac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\right) = -(\varepsilon^2 + |Du^\varepsilon|^2)^{-\frac{1}{2k}} & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

which has a unique smooth solution u^ε for sufficiently small $\varepsilon > 0$, cf. [51, Section 4]; moreover, there is $c_0 > 0$ such that

$$\|u^\varepsilon\|_{C^1(\bar{\Omega})} \leq c_0 \tag{2.3}$$

uniformly in ε and (for a subsequence)

$$u^\varepsilon \to u \in C^{0,1}(\bar{\Omega}) \tag{2.4}$$

in $C^0(\bar{\Omega})$. We call u a weak solution of (2.1), which is unique for $n \leq 6$. All the above facts are proved in [51, Section 4] under the assumption that $k \geq 1$. A weak solution of (2.1) satisfies (2.1) in the viscosity sense, cf. Section [40, Section 2].

3. Discretization and error estimate for regularized level set PMCF. In this section we present a finite element discretization of the regularized equation (2.2) and prove error estimates. We will restrict us to the case that the space dimension $n+1$ is 2 or 3 and that Ω is convex. The latter is only a restriction if $n+1=3$ since $\partial\Omega$ has positive mean curvature by assumption, cf. Section 1.

3.1. Discretization. We fix some standard notation concerning finite elements. We denote the Euclidean norm of $\mathbb{R}^n$ by $|\cdot|$. For an open subset Ω of $\mathbb{R}^n$ and $m \in \mathbb{N}^*$, $p \geq 1$, we denote the corresponding Sobolev spaces by $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$. The dual spaces are denoted by $W^{-m,p}(\Omega) = W_0^{m,p}(\Omega)^*$ and the dual pairing by

$$W^{-m,p}(\Omega) \times W_0^{m,p}(\Omega) \ni (F,\varphi) \mapsto \langle F,\varphi\rangle = F\varphi \in \mathbb{R}. \tag{3.1}$$

Let (T_h,Ω_h) be a quasi-uniform triangulation of Ω with mesh size $0 < h < h_0$, h_0 sufficiently small, and $V_h \subset H^1(\Omega_h)$ the finite element space given by

$$V_h = \left\{ v \in C^0(\bar{\Omega}_h) : v_{|\partial\Omega_h} = 0,\ v_{|T} \text{ linear for all } T \in T_h \right\}. \tag{3.2}$$

In view of the convexity of Ω there holds $\Omega_h \subset \Omega$. A function $u_h \in V_h$ will be also considered as a function on Ω by extending it by zero in $\Omega \setminus \Omega_h$. Then $v_h \in H^1(\Omega)$. The variational formulation of (2.2) is given by

$$\int_{\Omega_h} \frac{\langle Du_h^\varepsilon, Dv_h\rangle}{\sqrt{\varepsilon^2 + |Du_h^\varepsilon|^2}} \mathrm{d}x = \int_{\Omega_h} (\varepsilon^2 + |Du_h^\varepsilon|^2)^{-\frac{1}{2k}} v_h \mathrm{d}x \quad \text{for all } \ v_h \in V_h \tag{3.3}$$

where we fix $\varepsilon > 0$ from now on and denote the finite element solution by u_h^ε. For formal reason we will consider boundary tetrahedrons (boundary triangles in case $d = 2$) to be extended to a boundary tetrahedron with one 'curved face'. Therefore we will replace a boundary element $T \in T_h$ (i.e. $n+1$ vertexes of T lie on $\partial\Omega$) by $\tilde{T} = T \cup B$ with

$$B = \{tp + (1-t)Pp \mid 0 \leq t \leq 1, p \in bf\}, \tag{3.4}$$

where bf is the boundary face of T, i.e. $n+1$ vertexes of bf lie on $\partial\Omega$, and Pp is the unique minimizer of $\operatorname{dist}(p,\cdot)_{|\partial\Omega}$. We denote the resulting triangulation by $\tilde{T}_h$. This leaves the space of finite element functions we use (namely V_h) unchanged. Note, that the boundary strip $\Omega \setminus \Omega_h$ has measure $O(h^2)$.

3.2. The linearized operator. We define the linear operator L_ε and its dual and state some properties. Let $p > 1$. We define for $\varepsilon > 0$ and $z \in \mathbb{R}^n$

$$|z|_\varepsilon := f_\varepsilon(z) := \sqrt{|z|^2 + \varepsilon^2} \tag{3.5}$$

and denote derivatives of f_ε with respect to z^i by $D_{z^i} f_\varepsilon$. We have

$$D_{z^i} f_\varepsilon(z) = \frac{z_i}{|z|_\varepsilon}, \quad D_{z^i} D_{z^j} f_\varepsilon(z) = \frac{\delta_{ij}}{|z|_\varepsilon} - \frac{z_i z_j}{|z|_\varepsilon^3}. \tag{3.6}$$

We define the operator Φ_ε by

$$\Phi_\varepsilon : W_0^{1,p}(\Omega) \to W^{-1,p^*}(\Omega), \quad \Phi_\varepsilon(v) = -D_i\left(\frac{D_i v}{|Dv|_\varepsilon}\right) - \frac{1}{|Dv|_\varepsilon^{\frac{1}{k}}}, \tag{3.7}$$

where $\frac{1}{p}+\frac{1}{p^*}=1$, so that (2.2) can be written as

$$\Phi_\varepsilon(u^\varepsilon)=0. \tag{3.8}$$

We denote the derivative of Φ_ε in u^ε by

$$L_\varepsilon := D\Phi_\varepsilon(u^\varepsilon) \tag{3.9}$$

and have for all $\varphi \in W_0^{1,p}(\Omega)$ that

$$\begin{aligned} L_\varepsilon\varphi &= -D_i\left(D_{z^i}D_{z^j}f_\varepsilon(Du^\varepsilon)D_j\varphi\right)+\frac{1}{k}f_\varepsilon(Du^\varepsilon)^{-1-\frac{1}{k}}D_{z^j}f_\varepsilon(Du^\varepsilon)D_j\varphi \\ &=: -D_i(a^{ij}D_j\varphi)+b^iD_i\varphi, \end{aligned} \tag{3.10}$$

where we use the convention to sum over repeated indices. The coefficients a^{ij} and b^i are in $C^\infty(\bar\Omega)$. Note, that the estimate (2.3) is not available for higher order derivatives of u^ε uniformly in ε but since ε is fixed in the present section, this does not have an effect on the following considerations. The linear operator

$$L_\varepsilon : W_0^{1,p}(\Omega)\to W^{-1,p^*}(\Omega) \tag{3.11}$$

and its adjoint operator L_ε^* are topological isomorphism, cf. Corollary A.2 in Section 7. From [14, Theorem 8.5.3] we deduce for $L=L_\varepsilon$ or $L=L_\varepsilon^*$ and $F\in W^{-1,p^*}(\Omega)$ that there is a unique solution $u_h\in V_h$ of

$$\langle Lu_h,\varphi_h\rangle = F\varphi_h \quad \text{for all } \varphi_h\in V_h, \tag{3.12}$$

where $u\in H^1(\Omega)$ is the unique solution of $Lu=F$ and we have the estimate

$$\|u_h\|_{W^{1,p}(\Omega)}+\|u-u_h\|_{W^{1,p}(\Omega)}\le c\|u\|_{W^{1,p}(\Omega)}. \tag{3.13}$$

Furthermore, if $F\in L^p(\Omega)$ we have

$$\|u-u_h\|_{W^{1,p}(\Omega)}+h\|u-u_h\|_{L^p(\Omega)}\le ch^2\|F\|_{L^p(\Omega)}. \tag{3.14}$$

Remark 3.1. Note, that we used the assertion of [14, Theorem 8.5.3] under slightly different assumptions, namely:
(i) We assume a right-hand side $F\in W^{-1,p^*}(\Omega)$ (instead $F\in L^p(\Omega)$).
(ii) We consider the equation on Ω (instead of a polygonal domain) and use as discretization the triple $(\tilde T_h,\Omega,V_h)$.

3.3. Error estimate. We have the following error estimate in the $W^{1,p}$-norm.

Theorem 3.2. *For every $p>n+1$ and small $h>0$ there exists a constant $0<c=c(\|u^\varepsilon\|_{W^{2,2}(\Omega)},p)$ such that (3.3) has a solution $u_h^\varepsilon\in V_h$ satisfying*

$$\|u^\varepsilon-u_h^\varepsilon\|_{W^{1,p}(\Omega)}\le ch. \tag{3.15}$$

This solution is unique in a small $W^{1,p}$-neighborhood of u^ε in V_h.

Proof. We set

$$\bar B_\rho^h=\{v_h\in V_h : \|u^\varepsilon-v_h\|_{W^{1,p}(\Omega)}\le\rho\}, \tag{3.16}$$

with

$$\rho=h^\lambda \tag{3.17}$$

for an arbitrary and now fixed $\frac{n+1}{p} < \lambda < 1$. We will obtain u_h as the unique fixed point in $\bar{B}^h_\rho$ of the operator $T : V_h \to V_h$ with

$$L_\varepsilon(w_h - Tw_h) = \Phi_\varepsilon(w_h), \quad w_h \in V_h. \tag{3.18}$$

We show that (i) $\bar{B}^h_\rho \neq \emptyset$, (ii) T is a contraction and (iii) $T(\bar{B}^h_\rho) \subset \bar{B}^h_\rho$.

(i) Let $I_h u^\varepsilon$ be the interpolation of u^ε, i.e. the continuous piecewise linear function on Ω_h which is equal to u^ε at all nodes of Ω_h. We extend $I_h u^\varepsilon$ by zero to a function on Ω. Since

$$\|I_h u^\varepsilon - u^\varepsilon\|_{W^{1,p}(\Omega)} \leq ch \tag{3.19}$$

we have $I_h u^\varepsilon \in \bar{B}^h_\rho$ for small h.

(ii) Let $v_h, w_h \in \bar{B}^h_\rho$ and $\xi_h = v_h - w_h$ then using (3.18) we conclude

$$\begin{aligned} L_\varepsilon(Tv_h - Tw_h) &= L_\varepsilon \xi_h + \Phi_\varepsilon(w_h) - \Phi_\varepsilon(v_h) \\ &= (L_\varepsilon - D\Phi_\varepsilon(v_h + \Theta\xi_h))\xi_h =: F \end{aligned} \tag{3.20}$$

In order to estimate $\|F\|_{W^{-1,p^*}(\Omega)}$ which leads to an estimate of $\|Tv_h - Tw_h\|_{W^{1,p}(\Omega)}$ in view of Corollary A.2 we choose $\psi \in W^{1,p^*}_0(\Omega)$ with $\|\psi\|_{W^{1,p^*}(\Omega)} \leq 1$ and estimate $\langle F, \psi \rangle$. To do so we use a mean value theorem for which we need the following auxiliary estimate

$$\begin{aligned} \|Du^\varepsilon - (Dv_h + \Theta D\xi_h)\|_{L^\infty(\Omega)} &\leq \|Du^\varepsilon - DI_h u^\varepsilon\|_{L^\infty(\Omega)} \\ &\quad + \|DI_h u^\varepsilon - D\tilde{v}_h\|_{L^\infty(\Omega)} \\ &\leq ch + c\rho h^{-\frac{n+1}{p}}, \end{aligned} \tag{3.21}$$

where $\tilde{v}_h = v_h + \Theta\xi_h \in \bar{B}^h_\rho$ and where we used an inverse estimate. The resulting estimate implies

$$\|Tv_h - Tw_h\|_{W^{1,p}(\Omega)} \leq c(h + \rho h^{-\frac{n+1}{p}})\|\xi_h\|_{W^{1,p}(\Omega)} \leq \frac{1}{4}\|\xi_h\|_{W^{1,p}(\Omega)} \tag{3.22}$$

for small h.

(iii) Let $w_h \in \bar{B}^h_\rho$. We have

$$\begin{aligned} \|Tw_h - u^\varepsilon\|_{W^{1,p}(\Omega)} &\leq \|Tw_h - TI_h u^\varepsilon\|_{W^{1,p}(\Omega)} + \|TI_h u^\varepsilon - I_h u^\varepsilon\|_{W^{1,p}(\Omega)} \\ &\quad + \|I_h u^\varepsilon - u^\varepsilon\|_{W^{1,p}(\Omega)} \\ &\leq \frac{\rho}{2} + \|TI_h u^\varepsilon - I_h u^\varepsilon\|_{W^{1,p}(\Omega)} + ch. \end{aligned} \tag{3.23}$$

It remains to estimate the norm on the right-hand side. We have

$$\begin{aligned} \|TI_h u^\varepsilon - I_h u^\varepsilon\|_{W^{1,p}(\Omega)} &\leq c\|\Phi_\varepsilon(I_h u^\varepsilon)\|_{W^{-1,p^*}(\Omega)} \\ &= c\|\Phi_\varepsilon(I_h u^\varepsilon) - \Phi_\varepsilon(u^\varepsilon)\|_{W^{-1,p^*}(\Omega)} \\ &\leq ch \end{aligned} \tag{3.24}$$

again by a mean value theorem estimate. In view of (3.17) we have

$$T(\bar{B}^h_\rho) \subset \bar{B}^h_\rho. \tag{3.25}$$

∎

Employing a duality argument as in [29] we obtain an L^p-error estimate in the following theorem.

THEOREM 3.3. *For $p > n+1$ we have*

$$\|u^\varepsilon - u_h^\varepsilon\|_{L^p(\Omega)} \le ch^2 \tag{3.26}$$

with $c = c(\|u^\varepsilon\|_{W^{2,2}(\Omega)}, p) > 0$.

Proof. From the definitions of u^ε and u_h^ε we get

$$\int_\Omega \left(\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|_\varepsilon} - \frac{\nabla u_h^\varepsilon}{|\nabla u_h^\varepsilon|_\varepsilon} \right) \cdot \nabla \varphi_h \mathrm{d}x + \int_\Omega \left(|\nabla u^\varepsilon|_\varepsilon^{\frac{1}{k}} - |\nabla u_h^\varepsilon|_\varepsilon^{\frac{1}{k}} \right) \varphi_h \mathrm{d}x = 0 \text{ for all } \varphi_h \in V_h. \tag{3.27}$$

This equation can be written equivalently as

$$\int_\Omega (A_h^\varepsilon \nabla e_h^\varepsilon) \cdot \nabla \varphi_h \mathrm{d}x + \int_\Omega (a_h^\varepsilon \cdot \nabla e_h^\varepsilon)\, \varphi_h \mathrm{d}x = 0 \quad \text{for all } \varphi_h \in V_h. \tag{3.28}$$

with

$$\begin{aligned} A_h^\varepsilon &:= \int_0^1 D^2 f_\varepsilon(\nabla u^\varepsilon + t\nabla(u_h^\varepsilon - u^\varepsilon))dt \\ a_h^\varepsilon &:= \frac{1}{k}\int_0^1 f_\varepsilon(\nabla u^\varepsilon + t\nabla(u_h^\varepsilon - u^\varepsilon))^{\frac{1}{k}-1} D f_\varepsilon(\nabla u^\varepsilon + t\nabla(u_h^\varepsilon - u^\varepsilon))dt \\ e_h^\varepsilon &:= u_h^\varepsilon - u^\varepsilon \end{aligned} \tag{3.29}$$

and for later purposes we set

$$\begin{aligned} \bar{A}_h^\varepsilon &:= D^2 f_\varepsilon(\nabla u^\varepsilon) \\ \bar{a}_h^\varepsilon &:= \frac{1}{k} f_\varepsilon(\nabla u^\varepsilon)^{\frac{1}{k}-1} D f_\varepsilon(\nabla u^\varepsilon). \end{aligned} \tag{3.30}$$

Let $\varphi \in W_0^{1,p^*}(\Omega)$ be given by

$$L_\varepsilon^* \varphi := |e_h^\varepsilon|^{p-1} \operatorname{sgn}(e_h^\varepsilon) \tag{3.31}$$

with sign-function sgn. Furthermore, let $\varphi_h \in V_h$ the corresponding finite element solution of this equation. We test (3.31) with e_h^ε and get by symmetry of $\bar{A}_h^\varepsilon$ that

$$\int_\Omega |e_h^\varepsilon|^p \mathrm{d}x = \int_\Omega (A_h^\varepsilon \nabla e_h^\varepsilon) \cdot \nabla \varphi_h \mathrm{d}x + \int_\Omega (a_h^\varepsilon \cdot \nabla e_h^\varepsilon)\varphi_h \mathrm{d}x. \tag{3.32}$$

By (3.28) we have further

$$\begin{aligned} \int_\Omega |e_h^\varepsilon|^p \mathrm{d}x &= \int_\Omega \left((A_h^\varepsilon - \bar{A}_h^\varepsilon)\nabla e_h^\varepsilon \right) \cdot \nabla \varphi_h \mathrm{d}x + \int_\Omega \left((a_h^\varepsilon - \bar{a}_h^\varepsilon) \cdot \nabla e_h^\varepsilon \right) \varphi_h \mathrm{d}x \\ &\le c \int_\Omega |\nabla e_h^\varepsilon|^2 |\nabla \varphi_h| \mathrm{d}x + c\int_\Omega |\nabla e_h^\varepsilon|^2 \varphi_h \mathrm{d}x \\ &\le c\|\varphi_h\|_{W^{1,p^*}(\Omega)} \|e_h^\varepsilon\|^2_{W^{1,2p}(\Omega)}. \end{aligned} \tag{3.33}$$

In view of Corollary A.2 and (3.31) we get

$$\|\varphi_h\|_{W^{1,p^*}(\Omega)} \le c \left(\int_\Omega |e_h^\varepsilon|^{(p-1)p^*} \mathrm{d}x \right)^{\frac{1}{p^*}} = c\|e_h^\varepsilon\|_{L^p(\Omega)}^{\frac{p}{p^*}}. \tag{3.34}$$

Using (3.32) and Theorem 3.2 to estimate $\|e_h^\varepsilon\|_{W^{1,2p}(\Omega)}$ we conclude. $\square$

The discretization errors in the H^1- and L^2-norm have been calculated numerically by solving Eqn. (2.1) for the case of a unit circle as initial curve with $\varepsilon = 0.1$ and $k = 1$ on meshes of various refinements ($h = 0.006, ..., 0.4$). The solutions have been calculated iteratively by linearizing Eqn. (3.3) in the following way:

$$\int_{\Omega_h} V_j \left\langle Du^\varepsilon_{h,j}, Dv_h \right\rangle \mathrm{d}x = \int_{\Omega_h} f_j v_h \mathrm{d}x \quad \text{for all} \quad v_h \in V_h \tag{3.35}$$

where $j \in \mathbb{N}$ is the iteration index and V_j and f_j have been updated in each iteration:

$$\begin{aligned} V_j &:= \begin{cases} 1 & j = 1, \\ \gamma V_{j-1} + (1-\gamma)\frac{1}{\sqrt{\varepsilon^2 + |Du^\varepsilon_{h,j}|^2}} & j > 1, \end{cases} \\ f_j &:= \begin{cases} 1 & j = 1, \\ \gamma f_{j-1} + (1-\gamma)(\varepsilon^2 + |Du^\varepsilon_{h,j}|^2)^{-\frac{1}{2k}} & j > 1. \end{cases} \end{aligned} \tag{3.36}$$

The value of γ affects the convergence of the iterations and was set to 0.1. In each iteration the linearized equation has been solved using Freefem [37]. These solutions where compared to a solution obtained by solving Eqn. (2.1) in radial coordinates. Note, that due to radial symmetry Eqn. (2.1) simplifies to an ODE which was solved on a one-dimensional grid of $h = 0.001$. Figure 1 shows the numerical discretization errors as well as the rates proven in Theorem 3.2 and Theorem 3.3. Figure 2 shows the discretization errors $u^\varepsilon_h - u^\varepsilon_{0.001}$ and the corresponding meshes.

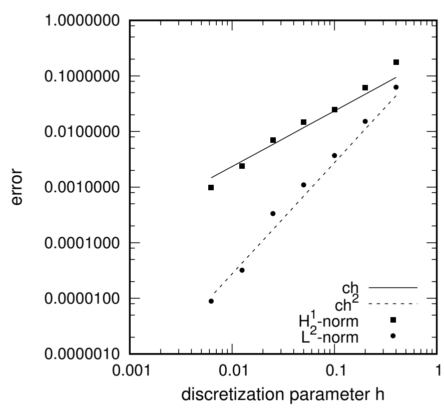

FIG. 1. *Discretization error for the unit circle as initial curve with $k = 1$, $\varepsilon = 0.1$.*

4. Numerical study of the regularization error for PMCF. This section shows some numerically obtained rates for the regularization error of regularized level set PMCF. Since small values of the regularization parameter ε are difficult to handle we restrict the setting in order to have very high accuracy to the rotational symmetric case. A similar case in a rotational symmetric setting which omits the important issue of artificial boundary values in the model from [39] was considered in [29, Figure 5] for regularized level set IMCF.

We state the following theoretical regularization error estimate.

THEOREM 4.1. *Let u be solution of (2.1) and u^ε of (2.2). Then for $\lambda > 2k$ the regularization error of the regularized level set PMCF satisfies*

$$|u - u^\varepsilon|_{C^0(\Omega)} = O(\varepsilon^{\frac{1}{\lambda}}). \tag{4.1}$$

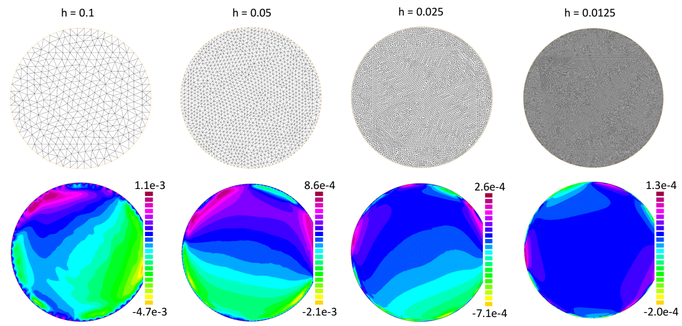

FIG. 2. *Top: mesh with decreasing meshsize h. Bottom: $u_h^\varepsilon - u_{0.001}^\varepsilon$*

Proof. See [40]. ∎

We recall a well-known interpolation lemma which enables us to deduce also an approximation error in Hölder-norms.

LEMMA 4.2. *For $0 < \beta < \alpha \le 1$ and a function $v \in C^0(\Omega)$ we have*

$$[v]_\beta \le 2^{1-\frac{\beta}{\alpha}} [v]_\alpha^{\frac{\beta}{\alpha}} \|v\|_{C^0(\Omega)}, \tag{4.2}$$

where these expressions might become infinity and

$$[v]_\alpha := \sup_{x \neq y} \frac{|v(x) - v(y)|}{|x-y|^\alpha}. \tag{4.3}$$

Since u^ε is uniformly bounded in the C^1-norm, cf. (2.3), we can use Lemma 4.2 to conclude

$$\|u - u^\varepsilon\|_{C^{0,\beta}(\Omega)} \le c(\beta) \varepsilon^{\lambda(1-\beta)} \tag{4.4}$$

for every $0 < \beta < 1$ and $0 < \lambda < \frac{1}{2k}$.

Our rotational symmetric example for which we study the regularization error is a shrinking circle. Let the circle $\partial B_{r_0}(0) \subset \mathbb{R}^2$ with radius $r_0 > 0$ serve as our initial curve so that the solution u of (2.1) is given as

$$u(r) = \frac{r_0^{k+1} - r^{k+1}}{k+1}, \tag{4.5}$$

where $r \ge 0$ denotes the radius variable in polar coordinates in $\mathbb{R}^2$ with center in 0. The regularized equation reduces to the following one-dimensional equation when formulated in radial coordinates

$$\begin{cases} \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{r\frac{\mathrm{d}}{\mathrm{d}r}u^\varepsilon}{\sqrt{\varepsilon^2 + \left|\frac{\mathrm{d}}{\mathrm{d}r}u^\varepsilon\right|^2}}\right) = -\left(\varepsilon^2 + \left|\dfrac{\mathrm{d}}{\mathrm{d}r}u^\varepsilon\right|^2\right)^{-\frac{1}{2k}} \quad \text{in } [0, r_0], \\ \dfrac{\mathrm{d}}{\mathrm{d}r}u^\varepsilon(0) = 0 \quad \text{and} \quad u^\varepsilon(r_0) = 0. \end{cases} \tag{4.6}$$

We solve this one-dimensional boundary value problem by a Newton-Algorithm combined with Thomas Algorithm as direct solver. In Figure 3(a) the error $\|u^\varepsilon - u\|_{L^\infty(\Omega_h)}$

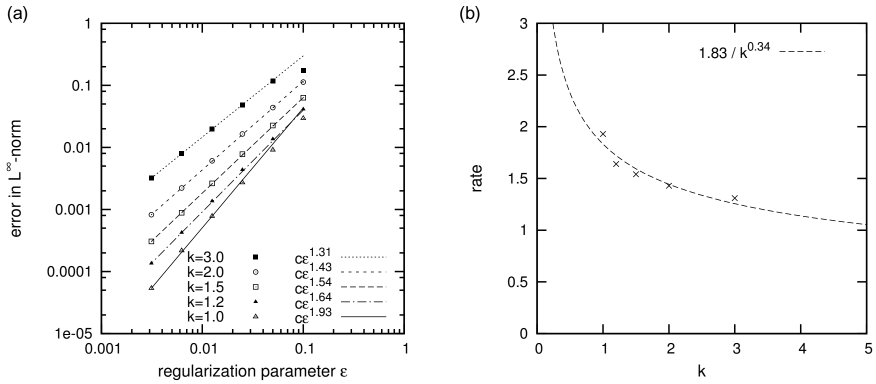

FIG. 3. *(a) regularization error in case of a circle as initial curve for $k = 1, 1.2, 1.5, 2, 3$; (b) rate of regularization error as function of k.*

is plotted for $k = 1.0$, $k = 1.2$, $k = 1.5$, $k = 2.0$, and $k = 3$. By fitting the function $c\varepsilon^{r_k}$ to the computed L^∞-errors we obtain the rate of regularization error r_k (Figure 3(b)) as function of k, namely $r_k \approx 1.83/k^{0.34}$, that means indeed that the rate depends on the power k — the larger k the smaller the rate. However, this is a little bit better than $1/(2k)$.

Figure 4 shows the corresponding solutions of the regularized equations.

5. Level set IMCF. We recall from [39] the following facts. Let $M \subset \mathbb{R}^{n+1}$ be a closed, embedded hypersurface. A classical solution of the inverse mean curvature flow is a smooth family $x : M \times [0,T] \to \mathbb{R}^{n+1}$ of hypersurfaces $M_t := x(M,t)$ satisfying the parabolic evolution equation

$$\frac{\partial x}{\partial t} = \frac{\nu}{H}, \quad x \in M_t, \quad 0 \le t \le T, \tag{5.1}$$

where H, assumed to be positive, is the mean curvature of M_t at the point x and ν is the outward unit normal. If the flow is given by the level sets of a function $u : \mathbb{R}^{n+1} \to \mathbb{R}$ via

$$E_t := \{x \in \mathbb{R}^{n+1} : u(x) < t\}, \quad M_t := \partial E_t, \tag{5.2}$$

then, wherever u is smooth with $\nabla u \neq 0$, equation (5.1) is equivalent to

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u| \tag{5.3}$$

and the left-hand side of (5.3) is the mean curvature of the level set $\{u = t\}$ and the right-hand side is the inverse normal speed.

We set

$$v := (n-1)\log|x|, \quad F_L := \{v < L\}, \quad \Omega_L := F_L \setminus \bar{E}_0 \tag{5.4}$$

where $E_0 \subset \mathbb{R}^{n+1}$ is an open set with $\partial E_0 \in C^1$ and $E_0 \subset\subset F_0$. The regularized level

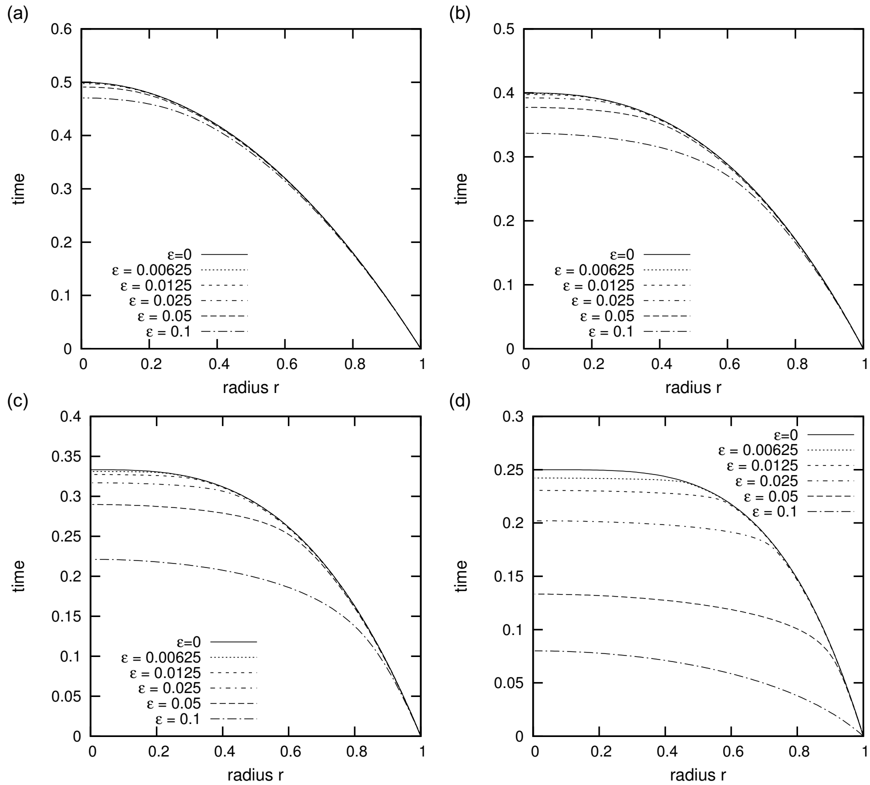

FIG. 4. *Radial solution for a circle as initial curve for (a) $k = 1$, (b) $k = 1.5$, (c) $k = 2$ and (d) $k = 3$.*

set equation is given by

$$\tag{5.5} \begin{cases} E^\varepsilon u^\varepsilon := \operatorname{div}\left(\dfrac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}}\right) - \sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2} = 0 & \text{in } \Omega_L, \\ u^\varepsilon = 0 & \text{on } \partial E_0, \\ u^\varepsilon = L-2 & \text{on } \partial F_L \end{cases}$$

where $L, \varepsilon > 0$ and in case there exists a (hence unique) solution u^ε we will denote it by $u^{\varepsilon,L}$ as well. From [39, Lemma 3.4] we know the following existence result.

LEMMA 5.1. *For every $L > 0$ there is $\varepsilon(L) > 0$ such that for $0 < \varepsilon \le \varepsilon(L)$ a smooth solution u^ε of (5.5) exists.*

Furthermore, [39, Example 2.3] shows that one expects that L is at most $\frac{c}{\varepsilon(L)}$, or equivalently,

$$\tag{5.6} \varepsilon(L) \le \frac{c}{L}.$$

From [39, page 365] we recall the definition of a weak solution of (5.3).

DEFINITION 5.2. *(i) Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set then $u \in C^{0,1}(\Omega)$ is a weak solution of (5.3) on Ω if*

$$\tag{5.7} J_u^K(u) \le J_u^K(v)$$

where

$$J_u^K(v) := \int_K |\nabla v| + v|\nabla u| \mathrm{d}x \tag{5.8}$$

for all $v \in C^{0,1}(\Omega)$ *with* $\{v \neq u\} \subset\subset \Omega$ *and compact* $\{v \neq u\} \subset K \subset \mathbb{R}^{n+1}$.

(ii) u *is a weak solution of (5.3) with initial condition* E_0 *if*

$$u \text{ satisfies (i) with } \Omega = \mathbb{R}^{n+1} \setminus \bar{E}_0 \text{ and } E_0 = \{u < 0\}. \tag{5.9}$$

We state the following existence theorem which holds due to [39, Theorem 3.1] and remark that the latter contains further information like a gradient estimate.

THEOREM 5.3. *For every open and bounded set* $\Omega \subset \mathbb{R}^{n+1}$ *there is a weak solution of (5.3) with initial condition* E_0 *which is unique on* $\mathbb{R}^{n+1} \setminus E_0$.

Furthermore, from the proof of [39, Theorem 3.1] we conclude the following. There exist $\mathbb{R} \ni L_i \to \infty$, $0 < \varepsilon_i \to 0$, solutions $u_i = u^{\varepsilon_i, L_i}$ of (5.5) (i.e. with $\varepsilon = \varepsilon_i$ and $L = L_i$) and $u \in C^{0,1}(\mathbb{R}^{n+1} \setminus E_0)$ so that

$$u_i \to u \tag{5.10}$$

locally uniformly on $\mathbb{R}^{n+1} \setminus E_0$ and u is a weak solution of (5.3). Furthermore, the proof of [39, Theorem 3.1] shows that we may assume that ε_i is small compared to $\frac{1}{L_i}$.

6. Barriers and regularization error for rotational symmetric level set IMCF. The regularized equation (5.5) depends on the triple $(\varepsilon, \Omega_L, L)$ where Ω_L and L are coupled explicitly. We derive in a first step upper and lower bounds for the solution of (5.5) depending on the data. Then we simplify the setting by assuming that E_0 is a ball and that the boundary values on $\partial\Omega_L$ coincide with (exact) IMCF starting from ∂E_0. Our general barriers imply in this special case that the regularization error is of order ε^2 which we confirm with a numerical example. We remark that in general the 'regularization error' is mainly dominated by the artificial boundary values. We think that it is interesting to study the regularization error for more general E_0.

Let positive $\bar{L}$ and $\bar{\varepsilon}$ be given so that problem (5.5) has a solution $\bar{u} = u^{\bar{\varepsilon}, \bar{L}}$. As stated above we want to estimate the regularization error which means here to estimate

$$|u - \bar{u}|_{C^0(\bar{\Omega}_l)} \tag{6.1}$$

for some fixed $0 < l < \bar{L}$, where u is a weak solution of (5.3). Since the boundary values on ∂F_L in (5.5) are rather artificial, (6.1) can only be expected to be small for $0 < l \ll \bar{L}$. The idea to derive an estimate for (6.1) is as follows. Let $\delta > 0$ be arbitrary then there exists $i = i(\delta) \in \mathbb{N}$ so that

$$|u - u^{\varepsilon_i, L_i}|_{C^0(\Omega_l)} \leq \delta \quad \text{and} \quad \varepsilon_i \ll \bar{\varepsilon} \quad \text{and} \quad L_i \gg \bar{L}. \tag{6.2}$$

We then construct barriers for (5.5) for general $\varepsilon > 0$ and $L > 0$ for which (5.5) has a solution and deduce from these bounds in the cases $(\varepsilon, L) = (\bar{\varepsilon}, \bar{L})$ and $(\varepsilon, L) = (\varepsilon_i, L_i)$ bounds for $\bar{u}$ and u^{ε_i, L_i}, respectively, which imply an estimate for $\bar{u} - u^{\varepsilon_i, L_i}$.

We assume that $\varepsilon > 0$ and $L > 0$ are so that (5.5) has a solution. We use the ansatz $\varphi(v)$ where $\varphi \in C^\infty(\mathbb{R})$ will be chosen appropriately. We have

$$D_i v(x) = (n-1)\frac{x_i}{|x|^2}, \quad D_i D_j v(x) = (n-1)\left(\frac{\delta_{ij}}{|x|^2} - 2\frac{x_i x_j}{|x|^4}\right) \tag{6.3}$$

and

$$\Delta v(x) = \frac{(n-1)(n-2)}{|x|^2}. \tag{6.4}$$

Setting $r = |x|$ and $w = (|\varphi'|^2|Dv|^2 + \varepsilon^2)^{\frac{1}{2}}$ we have

$$\begin{aligned} E^\varepsilon\varphi(v) &= \operatorname{div}\left(\frac{\varphi' Dv}{(|\varphi'|^2|Dv|^2+\varepsilon^2)^{\frac{1}{2}}}\right) - \left(|\varphi'|^2|Dv|^2+\varepsilon^2\right)^{\frac{1}{2}} \\ &= w^{-3}(\varphi''|Dv|^2 w^2 + \varphi'\Delta v w^2 - |\varphi'|^2\varphi''|Dv|^4 \\ &\quad - (\varphi')^3 D_i v D_j v D_i D_j v - w^4). \end{aligned} \tag{6.5}$$

Using $w = (\frac{(n-1)^2}{r^2}|\varphi'|^2 + \varepsilon^2)^{\frac{1}{2}}$ we obtain further

$$\begin{aligned} E^\varepsilon\varphi(v) = w^{-3}\Bigg(&\varphi''\frac{(n-1)^2}{r^2}w^2 + \varphi'\frac{(n-1)(n-2)}{r^2}w^2 - |\varphi'|^2\varphi''\frac{(n-1)^4}{r^4} \\ &+ (\varphi')^3\frac{(n-1)^3}{r^4} - w^4\Bigg) \\ = w^{-3}\Bigg(&\frac{(\varphi')^3}{r^4}(n-1)^4 + \frac{\varepsilon^2}{r^2}(\varphi''(n-1)^2 + \varphi'(n-1)(n-2)) \\ &- \frac{(n-1)^4}{r^4}|\varphi'|^4 - 2\varepsilon^2\frac{(n-1)^2}{r^2}|\varphi'|^2 - \varepsilon^4\Bigg). \end{aligned} \tag{6.6}$$

ASSUMPTION 6.1. *We assume*

$$\varepsilon = \alpha L^{-1}, \tag{6.7}$$

with $0 < \alpha < \frac{1}{2}$.

In view of (5.6) this assumption is a not too restrictive and will be assumed in the following. Our aim is that the leading term of the right-hand side of (6.5) is

$$\frac{(n-1)^4}{r^4}(\varphi')^3(1-\varphi') \tag{6.8}$$

and this is enforced by exploiting the fact that all remaining terms

$$\frac{\varepsilon^2}{r^2}(\varphi''(n-1)^2 + \varphi'(n-1)(n-2) - 2(n-1)^2|\varphi'|^2 - \varepsilon^2 r^2) \tag{6.9}$$

have the factor ε^2. We will choose φ with $|\varphi''| < 1$ and $0 < |1-\varphi'| = \delta < 1$ so that the term (6.8) becomes leading if

$$\begin{aligned} |(\varphi')^3(1-\varphi')| &\geq (1-\delta)^3\delta \\ &\geq 12\alpha^2(n-1)^{-2} \\ &\geq \alpha^2((n-1)^{-2} + (1+\delta)(n-1)^{-2} \\ &\quad + 2(1+\delta)^2(n-1)^{-2} + \frac{1}{8}(n-1)^{-4}). \end{aligned} \tag{6.10}$$

We assume that $\partial E_0 \subset B_{r_2}(0) \setminus B_{r_1}(0)$ with $0 < r_1 < r_2$. Our ansatz for the upper barrier is $\varphi_1(v)$ and for the lower barrier $\varphi_2(v)$ with φ_i, $i = 1, 2$, a linear function with slope $1+(-1)^{i+1}\delta$ which lies above (case $i=1$)/below (case $i=2$) and touches the line segment which connects the points $((n-1)\log r_i, 0)$ and $(L, L-2)$ and δ satisfies (6.10). From a comparison principle we know that these barriers are bounds for u^ε from above and below in Ω_L, furthermore, these bounds are obtained explicitly for given data L, α as follows. For the upper bound we obtain

$$\varphi_1(x) = \begin{cases} s_1(x-(n-1)\log r_1) & \text{if} \quad s_1 > \frac{L-2}{L-(n-1)\log r_1}, \\ s_1(x-L)+L-2 & \text{else}, \end{cases} \tag{6.11}$$

where $x \in \mathbb{R}$. Analogously, we have for the lower bound

$$\varphi_2(x) = \begin{cases} s_2(x-(n-1)\log r_2) & \text{if} \quad s_2 < \frac{L-2}{L-(n-1)\log r_2}, \\ s_2(x-L)+L-2 & \text{else}. \end{cases} \tag{6.12}$$

These barriers provide good estimates in Ω_l, $0 < l \ll L$, if L is large, α small and $r_i < 1$ are both close to 1, then especially the initial hypersurface has 'small oscillation'. Note, that since the boundary values $L-2$ of u^ε on ∂F_L are rather artificial we also expect good estimates only in Ω_l (and not in Ω_L). We recall that applying this construction of barriers for the pairs $(\bar{\varepsilon}, \bar{L})$ and (ε_i, L_i) leads to barriers $\bar{\varphi}_1(v)$, $\bar{\varphi}_2(v)$, $\varphi_1^i(v)$, and $\varphi_2^i(v)$ with obvious notation and that we have then the inequality

$$\varphi_2^i(v) - \bar{\varphi}_1(v) \le u - u^{\bar{\varepsilon}} \le \varphi_1^i(v) - \bar{\varphi}_2(v) \tag{6.13}$$

which is an estimate of the regularization error.

We will exploit this way to construct barriers in the special situation when E_0 is a ball with center in the origin and radius r and when we choose as boundary values on $\partial\Omega_L$ those from (exact) level set IMCF. Then the requirement for the barriers is that φ_i, $i=1,2$, is a linear function with slope $1+(-1)^{i+1}\delta$ which lies above (case $i=1$)/below (case $i=2$) and touches the line segment which connects the points $((n-1)\log r, 0)$ and $(L, \tilde{L})$ where $\tilde{L} \in \mathbb{R}$ suitable (i.e. equal to the 'arrival time' of IMCF at the boundary of $B_L(0)$) and δ satisfies (6.10). It is clear that this yields a regularization error which is 'purely due to ε' and which is given by ε^2. To see this note that the constant δ involved in the definition of the slopes of the barriers is of size $\delta \approx \alpha^2 \approx \varepsilon^2$.

In the corresponding implementation we solved the equivalent 1D-problem on the interval $[0, \log L]$ given by

$$\frac{(\varphi')^3}{e^{4x}} + \frac{\varepsilon^2}{e^{2x}}\varphi'' - \frac{1}{e^{4x}}|\varphi'|^4 - 2\varepsilon^2\frac{1}{e^{2x}}|\varphi'|^2 - \varepsilon^4 = 0 \tag{6.14}$$

with boundary values $\varphi(0)=0$, $\varphi(\log L) = \log L$ which is solved by $\varphi(x) = x$ if $\varepsilon = 0$. This is illustrated in Figure 5. Note that [29, Table 5] differs by the obtained rate $O(\varepsilon)$ instead of $O(\varepsilon^2)$ as in our case. This might be due to the fact that the example in [29, Table 5] simulates IMCF backward in time, i.e. the boundary of the domain is the hypersurface which is reached by the IMCF starting from level sets in the interior of the domain. Then the solution has stationary points. We study the forward problem, i.e. we calculate what happens with a given initial hypersurface in the future and hence we don't have stationary points in the rotational symmetric case.

7. Simulations and further remarks. In this section we present some simulations in the non-rotational symmetric case for PMCF. A short description of the implementation used for the numerical examples is presented, and finally we provide an alternative level set formulation.

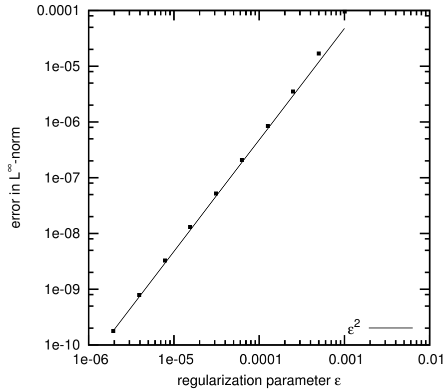

FIG. 5. *Regularization error in case of a circle as initial curve.*

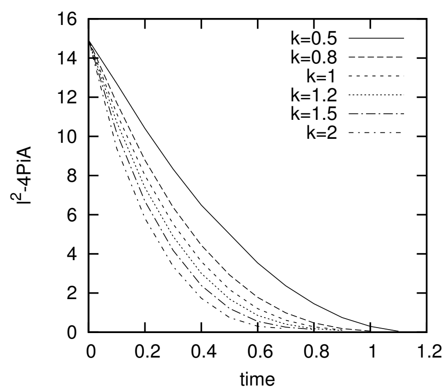

FIG. 6. *Isoperimetrical deficit.*

7.1. Simulations in a non-rotational symmetric case for PMCF. The phenomenon of becoming round can be measured by the isoperimetrical deficit

$$l(t)^2 - 4\pi a(t), \tag{7.1}$$

where $l(t)$ denotes the length of the curve and $a(t)$ the enclosed area at time t. According to theoretical results in [51] we confirm the monotonicity of this deficit during the evolution in the special case of the ellipse as initial curve, see Figure 6. Furthermore, we see that with increasing k and $\varepsilon = 0.05$ the curves transform faster into a circle except for $k = 0.5$, see Figure 8, where the 'point' is reached quite well). In Figure 4 and Figure 7 we see when comparing the exact solutions for the circle for different values of k and the approximate solutions for the ellipse with $\varepsilon = 0.15$ for different values of k, respectively, that the flow reaches the singularity for larger k earlier.

In our previous Figure 4 we plot a section (along the long and short half axes of the initial curve) of the solution u^ε in the case of the circle and in Figure 7 in the case of an ellipse as initial curve for different values of ε.

Figure 8 shows level sets of $u^{0.1}$ for the case of the ellipse as initial curve and different values of k. We remark that our theory covers only the case $k \geq 1$ but there

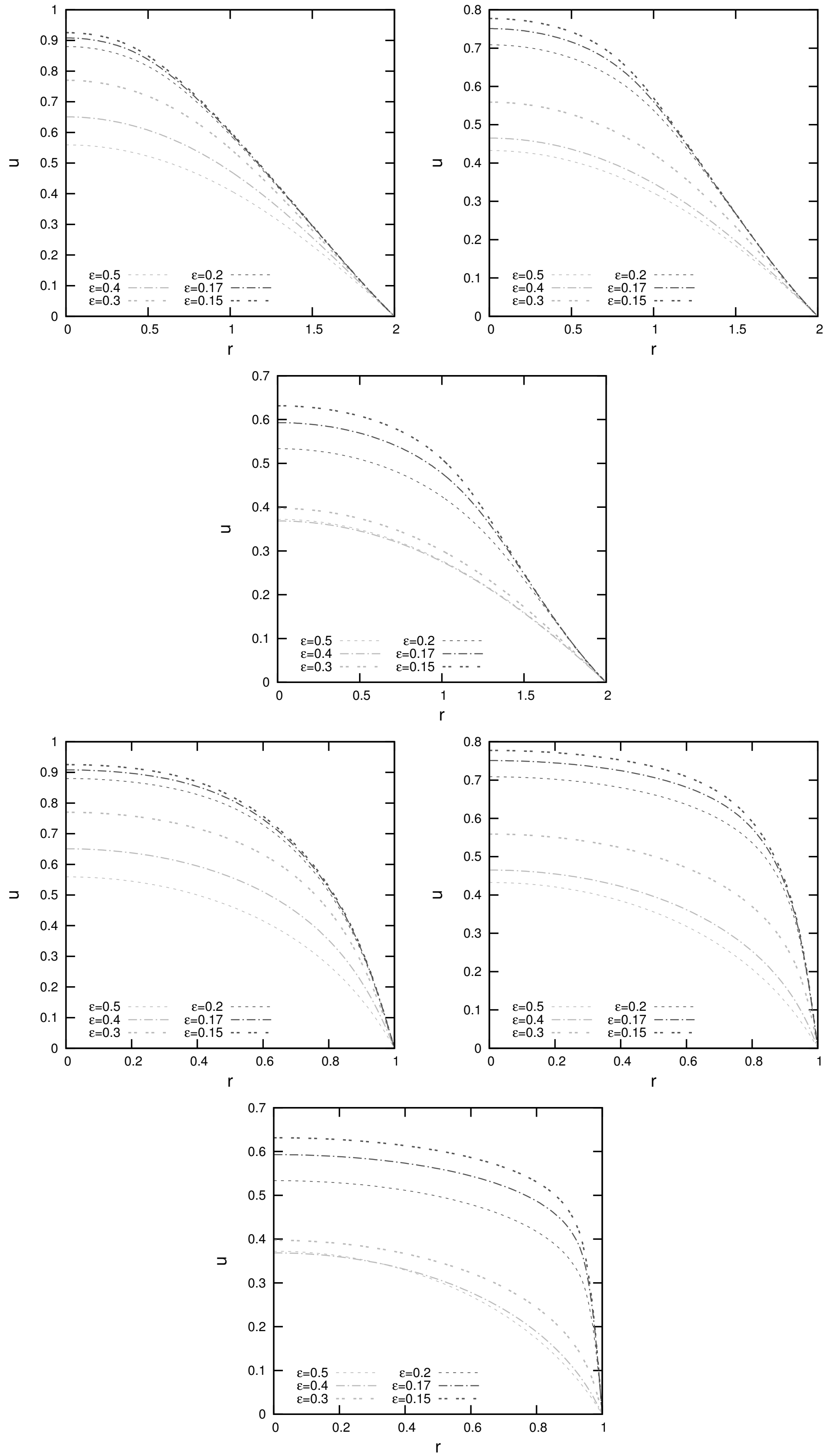

FIG. 7. *Solution for ellipse as initial curve. Picture 1–3: Section in direction of the long half axis of the initial curve for* $k = 1, 1.5, 2$*; Picture 4–6: the same for the short half axis.*

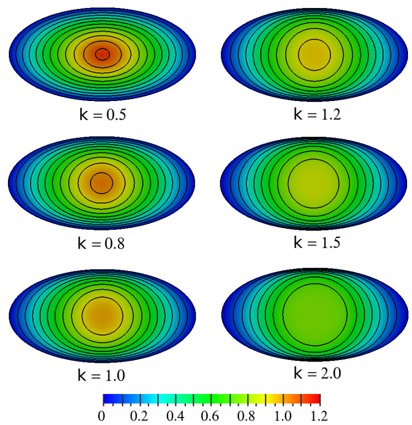

FIG. 8. *Solution for ellipse for* $\varepsilon = 0.1$.

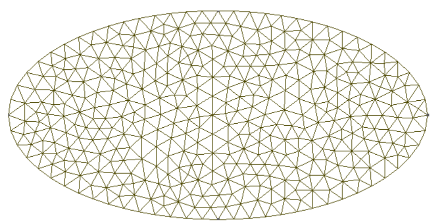

FIG. 9. *Mesh for the discretization with size* $h = 0.15$ *for ellipse with half axes 1 and 2.*

is a well-defined and well-known behavior for the flow of convex curves with speeds given by general positive powers of the curvature, see [4]. Our observations are as follows. For $k = 0.5$ we see for $\varepsilon = 0.1$ a quite good approximation of the phenomenon of shrinking to a 'round point' and further lessening of ε does not show significant improvements. For all k the inner level line for $\varepsilon = 0.1$ seems to be already 'round' while for $k = 2$ this seems to be far from a 'point'.

7.2. On the implementation. To calculate the finite element approximation u_h^ε of u^ε we used a discretization with unstructured grids, see Figure 9. These were generated by the mesh generator Gmsh, see [34]. We solved the non-linear equation (3.3) with a Newton method which uses a bi-conjugate gradient stabilized solver (BiCGSTAB) and SSOR preconditioning. For the implementation we used PDELab, a discretization module for solving PDEs which depends on the Distributed and Unified Numerics Environment (DUNE). As further references concerning PDELab we refer to [48, 11], information about DUNE can be found in [13, 9, 10, 26]. In order to get solutions for small ε we used a warm-start, i.e. we decreased ε stepwise to the desired small value and performed on each stage a calculation with the solution for the previous ε as initial value.

7.3. Alternative level set formulation. For completeness we mention an alternative formulation of the motion by powers of the mean curvature which uses a level set formulation with a level set function which depends on the time. Let $M_0 \subset \mathbb{R}^{n+1}$

be a given initial hypersurface and $u_0 : \mathbb{R}^{n+1} \to \mathbb{R}$ a continuous function such that

$$M_0 = \{x \in \mathbb{R}^{n+1} : u_0(x) = 0\}. \tag{7.2}$$

Let $u : [0, \infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ be the unique viscosity solution of

$$\frac{d}{dt} u = |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right)^k \tag{7.3}$$

in $\mathbb{R}^{n+1} \times (0, \infty)$ with $u(0, \cdot) = u_0$ in $\mathbb{R}^{n+1}$, we call the family of the

$$M(t) = \{x \in \mathbb{R}^{n+1} : u(t, x) = 0\}, \quad t > 0, \tag{7.4}$$

a (time dependent) level set PMCF. Equation (7.3) is a fully nonlinear, degenerate and possibly singular (if $Du = 0$) parabolic equation. In the case $k > 1$ the elliptic main part of (7.3) is not in divergence form and fully nonlinear in the second spatial derivatives which is of disadvantage having our finite element approach in mind. Furthermore, it is higher dimensional than the equation we used. Nevertheless, this formulation is quite common in the literature in the cases $0 < k \le 1$ and in general also available when the speed is not necessarily positive. We give a short overview. Existence and uniqueness of a solution for (7.3) is proved in [18, 19, 28] in the case $k = 1$. In [44] equation (7.3) in case $0 < k \le 1$ is approximated by a family of regularized equations and rates of convergence of the corresponding solutions are obtained. Concerning regularizations of equations we also refer to [12]. The time dependent formulation (7.3) in the case $k = \frac{1}{3}$, i.e. the affine curvature equation, is used for image processing, cf. [1, 31]. In the case $k = 1$, i.e. mean curvature flow, equation (7.3) has been studied intensively analytically and numerically, cf., e.g., [15, 21, 23, 24, 41]. We want to point out the paper [22] by Deckelnick, where the solution u^ε of a regularized version of (7.3) is approximated by a finite difference scheme which was originally proposed by Crandall and Lions [21]. In Deckelnick's paper rates for the convergence of the discrete solution to the solution u of the (not regularized) level set equation are proved. The total error consists of a regularization error of the form

$$\|u - u^\varepsilon\|_{L^\infty(\Omega)} \le c_\alpha \varepsilon^\alpha \tag{7.5}$$

with $\alpha \in (0, \frac{1}{2})$ arbitrary and c_α a positive constant, see [22, Theorem 1.2] for details, and a discretization error which is a *polynomial* expression in the numerical parameter and the reciprocal regularization parameter. Furthermore, the value for the convergence order of the discretization error (and hence for the total approximation error) is very low; the main point here is that this rate is of polynomial order. A calculation showed that in the setting of our paper the constants involved in the L^2- and H^1-error estimate for the discretization error depend at most exponentially on $\frac{1}{\varepsilon}$. Such an exponential dependence is not unusual as can be seen in the paper [25]. There the viscosity solution u of (7.3) is approximated by a solution u_ε of the regularized equation and then the regularized equation is approximated by a solution $u_{\varepsilon,h}$ of a semi-discrete problem. The regularization error is again of the form (7.5) but the error $u_\varepsilon - u_{\varepsilon,h}$ measured in a certain energy norm, cf. [24, Theorem 6.4], is only of order $c_\varepsilon h$, where, and this is the important point, the constant c_ε depends *exponentially* on $\frac{1}{\varepsilon}$. Numerical tests from that reference, however, suggest that the resulting bound overestimates the error. In the special case of two dimensions, i.e. the moving hypersurfaces are curves, Deckelnick and Dziuk [25] prove L^∞-convergence (without rates) of the discrete solution provided $h = h(\varepsilon)$ sufficiently small, where 'sufficiently small' is not given by an explicit formula or polynomial dependence.

Appendix A. Some auxiliary observations. Since $L_\varepsilon : H^1_0(\Omega) \to H^{-1}(\Omega)$ is a topological isomorphism by classical L^2-theory this also holds for $L^*_\varepsilon : H^1_0(\Omega) \to H^{-1}(\Omega)$. We define the to L_ε associated uniformly, elliptic regular Dirichlet bilinear form of order 1 by

$$B : W^{1,p}_0(\Omega) \times W^{1,p^*}_0(\Omega) \to \mathbb{R}, \quad B[u,v] = \int_\Omega a^{ij} D_i u D_j v + b^i D_i u v \, \mathrm{d}x \tag{A.1}$$

and set

$$\begin{aligned} N_{p^*} =& \{v \in W^{1,p^*}_0(\Omega) : B[\psi, v] = 0 \text{ for every } \psi \in C^\infty_0(\Omega)\} \\ N_p =& \{v \in W^{1,p}_0(\Omega) : B[v, \phi] = 0 \text{ for every } \phi \in C^\infty_0(\Omega)\} \end{aligned} \tag{A.2}$$

From Fredholm's alternative, cf. [55, Theorem 10.7], we deduce that for every $F \in W^{-1,p^*}(\Omega)$ the equation

$$B[u, \varphi] = F\varphi \quad \text{for all } \varphi \in W^{1,p^*}_0(\Omega) \tag{A.3}$$

has a solution $u \in W^{1,p}_0(\Omega)$ if and only if

$$v \in N_{p^*} \quad \textit{implies} \quad Fv = 0. \tag{A.4}$$

If $\dim N_{p^*} = \dim N_p = 0$ then for every $F \in W^{-1,p^*}(\Omega)$ equation (A.3) has a unique solution.

Lemma A.1. $\dim N_{p^*} = \dim N_p = 0$.

Proof. Let $v \in N_{p^*}$. From [55, Theorem 7.6] we get $v \in W^{1,p'}_0(\Omega)$ for all $1 < p' < \infty$, especially for $p' = 2$. Since we know from L^2-theory that (A.3) has a unique solution $u \in W^{1,2}_0(\Omega)$ if $p = 2$ and $F = 0$ we deduce that $v = 0$. Analogously we obtain the remaining claim. ∎

By bounded inverse theorem we conclude the following result.

Corollary A.2. *$L_\varepsilon, L^*_\varepsilon$ are topological isomorphisms.*

Acknowledgment. Parts of the work were carried out while the third author benefited from a Weierstrass Stipendium of the Weierstrass Institute-Berlin (WIAS).

REFERENCES

[1] L. Alvarez, F. Guichard, P.-L. Lions and J.-M. Morel. Axioms and fundamental equations of image processing. *Arch. Rational Mech. Anal.*, **123** (3): 199–257, 1993.

[2] L. Alvarez and J.-M. Morel. *Formalization and computational aspects of image analysis.* Acta Numer., Cambridge University Press, Cambridge, UK, pp. 200–257, 1993.

[3] B. Andrews. Evolving convex curves. *Calc. Var. Partial Differential Equations*, **7**: 315–371, 1998.

[4] B. Andrews. Classification of limiting shapes for isotropic curve flows. *J. Amer. Math. Soc.*, **16**: 443–459, 2003.

[5] S. B. Angement, G. Sapiro and A. Tannenbaum. On affine heat equation for non-convex curves. *J. Amer. Math. Soc.*, **11**: 601–634, 1998.

[6] J. W. Barrett, H. Garcke and R. Nürnberg. On the Variational Approximation of Combined Second and Fourth Order Geometric Evolution Equations. *SIAM J. Sci. Comput*, **29** (3): 1006–1041, 2007.

[7] J. W. Barrett, H. Garcke and R. Nürnberg. On the parametric finite element approximation of evolving hypersurfaces in $\mathbb{R}^3$. *J. Comp. Phys.*, **227** (9): 4281–4307, 2008.

[8] J. W. Barrett, H. Garcke and R. Nürnberg. Parametric approximation of isotropic and anisotropic elastic flow for closed and open curves. *Numer. Math.*, **120**: 489–542, 2012.

[9] P. Bastian, M. Blatt, A. Dedner, C. Engwer, R. Klöfkorn, R. Kornhuber, M. Ohlberger and O. Sander. A generic grid interface for parallel and adaptive scientific computing. Part ii: Implementation and tests in DUNE. *Computing*, **82** (2-3): 121–138, 2008.

[10] P. Bastian, M. Blatt, A. Dedner, C. Engwer, R. Klöfkorn, M. Ohlberger and O. Sander. A generic grid interface for parallel and adaptive scientific computing. part i: abstract framework. *Computing*, **82** (2-3): 103–119, 2008.

[11] P. Bastian, F. Heimann and S. Marnach. Generic implementation of finite element methods in the distributed and unified numerics environment (dune). *Kybernetika*, **46** (2): 294–315, 2010.

[12] G. Bellettini, I. Fragalá. Elliptic approximations pf prescribed mean curvature surfaces in Finsler geometry. *Asymptotic Anal.*, **22**, No. 2, 87-111, 2000.

[13] M. Blatt and P. Bastian. The Iterative Solver Template Library. *Springer, New York, USA* 2007.

[14] J. W. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods.* Texts in Applied Mathematics 15, Springer, Berlin, 1996.

[15] E. Carlini, M. Falcone and Ferretti. Convergence of a large time-step scheme for mean curvature motion. *Interfaces Free Bound.*, **12**: 409–441, 2010.

[16] V. Caselles, F. Catte, T. Coll and F. Dibos. A geometric model for active contours in image processing. *Numer. Math.*, **66**: 1–31, 1993.

[17] T. Chan and L. Vese. *An Active Contour Model without Edges.* M. Nielsen et al. (Eds.): Scale-Space'99, LNCS 1682, 141–151, Springer, Berlin, 1999.

[18] Y. G. Chen, Y. Giga and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *Proc. Japan. Acad.*, **65**: 207–210, 1989.

[19] Y. G. Chen, Y. Giga and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differ. Geom.*, **33** (3): 749–786, 1991.

[20] M. G. Crandall, H. Ishii. and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin (new Series) of the American Mathematical Society*, **27** (1): 1–67, 1992.

[21] M. G. Crandall and P.-L. Lions. Convergent difference schemes for nonlinear parabolic equations and mean curvature flow. *Numer. Math.*, **75** (3): 17–41, 1996.

[22] K. Deckelnick. Error bounds for a difference scheme approximating viscosity solutions of mean curvature flow. *Interfaces Free Bound.*, **2**: 117–142, 2000.

[23] K. Deckelnick and G. Dziuk. Convergence of a finite element method for non–parametric mean curvature flow. *Numer. Math.*, **72**: 197–222, 1995.

[24] K. Deckelnick, G. Dziuk and C. M. Elliott. Computation of geometric partial differential equations and mean curvature flow. *Acta Numer.*, **14**: 139–232, 2005.

[25] K. Deckelnick and G. Dziuk. *Convergence of numerical schemes for the approximation of level set solutions to mean curvature flow.* M. Falcone and C. Makridakis (Eds.): Numerical Methods for Viscosity Solutions and Applications, Vol. 59, Series Adv. Math. Appl. Sciences, Springer, pp 77–94, 2001.

[26] DUNE, Distributed and unified numerics environment. URL: http://www.dune-project.org 12/01/2014.

[27] K. Ecker and G. Huisken. Mean Curvature Evolution of Entire Graphs. *Ann. Math.*, 2nd Ser., Vol. **130**: 453–471, 1989.

[28] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. *J. Differ. Geom.*, **33**: 635–681, 1991.

[29] X. Feng, M. Neilan and A. Prohl. Error analysis of finite element approximations of the inverse mean curvature flow arising from the general relativity. *Numer. Math.*, **108** (1): 93–119, 2007.

[30] M. E. Gage. Curve shortening makes convex curves circular. *Invent. Math.*, **76** (2): 357–364, 1984.

[31] Y. Giga. *Surface evolution equations. A level set approach.* Monographs in Mathematics 99, Birkhäuser Verlag, Basel, 2006.

[32] M. E. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *J. Diff. Geom.*, **23** (1): 69–96, 1986.

[33] C. Gerhardt. Flow of nonconvex hypersurfaces into spheres. *J. Differential Geom.*, **32** (1): 299–314, 1990.

[34] C. Geuzaine and J.-F. Remacle,. Gmsh: a 3-d finite element mesh generator with built-in pre-and post-processing facilities. *Int. J. Numer. Methods Eng.*, **79** (11): (2009) 1309–1331.

[35] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Classics in Mathematics 224, Springer, Berlin, Heidelberg, New York etc., 2001.

[36] M. Grayson. The heat equation shrinks embedded plane curves to points. *J. Diff. Geom.*, **26**: 285–314, 1987.

[37] F. Hecht. New development in FreeFem++. *J. Numer. Math.* **20**, No. 3-4, 251-265, 2012.

[38] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differ. Geom.*, **20** (3): 117–138, 1984.

[39] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differ. Geom.*, **59** (3): 353–437, 2001.

[40] H. Kröner. Approximation rates for regularized level set power mean curvature flow, *Port. Math.*, to appear.

[41] R.V. Kohn and S. Serfaty. A deterministic-control-based approach to motion by curvature. *Communications on Pure and Applied Mathematics* **59**: 344-407, 2006.

[42] R. Malladi, J. A: Sethian. *Level set methods for curvature flow, image enhancement, and shape recovery in medical images.* H. C. Helde, K. Polthier (Eds.): Visualization and Mathematics. Experiments, Simulation and Environments, 329–345, Springer, Berlin, 1997.

[43] K. Mikula, D. Ševčovč. Evolution of plane curves driven by a nonlinear function of curvature and anisotropy. *SIAM Journal on Applied Mathematics*, **61** (5): 1473–1501, 2001.

[44] H. Mitake. On convergence rates for solutions of approximate mean curvature equations. *Proceedings of the American Mathematical Society*, **139** (10): 3691–3696, 2011.

[45] R. H. Nochetto and C. Verdi. Convergence Past Singularities for a Fully Discrete Approximation of Curvature-Driven Interfaces. *SIAM J. Numer. Anal.*, **34** (2): 490–512, 1997.

[46] K. Osher and R. Fedkiw. *Level Set Methods and Dynamic Implicit Surfaces.* Applied Mathematical Sciences Vol. 153, Springer, 2003.

[47] K. Osher, J. A. Sethian. Fronts Propagating with Curvature Dependent speed: Algorithms Based on Hamilton-Jacobi Formulation. *Journal of Computational Physics*, **79**: 12–49, 1988.

[48] PDELab. URL: http://www.dune-project.org/pdelab/index.html 12/01/2014.

[49] G. Sapiro and A. Tannenbaum. On affine plane curve evolution. *J. Funct. Anal.*, **119**: 79–120, 1994.

[50] F. Schulze. Evolution of convex hypersurfaces by powers of the mean curvature. *Math. Z.*, **251** (4): 721–733, 2005.

[51] F. Schulze. Nonlinear evolution by mean curvature and isoperimetrical inequalities. *J. Diff. Geom.*, **79**: 197–241, 2008.

[52] F. Schulze, appendix with Oliver Schnürer. Convexity estimates for flows by powers of the mean curvature. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **5** (5), no 2: 261–277, 2006.

[53] J. A. Sethian. Theory, Algorithms, and Applications of Level Set Methods for Propagating Interfaces. *Acta Numerica*, **5**: 309–395, 1996.

[54] J. A. Sethian. *Level Set Methods and Fast Marching Methods.* Cambridge Monographs on Applied and Computational Mathematics, Vol. 3, Cambridge University Press, 1999.

[55] C. G. Simader. *On Dirichlet's Boundary Value Problem.* Lecture Notes in Math. v. 268, Springer, Berlin 1972.

[56] I. Toulopoulos, T. Wick. Numerical methods for power-law diffusion problems. *SIAM J. Sci. Comput.*, **39** (3), A681-A710, 2017.

[57] N. J. Walkington. Algorithms for computing motion by mean curvature. *SIAM J. Numer. Anal*, **33**: 2215–2238, 1996.