A normal form approach for nonlinear normal modes

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A NORMAL FORM APPROACH FOR NON-LINEAR NORMAL MODES

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Résumé

The definition of a non-linear normal mode (NNM) is considered through the framework of normal form theory. Following Shaw and Pierre, a NNM is defined as an invariant manifold which is tangent to its linear counterpart at the origin. It is shown that Poincaré and Poincaré-Dulac's theorems define a non-linear change of variables which permits to span the phase space with the coordinates linked to the non-linear invariant manifolds. Hence, the equations governing the geometry of the NNMs are contained within the coordinate transformation. Moreover, the attendant dynamics onto the manifolds are given by the normal form of the problem. General calculations for a conservative N-degrees of freedom system are provided. The relevance of the method for the study of non-linear vibrations of continuous damped structures is discussed.

1 Introduction

The main motivation for defining non-linear normal modes (NNMs) relies upon the dependence of the mode shapes on the vibration amplitude. When increasing the vibration amplitude of a continuous structure, it is observed that single-mode motions (the dynamics of which is still described by a single displacement-velocity pair) exhibit a spatial deformation and differ more and more from the linear mode shape. This fact has been reported for example by Bennouna, Benamar *et al.* [6, 7].

On the theoretical viewpoint, the review article of Rosenberg [1] accounts for the first developments on the subject. Subsequent efforts have been made in the same direction in order to define a NNM adequately. Within the framework of conservative non-linear systems, significant developments have been realized by Rand, Vakakis and coworkers, with the objective of taking the stability of the NNMs and their bifurcations into account [2, 3, 4, 5].

A decisive contribution has been brought on the subject by the work of S. Shaw and C. Pierre [8, 9, 10]. These authors define a NNM as an invariant manifold which is tangent to its linear counterpart, the planar eigenspaces, at the origin. This definition allows them to use efficient techniques developed for dynamical systems (see for example [11, 12], or [13] for the relevance of the approach in the context of hydrodynamics). The method used for demonstrating the center manifold theorem [11, 14] has been applied by Shaw and Pierre to solve the vibratory problem at hand. The invariance property of the manifolds has been underlined. It permits to describe the motion along a NNM with a single displacement-velocity pair, and thus to explain the amplitude dependence of the mode shapes. This is a key point for generating reduced order models. This *operative definition* was shown to be effective for discrete [9] as well as for continuous systems [10].

A quite similar viewpoint has been developed by Nayfeh *et al.* [15]. The crucial point demonstrated in [15], is the equivalence of different approaches of current use for solving the partial differential equations that govern the dynamics under study. The methods developed in [8, 9, 10], in [15], and, more recently, in [16], enable one to compute a single NNM. Once the geometry of the non-linear manifold has been calculated, the dynamics is projected onto it, thus leading to a second-order differential problem. This method works especially well when no internal resonance is considered. Although this problem is well understood theoretically, challenging technical problems arise in the computation of invariant manifolds when internal resonances are present. This leads to complicated and tedious calculations, see [17, 18, 19, 20].

In this contribution, the definition of a NNM as an invariant manifold is used. Normal form theory yields a non-linear change of variables which allows to go from the phase space spanned by the linear eigenspaces to the one spanned by the NNMs. The geometrical equations of the NNMs are thus included in the change of variables. The same idea, in the context of vibratory systems, has already been used by Jezequel and Lamarche [21]. However they didn't use the oscillator-form decoupling at linear order. Thus comparisons with other methods were not made explicit. Pellicano and Mastroddi [32] developp similar calculations in order to correct a mistake present in [9]. Once again, complex-form representation is used at linear stage.

Here, general quadratic and cubic non-linearities, as well as N degrees of freedom, are taken into account. All invariant manifolds are calculated in a single computation, and the dynamics onto the manifolds are made explicit. Roughly speaking, this demonstrates that the normal form is the key point for the attendant dynamics onto the manifolds, and thus that the problem at hand finds its ground basis in the theorems of Poincaré and Poincaré-Dulac [12, 11, 13]. Finally, a simple definition of a NNM in terms of the new variables will be given.

2 FRAMEWORK

2.1 General assumptions

In this paper, general computations for a N degrees-of-freedom non-linear structural system is given. The system is assumed to be decoupled at linear order. If this is not the case, eventual linear coupling terms can be easily eliminated. For example, one can use a procedure presented in Appendix 1 of [9], which allows to put the system under "oscillatory forms" at the linear stage. The non-linearities are of second and third order in displacement. Thus the equations of motion take the form :

$$\forall\, p = 1...N \ : \quad \ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{N} \sum_{j\geq i}^{N} g_{ij}^p X_i X_j + \sum_{i=1}^{N} \sum_{j\geq i}^{N} \sum_{k\geq j}^{N} h_{ijk}^p X_i X_j X_k = 0, \qquad (1)$$

where X_p stands for the modal amplitude of linear mode p, with eigenfrequency ω_p. In what follows, X_p is called the *physical coordinate*.

Damping is not considered in (1). However, for lightly damped systems, for which the damping is much smaller than the eigenfrequency, damping may be added after the non-linear change of variables, as a small perturbation to the conservative case. A specific discussion on damping can be found in section 7.2.

Assembly of N discrete non-linear oscillators produces equations of motion of the form presented in Eq. (1). However, the method proposed here can also be applied to continuous structures such as strings, beams, arches, plates or shells, with weak geometric non-linearities. The projection onto the linear eigenspaces of the partial differential equations which governs the vibration of such structures (*e.g.* Von Kármán type equations) leads to a problem of the form (1) (see for example [22], or [23, 24, 25]).

2.2 Invariant manifolds

As was pointed out by S. Shaw and C. Pierre, invariance is the central property of linear modes that can be extended to the non-linear range. By invariance we mean that a motion initiated along a manifold at time $t = 0$ will always be contained within the manifold, for every $t > 0$. For a linear system, the eigenspaces are planar invariant surfaces graduated by the physical coordinates. A motion initiated along the first mode, for example, will be such that :

$$\forall\, p\ \geq 2;\ \forall\, t > 0,\ X_p(t) = 0.$$

It is this property that one is interested in extending to the non-linear range. When the motion is getting far from the equilibrium point, and when the non-linearities cannot be neglected anymore, the invariant surfaces become curved manifolds. The effect of this bending leads to residual excitation and contamination of higher-frequency modes through non-resonant non-linear terms in (1), which could be viewed as "source terms". This fact was observed for a long time ago, see for example [26]. Defining a NNM as an invariant manifold, which is tangent to the linear eigenspace at the origin, allows to recover a dynamics governed by a single displacement-velocity pair [8, 9, 10].

In this paper, a non-linear change of variables will be defined with the help of normal form theory. This will span the phase space with the NNMs. The situation is sketched in Figure 1. The horizontal and vertical axis represent linear eigenspaces. These are in fact planes but are represented here by lines for convenience. The curved invariant manifolds are also represented. The idea behind this non-linear change of variables is to use the coordinates linked to the invariant manifolds in order to express the dynamics, as it is represented in figure 1 with a curved grid. This non-linear change of variables will be properly defined and calculated in the next sections. We will show that it is consistent with all previous approaches [9, 15, 18, 19].

3 Normal form theory

3.1 Poincaré and Poincaré-Dulac's theorems

In this section, we start by recalling the theorems of Poincaré and Poincaré-Dulac, which are the cornerstones of the normal form theory. The interested reader can refer to [12, 11] for a more formal presentation.

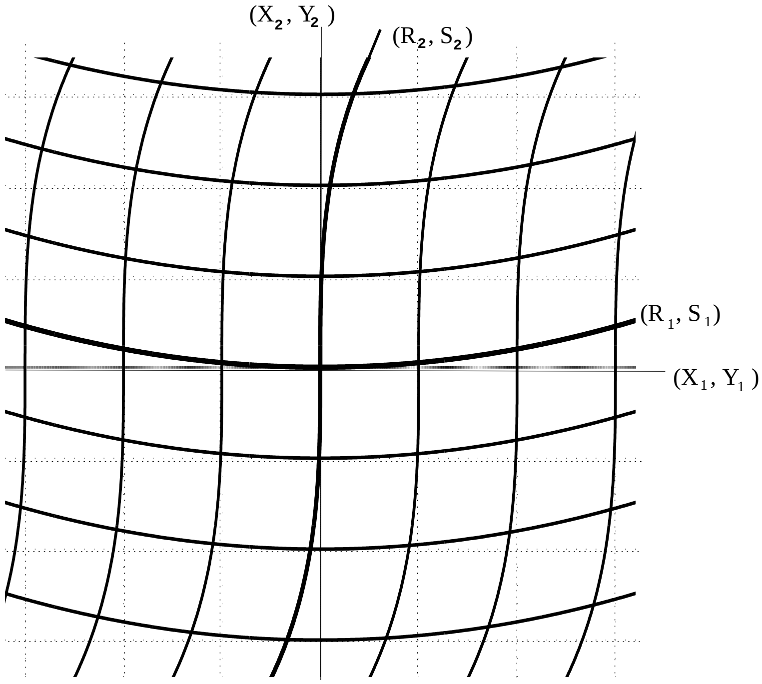

FIG. 1 – Sketch of the phase space of system (1) in the vicinity of the origin. Horizontal and vertical axis : linear eigenspaces (they are in fact planes), graduated by the physical coordinates : displacement X and velocity Y. For clarity, only two eigenplanes have been represented, although N are present. Curved heavy lines : invariant manifolds, graduated by the new variables : displacement R and velocity S. The level lines associated with those variables, which are related to (X, Y) by a non-linear relation, are also represented.

The leading idea consists of simplifying all the non-linear terms, which are not dynamically important, in the vicinity of a fixed point. Consider a dynamical system :

$$\dot{\mathbf{x}} = \mathbf{L}\mathbf{x} + \mathbf{N}(\mathbf{x}), \tag{2}$$

which has a fixed point at the origin $\mathbf{x} = 0$, where it is assumed that $\mathbf{N}(\mathbf{x})$ represents the non-linear terms, expanded in polynomial series, with $\mathbf{N}(0) = 0$. Poincaré's theorem states that, if there are no resonance relation between the eigenvalues of $\mathbf{L}$, then it is possible to find a non-linear change of variables :

$$\mathbf{x} = \mathbf{y} + h(\mathbf{y}), \tag{3}$$

such that the system (2) reduces to its linear part :

$$\dot{\mathbf{y}} = \mathbf{L}\mathbf{y}. \tag{4}$$

In equation (3), the non-linear coordinates transformation is tangent to the identity (and thus $h(0) = 0$), and $h(\mathbf{y})$ represents polynomial higher-order terms.

The resonance relations between the eigenvalues $\{\lambda_k\}_{k=1...N}$ of the linear operator $\mathbf{L}$ are defined by :

$$\forall\, s = 1...N : \lambda_s = \sum_{i=1}^{N} m_i \lambda_i, \quad m_i \geq 0, \quad \sum m_i = p \geq 2, \tag{5}$$

where p is called the order of resonance [11, 12, 13], and N the number of eigenvalues of the system. Poincaré-Dulac's theorem states that if there are resonance relations, then all monomial terms in $\mathbf{N}(\mathbf{x})$ that are resonant cannot be eliminated. Those theorems are demonstrated by successive eliminations of the quadratic terms, cubic terms... in $\mathbf{N}(\mathbf{x})$, by introducing a polynomial $h(\mathbf{y})$ of the appropriate degree, and eliminating the terms that can be cancelled [11, 12, 13].

3.2 Application to vibratory systems

The remainder of the article consists in reproducing the same demonstration, which means finding h. It is applied here to the particular case of a non-linear vibratory system of the form (1), characterized by an eigenvalue spectrum : $\{\pm i\omega_k\}_{k=1...N}$. The vector field under study will be reduced to its normal form. It will be shown that the new coordinates are linked with a grid defined by the invariant manifolds.

The strength of the normal form theory follows from the fact that one is able to know all important dynamical non-linear terms by simply looking at the eigenvalue spectrum of the linear operator. Here, we have to make a distinction between *trivial* resonance relations and internal resonance. Trivial resonance relations arise from the fact that, for a purely imaginary eigenspectrum, equation (5) is always fulfilled with relations of the form :

$$\forall\ k,\ \ \forall\ j: \quad i\omega_k = i(\omega_j - \omega_j) + i\omega_k. \tag{6}$$

Hence there are cubic terms that cannot be eliminated from (1). For example, equation (6) corresponds to a cubic term of the form $X_j^2 X_k$ in the k^{th} oscillator.

Internal resonance are all the other relations contained in (5) and which are not trivial. For example, if there exist a relation of the form : $\omega_2 = 2\omega_1$, then a second-order internal resonance is said to exist, and a monomial term of the form X_1^2 will be unremovable (resonant) in the evolution equation for X_2 [11, 12, 13, 22].

It is here assumed that there are no internal resonance relations between the eigenvalues $\{\pm i\omega_k\}_{k=1...N}$ of the system (1). Hence all quadratic terms are non-resonant and can be eliminated. Cubic terms arising from the trivial resonance relations will not be cancelled. This is demonstrated below.

4 Quadratic terms

In this section, we define the non-linear change of variables which cancels all quadratic terms. As the processing of the quadratic terms does not involve cubic terms, system (1) is truncated to second order, and is written :

$$\dot{X}_p = Y_p, \tag{7a}$$

$$\dot{Y}_p = -\omega_p^2 X_p - \sum_{i=1}^{N}\sum_{j\geq i}^{N} g_{ij}^p X_i X_j. \tag{7b}$$

(the mention : $\forall\ p = 1...N$ will be omitted when not confusing).

The first step consists of defining a second-order polynom with $2N$ variables (N pairs displacement-velocity). It is chosen tangent to the identity, and is written :

$$X_p = U_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}(a^p_{ij}U_iU_j + b^p_{ij}V_iV_j) + \sum_{i=1}^{N}\sum_{j=1}^{N} c^p_{ij}U_iV_j, \tag{8a}$$

$$Y_p = V_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}(\alpha^p_{ij}U_iU_j + \beta^p_{ij}V_iV_j) + \sum_{i=1}^{N}\sum_{j=1}^{N} \gamma^p_{ij}U_iV_j. \tag{8b}$$

(U_p, V_p) are the new variables. The polynoms (8) are written in this form to take commuting and non-commuting terms into account. Some practical formulaes linked to the manipulation of such polynoms are given in appendix A.

The unknown of the problem are now the $2N^2(2N+1)$ coefficients $\{a^p_{ij}, b^p_{ij}, c^p_{ij}, \alpha^p_{ij}, \beta^p_{ij}, \gamma^p_{ij}\}$. They are determined by introducing (8) in (7). This generates terms of the form $\dot{U}_iU_j, \dot{V}_iV_j, \dot{U}_iV_j$... They are cumbersome for the identification of the different monomials, because involving the derivative of a variable against time. This is remedied by observing that, at lower order :

$$\dot{U}_p = V_p + O(U_i^2, V_i^2), \tag{9a}$$

$$\dot{V}_p = -\omega_p^2 U_p + O(U_i^2, V_i^2), \tag{9b}$$

where $O(U_i^2, V_i^2)$ stands for all quadratic quantity involving the variables. Hence, for example, we have :

$$\dot{U}_iU_j = V_iU_j + O(U_i^3, V_i^3),$$

where the accuracy is now at cubic order. Thus all the terms of the form $\dot{U}_iU_j, \dot{V}_iV_j, \dot{U}_iV_j$... can be replaced by $V_iU_j, -\omega_i^2U_iV_j, V_iV_j$..., and the identification is possible at order two.

When truncating all the above developments to second order, (7) is now written as :

$$\begin{aligned}\dot{U}_p + &\sum_{i=1}^{N}\sum_{j\geq i}^{N}\left[(a^p_{ij} - \omega_j^2 b^p_{ij})V_iU_j + (a^p_{ij} - \omega_i^2 b^p_{ij})U_iV_j\right] + \sum_{i=1}^{N}\sum_{j=1}^{N} c^p_{ij}(V_iV_j - \omega_j^2 U_iU_j)\\ &= V_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}(\alpha^p_{ij}U_iU_j + \beta^p_{ij}V_iV_j) + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma^p_{ij}U_iV_j,\end{aligned} \tag{10a}$$

$$\begin{aligned}\dot{V}_p + &\sum_{i=1}^{N}\sum_{j\geq i}^{N}\left[(\alpha^p_{ij} - \omega_j^2 \beta^p_{ij})V_iU_j + (\alpha^p_{ij} - \omega_i^2 \beta^p_{ij})U_iV_j\right] + \sum_{i=1}^{N}\sum_{j=1}^{N} \gamma^p_{ij}(V_iV_j - \omega_j^2 U_iU_j)\\ &= -\omega_p^2\left[U_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}(a^p_{ij}U_iU_j + b^p_{ij}V_iV_j) + \sum_{i=1}^{N}\sum_{j=1}^{N} c^p_{ij}U_iV_j\right] - \sum_{i=1}^{N}\sum_{j\geq i}^{N} g^p_{ij}U_iU_j.\end{aligned} \tag{10b}$$

The last step is the identification of the different monoms in (10). This is realized after noticing that full sums (of the form $\sum_{i=1}^N\sum_{j=1}^N$) with commuting terms have been introduced through the process. This is simply resolved by recombining the terms, according to a formula, given in Appendix A.

The coefficients of like power terms are equated. This will produce linear subsystems in the unknowns of the problem. The solution of these subsystems will provide the coefficients that permits to cancel the quadratic terms. The identification for equation (10a) allows one to express $\{\alpha^p_{ij}, \beta^p_{ij}, \gamma^p_{ij}\}$ as functions of $\{a^p_{ij}, b^p_{ij}, c^p_{ij}\}$:

$\forall\ i = 1...N$:

$$2(a^p_{ii} - \omega_i^2 b^p_{ii}) = \gamma^p_{ii} \tag{11a}$$

$$-c^p_{ii}\omega_i^2 = \alpha^p_{ii} \tag{11b}$$

$$c^p_{ii} = \beta^p_{ii} \tag{11c}$$

$\forall\ i = 1...N-1,\ \ \forall\ j\ >\ i...N$:

$$-c^p_{ij}\omega_j^2 - c^p_{ji}\omega_i^2 = \alpha^p_{ij} \tag{11d}$$

$$c^p_{ij} + c^p_{ji} = \beta^p_{ij} \tag{11e}$$

$$a^p_{ij} - \omega_i^2 b^p_{ij} = \gamma^p_{ij} \tag{11f}$$

$$a^p_{ij} - \omega_j^2 b^p_{ij} = \gamma^p_{ji} \tag{11g}$$

For equation (10b), the identification of like power terms leads to :

$\forall\ i = 1...N$:

$$2(\alpha^p_{ii} - \omega_i^2 \beta^p_{ii}) = -\omega_p^2 c^p_{ii} \tag{12a}$$

$$\gamma^p_{ii}\omega_i^2 = \omega_p^2 a^p_{ii} + g^p_{ii} \tag{12b}$$

$$\gamma^p_{ii} = -\omega_p^2 b^p_{ii} \tag{12c}$$

$\forall\ i = 1...N-1,\ \ \forall\ j\ >\ i...N$:

$$-\gamma^p_{ij}\omega_j^2 - \gamma^p_{ji}\omega_i^2 = -\omega_p^2 a^p_{ij} - g^p_{ij} \tag{12d}$$

$$\gamma^p_{ij} + \gamma^p_{ji} = -\omega_p^2 b^p_{ij} \tag{12e}$$

$$\alpha^p_{ij} - \omega_i^2 \beta^p_{ij} = -\omega_p^2 c^p_{ij} \tag{12f}$$

$$\alpha^p_{ij} - \omega_j^2 \beta^p_{ij} = -\omega_p^2 c^p_{ji} \tag{12g}$$

The two systems (11) and (12) are solved to give all the searched coefficients :

$\forall\ i = 1...N\,,\ \forall\ j \geq i...N$:

$$a^p_{ij} = \frac{\omega_i^2 + \omega_j^2 - \omega_p^2}{D_{ijp}} g^p_{ij} \tag{13a}$$

$$b^p_{ij} = \frac{2}{D_{ijp}} g^p_{ij} \tag{13b}$$

$$c^p_{ij} = 0 \tag{13c}$$

$$\alpha^p_{ij} = 0 \tag{13d}$$

$$\beta^p_{ij} = 0 \tag{13e}$$

$$\gamma^p_{ii} = \frac{2}{4\omega_i^2 - \omega_p^2} g^p_{ii} \tag{13f}$$

$\forall\ i = 1...N\,,\ \forall\ j > i...N$:

$$\gamma^p_{ij} = \frac{\omega_j^2 - \omega_i^2 - \omega_p^2}{D_{ijp}} g^p_{ij} \tag{13g}$$

$$\gamma^p_{ji} = \frac{\omega_i^2 - \omega_j^2 - \omega_p^2}{D_{ijp}} g^p_{ij} \tag{13h}$$

where $D_{ijp} = (\omega_i + \omega_j - \omega_p)(\omega_i + \omega_j + \omega_p)(-\omega_j + \omega_i + \omega_p)(\omega_i - \omega_j - \omega_p)$.

One can notice that D_{ijp} contains all possible second-order resonance relations, which indicates that the calculation breaks down in case of internal resonance. This is completely usual and consistent with previous approaches [8, 18, 19, 15].

At this stage, we have exhibited a non-linear change of variables which allows to cancel all the quadratic terms in (1), when no internal resonance are present. The methodology sketched in this section is general and will be applied in the next section in order to eliminate the cubic terms. Since the calculations are tedious, only the important steps of the procedure will be explained. The only difference with the case treated in this section lies in the fact that *trivial* resonant terms exists at cubic order and will not be removed.

5 Cubic terms

5.1 Elimination of the quadratic terms

The first step consists in eliminating the quadratic terms from Eq. (1). We use the results of the previous section, that is we do the replacement :

$$X_p = U_p + \sum_{i=1}^{N} \sum_{j \geq i}^{N} (a_{ij}^p U_i U_j + b_{ij}^p V_i V_j), \tag{14a}$$

$$Y_p = V_p + \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}^p U_i V_j, \tag{14b}$$

in Eq. (1), beforehand written as $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$. As we are now interested in cubic terms, all developments are truncated at order 3. After some algebraic transformation, the system is written :

$$\dot{U}_p = V_p, \tag{15a}$$

$$\dot{V}_p = -\omega_p^2 U_p - \sum_{i=1}^{N} \sum_{j \geq i}^{N} \sum_{k \geq j}^{N} h_{ijk}^p U_i U_j U_k - \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k \geq j}^{N} \left[A_{ijk}^p U_i U_j U_k + B_{ijk}^p U_i V_j V_k \right], \tag{15b}$$

which is correct up to order 3 included.

The coefficients A_{ijk}^p, B_{ijk}^p are due to the introduction of the non-linear change of variables (14) which eliminates the quadratic terms but introduces cubic-order terms, of which some are velocity-dependent. Their expressions are :

$$A_{ijk}^p = \sum_{l \geq i}^{N} g_{il}^p a_{jk}^l + \sum_{l \leq i} g_{li}^p a_{jk}^l, \tag{16a}$$

$$B_{ijk}^p = \sum_{l \geq i}^{N} g_{il}^p b_{jk}^l + \sum_{l \leq i} g_{li}^p b_{jk}^l. \tag{16b}$$

5.2 Processing of the cubic terms

In this subsection, the cubic non-resonant terms of equation (15) will be cancelled through the same process as that used in section 4. The only difference is the presence of trivially resonant terms that cannot be eliminated. The following cubic polynoms are introduced :

$$U_p = R_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N}(r^p_{ijk}R_iR_jR_k + s^p_{ijk}S_iS_jS_k) \\ + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N}(t^p_{ijk}S_iR_jR_k + u^p_{ijk}R_iS_jS_k), \tag{17a}$$

$$V_p = S_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N}(\lambda^p_{ijk}R_iR_jR_k + \mu^p_{ijk}S_iS_jS_k) \\ + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N}(\nu^p_{ijk}S_iR_jR_k + \xi^p_{ijk}R_iS_jS_k), \tag{17b}$$

written in such a form to take the commuting and non-commuting terms into account. The unknowns of the problem are now the $\frac{4}{3}N^2(N+1)(2N+1)$ coefficients that are introduced : $\{r^p_{ijk},\ s^p_{ijk},\ t^p_{ijk},\ u^p_{ijk},\ \lambda^p_{ijk},\ \mu^p_{ijk},\ \nu^p_{ijk},\ \xi^p_{ijk}\}$. At the end of this step, the displacement-velocity pairs will be the (R_p, S_p) variables.

Equations (17) are differentiated with respect to time and introduced in (15). Products of the form $\dot{R}_iS_jR_k,\ R_i\dot{R}_jS_k$ ... are eliminated by noticing that, at lower order :

$$\dot{R}_p = S_p + O(R_i^3, S_i^3), \tag{18a}$$

$$\dot{S}_p = -\omega_p^2 R_p + O(R_i^3, S_i^3). \tag{18b}$$

The precision is now at cubic order, because the quadratic terms have been processed. This allows to replace the terms involving a derivative with respect to time, and leads to the identification of power-like cubic order terms. This step is not reproduced in detail here for a sake of conciseness. Two tricks are used in order to simplify a little bit the calculations. First the terms have to be ordered, and one has to make a distinction between commuting and non-commuting terms. A formula given in the appendix A helps in this tedious step. Secondly, the complete substitution can be performed only for equation (15a). For (15b), one has just to notice that the left-hand side of the equation is exactly the same, provided the substitution of $\{r^p_{ijk},\ s^p_{ijk},\ t^p_{ijk},\ u^p_{ijk}\}$ for $\{\lambda^p_{ijk},\ \mu^p_{ijk},\ \nu^p_{ijk},\ \xi^p_{ijk}\}$. Then, The first term of the right-hand side is calculated by doing the inverse substitution : $(\{\lambda^p_{ijk},\ \mu^p_{ijk},\ \nu^p_{ijk},\ \xi^p_{ijk}\}$ is replaced by $\{r^p_{ijk},\ s^p_{ijk},\ t^p_{ijk},\ u^p_{ijk}\})$. The two last terms are computed by simply substituting (U_i, V_j) for (R_i, S_j) (other terms leads to higher-order polynomials).

The power-like coefficients are then equated. This leads to two sets of 24 equations given in appendix B and appendix C, respectively from eq. (15a) and (15b). Combining together these equations produces closed subsystems for $\{r^p_{ijk},\ s^p_{ijk},\ t^p_{ijk},\ u^p_{ijk}\}$. Those systems have either a solution (which is the case for non-resonant terms), or not (which is the case for the trivially resonant terms). When a solution exist, then the monoms can be cancelled. When not, all the involved coefficients are set to zero, and the associated monom stay in the dynamical equation.

This calculation leads to the following results :

$$\forall\, i = 1...N,\ \forall\, j \geq i...N,\ \forall k \geq j...N :\ s^p_{ijk} = \lambda^p_{ijk} = 0 \tag{19a}$$

$$\forall\, i = 1...N,\ \forall\, j = 1...N,\ \forall k \geq j...N :\ t^p_{ijk} = \xi^p_{ijk} = 0 \tag{19b}$$

And thus the non-linear change of variables reduces to :

$$U_p = R_p + \sum_{i=1}^{N} \sum_{j \geq i}^{N} \sum_{k \geq j}^{N} r^p_{ijk} R_i R_j R_k + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k \geq j}^{N} u^p_{ijk} R_i S_j S_k, \tag{20a}$$

$$V_p = S_p + \sum_{i=1}^{N} \sum_{j \geq i}^{N} \sum_{k \geq j}^{N} \mu^p_{ijk} S_i S_j S_k + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k \geq j}^{N} \nu^p_{ijk} S_i R_j R_k. \tag{20b}$$

In these sums, some terms are equal to zero, which corresponds to the trivial resonance relations. These are :

$$\begin{aligned}
\forall p = 1...N: \qquad & u^p_{ppp} = r^p_{ppp} = \mu^p_{ppp} = \nu^p_{ppp} = 0 \\
\forall\, j > p...N: \ & r^p_{pjj} = u^p_{pjj} = u^p_{jpj} = 0 \\
& \mu^p_{pjj} = \nu^p_{pjj} = \nu^p_{jpj} = 0 \\
\forall\, i < p: \ & r^p_{iip} = u^p_{iip} = u^p_{pii} = 0 \\
& \mu^p_{iip} = \nu^p_{iip} = \nu^p_{pii} = 0
\end{aligned} \tag{21}$$

The non-zero coefficients are reported in Appendix D. The denominators of the coefficients given in Appendix D vanish when third-order internal resonances are present.

We are now in position to define the normal dynamics, thanks to the non-linear change of variables (20). This will be done in the next section, and will enable us to precise the definition of a NNM with the formalism of normal form theory.

6 Recapitulation of the results

6.1 Normal dynamics

The objective of the previous mathematical developments was to define a non-linear change of variables which eliminates the non-resonant terms in (1). This has been made with the help of normal form theory together with Poincaré and Poincaré-Dulac's theorems. Assembling the two calculations in section 4 and 5 gives the following relations, for all p :

$$\begin{aligned}
X_p = {} & R_p + \sum_{i=1}^{N} \sum_{j \geq i}^{N} (a^p_{ij} R_i R_j + b^p_{ij} S_i S_j) \\
& + \sum_{i=1}^{N} \sum_{j \geq i}^{N} \sum_{k \geq j}^{N} r^p_{ijk} R_i R_j R_k + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k \geq j}^{N} u^p_{ijk} R_i S_j S_k,
\end{aligned} \tag{22a}$$

$$Y_p = S_p + \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma^p_{ij} R_i S_j + \sum_{i=1}^{N} \sum_{j \geq i}^{N} \sum_{k \geq j}^{N} \mu^p_{ijk} S_i S_j S_k + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k \geq j}^{N} \nu^p_{ijk} S_i R_j R_k. \tag{22b}$$

(X_p, Y_p) are the *physical coordinates*; we decide to call (R_p, S_p) the *normal coordinates.* These coordinates allow one to define a kind of "decoupling" between the variables at the non-linear stage. This decoupling does not share all the properties of a linear decoupling with eigenspaces. Specifically, superposition is not possible. This result generalize the answer given in [32]. However, these new variables are linked with the invariant manifolds and allow a simple definition of a NNM, as will be shown in the next subsection, especially because the invariance property is now explicited.

The normal dynamics is defined by substituting for (22) into (1). It describes the dynamics of the system, up to order 3, in a space spanned by the invariant manifolds (recall figure 1). It reads :

$$
\begin{aligned}
\forall\, p =& 1 \dots N : \\
\dot{R}_p =& S_p, \qquad\qquad (23a)\\
\dot{S}_p =& -\omega_p^2 R_p - (A^p_{ppp} + h^p_{ppp}) R_p^3 - B^p_{ppp} R_p S_p^2 \\
& - R_p \left[\sum_{j>p}^{N} \left[(A^p_{jpj} + A^p_{pjj} + h^p_{pjj}) R_j^2 + B^p_{pjj} S_j^2 \right] + \sum_{i<p} \left[(A^p_{iip} + A^p_{pii} + h^p_{iip}) R_i^2 + B^p_{pii} S_i^2 \right] \right] \\
& - S_p \left[\sum_{j>p}^{N} B^p_{jpj} R_j S_j + \sum_{i<p} B^p_{iip} R_i S_i \right]. \qquad\qquad (23b)
\end{aligned}
$$

One can notice in (23b) all the trivially resonant terms that cannot be cancelled. These terms correspond to the coefficients set to zero in (21). Moreover, this normal dynamics shows that velocity-dependent terms have to be considered, a point that has already been noticed by S. Shaw and C. Pierre. It is demonstrated here that these terms are coming from the processing of the quadratic terms. Hence, in a purely cubic non-linear system (with $\alpha^p_{ij} = 0$), no velocity-dependent terms appear in the normal dynamics.

6.2 Definition of a NNM

With the *normal coordinates*, a non-linear normal mode is simply defined by the cancellation of all other variables, except the displacement-velocity pair considered. For example, one has just to set :

$$
\forall k \neq p : R_k = S_k = 0, \qquad (24)
$$

for investigating the NNM labelled p. Then the normal dynamics on the p^{th} non-linear manifold is governed by :

$$
\begin{aligned}
\dot{R}_p &= S_p, && (25a)\\
\dot{S}_p &= -\omega_p^2 R_p - (A^p_{ppp} + h^p_{ppp}) R_p^3 - B^p_{ppp} R_p S_p^2. && (25b)
\end{aligned}
$$

The equation of the manifold, expliciting its geometry in the phase space, is given in (22). Replacing (24) in (22) gives :

$$
\begin{aligned}
&\forall\, k \neq p : \\
&X_k = a^k_{pp} R_p^2 + b^k_{pp} S_p^2 + r^k_{ppp} R_p^3 + u^k_{ppp} R_p S_p^2 && (26a)\\
&Y_k = \gamma^k_{pp} R_p S_p + \mu^k_{ppp} S_p^3 + \nu^k_{ppp} S_p R_p^2 && (26b)
\end{aligned}
$$

This can be written with the physical coordinates, only. This will allow us to compare more precisely the method proposed here with [8, 9, 10], where the idea is to define a *master* displacement-velocity pair, and to express all the other pairs as functions of this one. The same operation is led here with equation (24). Since, from equation (22), we have :

$$
X_p = R_p + O(R_p^2, S_p^2)\ , \qquad Y_p = S_p + O(R_p^2, S_p^2),
$$

then, up to order 3 (the precision term $O(X_p^4, Y_p^4)$ is omitted) :

$$\forall\, k \neq p :$$

$$X_k = a_{pp}^k X_p^2 + b_{pp}^k Y_p^2 + r_{ppp}^k X_p^3 + u_{ppp}^k X_p Y_p^2, \tag{27a}$$

$$Y_k = \gamma_{pp}^k X_p Y_p + \mu_{ppp}^k Y_p^3 + \nu_{ppp}^k Y_p X_p^2. \tag{27b}$$

Coefficients appearing in equations (27), which define the geometry of the invariant manifold, have been compared with previous calculations of the same geometry led in [9, 18, 19, 15]. It has been verified that all the coefficients match, and thus the methods are completely equivalent. However, it seems more suitable to think about this non-linear change of variables as the definition of new coordinates linked to the invariant manifolds (see Figure 1). It allows one to have a more precise picture of the phase space in the vicinity of the origin, to take into account all the bendings of all the invariant manifolds, and finally, it seems more natural, in comparison with the linear case, to think about the NNMs as new coordinates, which are non-linearly linked with the physical ones. More specifically, it allows to explain easily the failure of Galerkin-based method followed by a severe truncature, without performing again many developments, as is done recently for example in [33].

Another interesting point is the generality of equations (23), which govern the dynamics onto the manifolds. These equations have been demonstrated in the most general case. Two remarks seems to be worth mentioning here.

First, the "decoupling" at the non-linear stage through the cancellation of all the non-resonant terms is now obvious, when simply looking at Eqs (23). Those non-resonant terms are not dynamically important and are only responsible of the curvature of the phase space (see figure 1). More specifically, what was defined as a "similar NNM" by Rosenberg correspond to a case where non-resonant terms are not present in the vibrations equations, and thus the manifolds are flat. The generic case give rise to "non-similar NNM" and a curvature of the manifolds. The invariance property is readable in equations (23) since all the variables $\{(R_j,\ S_j) | j \neq p\}$ are in factor of R_p and S_p. Thus "single-mode" dynamics along NNMs are now underlined : a motion initiated in one invariant manifold at time $t = 0$ will stay into it for every $t > 0$. This is again underlined in the exemple of subsection 7.4.

Second, it has been shown that the dynamics onto the manifolds is given by the normal form of the problem studied. This point is crucial since, through equations (23), one is able to write the normal form of ***any*** vibratory system, from the knowledge of its eigenspectrum $\{\pm i\omega_p\}_{p\geq 1}$, only. This is the strength of Poincaré's theorem to show that the dynamically important non-linear terms are given at the linear stage with the eigenvalues of the linear operator. Hence, it is now possible to study any dynamical vibratory system onto the manifolds with equations (23), and then to come back to the physical coordinates with the inverse operation of (22). In this manner, the operation stated here has some resemblance with the one led at the linear range (modal decomposition). Developments of such *ex-nihilo* models (only with the knowledge of the linear part), has already been successfully used in hydrodynamics, see for example [13, 27].

7 DISCUSSION

After having clarified the main points of the method proposed in the previous sections, some left-aside points are now discussed ; and a very simple example is given. This will allow us to state some open questions which asks for more work or experimental validations.

7.1 Asymptotic developments

The method proposed in this article for the computation of the NNMs relies upon a non-linear change of variables which permits to span the phase space with coordinates linked to the invariant manifolds. This non-linear coordinate transformation is calculated with an asymptotic development (see eqs. (22) for example). The main problem is that the radius of convergence of the asymptotic series is not given throughout the calculations. Moreover, all given equations are accurate up to order 3. Higher-order terms can cause disturbances when considering for example too large amplitudes. Example of the precision of the asymptotic development can be found in [28].

An alternative technique to overcome the failure of the asymptotic development is given in [16], where a polar form is used in order to calculate more precisely the invariant manifolds. The results seems very promising, and could perhaps be applied to the method proposed here. This will be tested in a near future. More generally, mathematical methods known as "Non-linear Galerkin Methods", introduced by M. Marion and R. Temam in [29], seem to be powerful tools to overcome the difficulties mentioned here. Finally, the developments written here could be done with a complex representation, as it is done for example in [15, 21, 32]. This could substantially simplify the calculations.

7.2 Damping

Damping has been ignored in all the previous developments. This is mainly motivated by the fact that dynamically important modes are the lightly damped ones, which have very long evolution time constants. Strongly damped modes are of less interest since their behaviour can be summarized as a quick exponential decay to zero.

Considering only lightly damped modes is also motivated by the two following facts. First, for metallic thin structures such as plates or shells for example, the lightly damped modes are those with eigenfrequencies below the coincidence frequency. They are generally numerous (of the order of 50, see for example [30]). For a comprehensive description of the steady state dynamics of such structures, at large time scales, it should be enough to consider those modes. Second, non-linear systems becomes more and more interesting when typical non-linear regime such as quasiperiodic or chaotic motions are encountered (see for example [31]). In this manner, only the lightly damped modes, which are "master" compared to the "slaved", strongly damped ones, are of prime importance. Slaved modes can be cancelled through adiabatic elimination, when the time constants are well apart one from each other [13].

Thus, for all these reasons, lightly damped modes are of prime interest. To proceed with them, it is possible to carry out the normal form calculations without the damping (as it has been made here) ; and to add the damping in the normal form, as a small perturbation of the conservative problem. Experimental and theoretical validations are needed to confirm this issue, and this will be the main point of the future research on this subject.

7.3 Internal resonance

One major advantage of the method proposed here is its ability to handle easily the case of internal resonance. The presence of an internal resonance will simply result in a failure in the computations of the coefficients resulting from the 48 linear equations listed in Appendix B and C. More precisely, only one subsystem will have no longer a solution. Then, as it has been done for the *trivially* resonant terms, all coefficients of the non-linear change of variables, corresponding to the subsystem under study, will be set to zero, and the corresponding monoms will remain in the normal form.

It is one advantage of the method proposed here, because all the calculations have been completely carried out. Considering an internal resonance does not give rise to other calculations.

7.4 A simple example

The tedious calculations of the general case are now illustrated through a very simple example. More complete examples are treated in [28].

Following Bennett and Eisley [26], the vibrations of a beam with clamped ends are investigated. The idea here is to point out some interesting features rather than to propose a complete treatment of the problem.

After projection of the partial differential equations of the motion on the linear normal mode basis, the restriction of the first three modes leads to the following temporal problem :

$$\ddot{X}_1 + \omega_1^2 X_1 + h^1_{111} X_1^3 + h^1_{122} X_2^2 X_1 + h^1_{133} X_3^2 X_1 + h^1_{113} X_1^2 X_3 + h^1_{333} X_3^3 = 0 \tag{28a}$$

$$\ddot{X}_2 + \omega_2^2 X_2 + h^2_{112} X_1^2 X_2 + h^2_{233} X_3^2 X_2 + h^2_{222} X_2^3 + h^2_{123} X_1 X_2 X_3 = 0 \tag{28b}$$

$$\ddot{X}_3 + \omega_3^2 X_3 + h^3_{111} X_1^3 + h^3_{122} X_2^2 X_1 + h^3_{133} X_3^2 X_1 + h^3_{113} X_1^2 X_3 + h^3_{333} X_3^3 = 0 \tag{28c}$$

In the paper of Bennett and Eisley, it was shown that forcing the system at a frequency near ω_1 leads to a residual contribution in the amplitude of the third mode (X_3). This was due to the term $h^3_{111} X_1^3$ in equation (28c), identified as a "source" term for the third mode. Initiating a motion along the first mode, with $X_2(0) = X_3(0) = 0$ shows that for $t > 0$, energy is transferred to X_3, and thus $X_3(t) \neq 0$. It is the same term which is responsible for the dependence of the first mode shape on amplitude, since a contribution of mode 3, vibrating at the forcing frequency near ω_1, is present in the response.

Applying the previous results to this system shows that the normal dynamics, expressed with the *normal coordinates* (R_p, S_p), is written :

$$\ddot{R}_1 + \omega_1^2 R_1 + R_1 (h^1_{111} R_1^2 + h^1_{122} R_2^2 + h^1_{133} R_3^2) = 0 \tag{29a}$$

$$\ddot{R}_2 + \omega_2^2 R_2 + R_2 (h^2_{112} R_1^2 + h^2_{233} R_3^2 + h^2_{222} R_2^2) = 0 \tag{29b}$$

$$\ddot{R}_3 + \omega_3^2 R_3 + R_3 (h^3_{113} R_1^2 + h^3_{333} R_3^2) = 0 \tag{29c}$$

Hence the "source term" in the third equation has been cancelled, and the invariance properties is maintained. This term is contained in the non-linear change of variables, and in the manifold equations which describe the geometry of the NNMs the phase space. Equations (29) can be directly deduced from (23). The study of non-linear normal motions can be realized with those equations, then one is able to come back to the physical problem given by (28).

8 CONCLUSION

In this paper, the concept of non-linear normal mode (NNM) has been reconsidered through normal form theory. It has been shown that a non-linear change of variables allows one to go from the phase space spanned by the linear eigenplanes to the one spanned by the invariant manifolds. The bending of the manifolds are given by the non-resonant terms, whereas the dynamics onto the manifolds are governed by the resonant ones. The expression of the geometry of all NNMs is contained in the non-linear change of variables. All these computations leads to a simple redefinition of a non-linear normal mode motion, which is simply obtained by the cancellation of all the other normal coordinates, as in the linear case. In fact, thinking of the NNMs as a curved span of the phase space, non-linearly connected to the orthogonal planar span given by the eigenmodes, allows to re-utilize some linear concepts, with adjustments, to the non-linear stage.

The main drawback of the method, which is based on an asymptotic development, has been underlined, and methods to overcome this difficulty has been suggested. However, general calculations led in this paper enables an easy implementation for computer simulations. This should allow to extend the range of linear modeling to the case of moderate non-linearities. The generality of equations (23), which capture all the possible dynamics up to order 3 for vibratory systems with discrete eigenspectrum of the form : $\{\pm i\omega_p\}_{p\geq 1}$, has been underlined. Moreover, it allows to define *ex-nihilo* models with the knowledge of the eigenvalues only. Experimental validations, as well as theoretical studies of the normal dynamics, featuring internal resonance or not, are now needed to complete the study of non-linear vibratory systems with moderate non-linearities.

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Références

[1] R. M. Rosenberg . On non-linear vibrations of systems with many degrees of freedom, *Advances in Applied Mechanics* **9**, 155-242, 1966.

[2] R. H. Rand. A direct method for non-linear normal modes, *Int. J. of Non-linear Mech.* **9**, 363-368, 1974.

[3] L. A. Month and R. H. Rand. The stability of bifurcating periodic solutions in a two-degree-of-freedom non-linear system, *ASME J. of Applied Mech.* **44**, 782-783, 1977.

[4] R. H. Rand, C. H. Pak and A . F. Vakakis. Bifurcation of non-linear normal modes in a class of two-degree-of-freedom systems, *Acta Mechanica* **3**, 129-145, 1992.

[5] A . F. Vakakis. Non-linear Normal Modes (NNMs) and their applications in vibration theory : an overview, *Mech. systems and Signal Proc.* **11**(1), 3-22, 1997.

[6] R. Benamar, M. M. Bennouna and R. G. White. The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures. Part I : simply supported and clamped-clamped beams, *J. Sound. Vib.* **149**(2), 179-195, 1991.

[7] R. Benamar, M. M. Bennouna and R. G. White. The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures. Part II : fully clamped rectangular plates, *J. Sound. Vib.* **164**, 399-424, 1993.

[8] S. W. Shaw and C. Pierre. Non-linear Normal Modes and invariant manifolds, *J. Sound. Vib.* **150**(1), 170-173, 1991.

[9] S. W. Shaw C. Pierre. Normal Modes for non-linear vibratory systems, *J. Sound. Vib.* **164**(1), 85-124, 1993.

[10] S. W. Shaw C. Pierre. Normal Modes of vibration for non-linear continuous systems, *J. Sound. Vib.* **169**(3), 319-347, 1994.

[11] J. Guckenheimer and P. J. Holmes. *Non-linear oscillations, dynamical systems and bifurcations of vector fields*, New-York, Springer-Verlag, 1986.

[12] V. I. Arnold. *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*, Mir-Librairie du Globe, 1980.

[13] P. Manneville. *Dissipative structures and weak turbulence*, Academic Press, 1990.

[14] J. Carr. *Applications of centre manifold theory*, New-York, Springer-Verlag, 1981.

[15] A. H. Nayhfeh and S. A. Nayfeh. On nonlinear modes of continuous systems, *Trans. ASME, Journal of Vibration and Acoustics* **116**, 129-136, 1994.

[16] E. Peshek, C. Pierre and S. W. Shaw. A new Galerkin-based approach for accurate non-linear normal modes through invariant manifolds, *J. Sound. Vib.* **249**(5), 971-993, 2002.

[17] N. Boivin, C. Pierre and S. W. Shaw. Non-linear modal analysis of structural systems featuring internal resonances, *J. Sound. Vib.* **182**(2), 336-341, 1995.

[18] E. Peshek, *Reduced order modeling of non-linear structural systems using non-linear normal modes and invariant manifolds*, PhD Thesis, University of Michigan, 2000.

[19] E. Peshek and C. Pierre. A Global methodology for the modal reduction of large non-linear systems containing quadratic and cubic non-linearities, *Proc. of DETC'97*, ASME Design Eng. Tech. Conf, 1997.

[20] A. H. Nayfeh, C. Chin and S. A. Nayfeh. On nonlinear modes of systems with internal resonance, *Trans. ASME, Journal of Vibration and Acoustics* **118**, 340-345, 1996.

[21] L. Jezequel and C. H. Lamarque. Analysis of non-linear dynamical systems by the normal form theory, *J. Sound. Vib.* **149**(3), 429-459, 1991.

[22] A. H. Nayfeh and D. T. Mook. *Non-linear oscillations.* John Wiley, New-York, 1979.

[23] J. M. Johnson and A. K. Bajaj. Amplitude modulated and chaotic dynamics in resonant motion of strings, *J. Sound. Vib.* **128**(1), 87-107, 1989.

[24] X. L. Yang and P. R. Sethna. Local and global bifurcations in parametrically excited vibrations of nearly square plates, *Int. J. of Non-Linear Mechanics* **26**(2), 199-220, 1991.

[25] C. Touzé, O. Thomas and A. Chaigne. Asymmetric non-linear forced vibrations of free-edge circular plates, part I : Theory. To appear in *J. Sound. Vib.*, 2002.

[26] J. A. Bennett and J. G. Eisley. A multiple degree-of-freedom approach to non-linear beam vibrations, *AIAA Journal* **8**(4), 734-739, 1970.

[27] P. Manneville and L. S. Tuckerman. Phenomenological modeling of the first bifurcations of a spherical Couette flow, *Journal de Physique* **48**, 1461-1469, 1987.

[28] C. Touzé and O. Thomas. Non-linear normal modes for conservative systems. *Proceedings of the ninth ICSV*, Orlando, Juillet 2002.

[29] M. Marion and R. Temam. Non-linear Galerkin methods, *SIAM J. Numer. Analysis* **26**(5), 1139-1157, 1989.

[30] A. Chaigne and C. Lambourg. Time domain simulation of damped impacted plates. Part I : theory and experiments, *Journal of the Acoustical Society of America*, **109**(4), 1422-1432, 2001.

[31] C. Touzé and A. Chaigne. Lyapunov exponents from experimental time series : applications to cymbal vibrations, *Acta Acustica*, **86**(3), 557-567, 2000.

[32] F. Pellicano and F. mastroddi. Applicability conditions of a non-linear superposition technique. *J. Sound. Vib.*, **200**(1), 3-14, 1997.

[33] A. H. Nayfeh and W. lacarbonara. On the discretization of distributed-parameters systems with quadratic and cubic nonlinearities. *Nonlinear dynamics*, **13**, 203-220, 1997.

A Algebraic formulas for some polynomial manipulations

We mention here some formulas which have been used in the derivation of the equations.

- Number of monoms in a polynom of degree N with p indeterminates :

$$\sum_{i=0}^{p-1} C_{N+i-1}^{i}$$

  (N.B. : without the constant term. Only the monoms with indeterminates have been counted.)

- Simplification of a quadratic polynom with commuting terms :

$$\sum_{i=1}^{N}\sum_{j\geq i}^{N}\Gamma_{ij}X_iX_j = \sum_{i=1}^{N}\Gamma_{ii}X_i^2 + \sum_{i=1}^{N-1}\sum_{j>i}^{N}(\Gamma_{ij}+\Gamma_{ji}X_iX_j)$$

- Reordering of a cubic polynom :

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N}\alpha_{ijk}X_iY_jZ_k &= \sum_{i=1}^{N}\alpha_{iii}X_iY_iZ_i \\ &+ \sum_{i=1}^{N-1}\sum_{j>i}^{N}\left[\alpha_{jii}X_jY_iZ_i + \alpha_{jij}X_jY_iZ_j + \alpha_{iij}X_iY_iZ_j + \alpha_{ijj}X_iY_jZ_j\right] \\ &+ \sum_{i=1}^{N-2}\sum_{j>i}^{N-1}\sum_{k>j}^{N}\left[\alpha_{kij}X_kY_iZ_j + \alpha_{jik}X_jY_iZ_k + \alpha_{ijk}X_iY_jZ_k\right]\end{aligned}$$

B Identification of equation (15a)

The 24 equations arising from the power-like identification of the terms in eq. (15a) are given here, with a mention of the monoms from which each term derives.

$\forall\ i = 1...N :$

$$
\begin{array}{rcll}
-\omega_i^2 t_{iii}^p & = & \lambda_{iii}^p & \qquad (R_i^3) \\
u_{iii}^p & = & \mu_{iii}^p & \qquad (S_i^3) \\
3r_{iii}^p - 2\omega_i^2 u_{iii}^p & = & \nu_{iii}^p & \qquad (R_i^2 S_i) \\
2t_{iii}^p - 3\omega_i^2 s_{iii}^p & = & \xi_{iii}^p & \qquad (R_i S_i^2)
\end{array}
$$

$\forall\ i = 1...N-1,\ \forall\ j > i...N :$

$$
\begin{array}{rcll}
r_{iij}^p - \omega_i^2 u_{iij}^p & = & \nu_{jii}^p & \qquad (R_i^2 S_j) \\
t_{jij}^p - \omega_i^2 s_{ijj}^p & = & \xi_{ijj}^p & \qquad (R_i S_j^2) \\
t_{iij}^p - \omega_j^2 s_{iij}^p & = & \xi_{jii}^p & \qquad (R_j S_i^2) \\
r_{ijj}^p - \omega_j^2 u_{jij}^p & = & \nu_{ijj}^p & \qquad (R_j^2 S_i) \\
-\omega_j^2 t_{jii}^p - \omega_i^2 t_{iij}^p & = & \lambda_{iij}^p & \qquad (R_i^2 R_j) \\
-\omega_j^2 t_{jij}^p - \omega_i^2 t_{ijj}^p & = & \lambda_{ijj}^p & \qquad (R_i R_j^2) \\
u_{jii}^p + u_{iij}^p & = & \mu_{iij}^p & \qquad (S_i^2 S_j) \\
u_{jij}^p + u_{ijj}^p & = & \mu_{ijj}^p & \qquad (S_i S_j^2) \\
2r_{iij}^p - 2\omega_i^2 u_{jii}^p - \omega_j^2 u_{iij}^p & = & \nu_{iij}^p & \qquad (R_i R_j S_i) \\
2r_{ijj}^p - 2\omega_j^2 u_{ijj}^p - \omega_i^2 u_{jij}^p & = & \nu_{jij}^p & \qquad (R_i R_j S_j) \\
-2\omega_i^2 s_{iij}^p + 2t_{jii}^p + t_{iij}^p & = & \xi_{iij}^p & \qquad (R_i S_i S_j) \\
-2\omega_j^2 s_{ijj}^p + t_{jij}^p + 2t_{ijj}^p & = & \xi_{jij}^p & \qquad (R_j S_i S_j)
\end{array}
$$

$\forall\ i = 1...N-2,\ \forall\ j > i...N-1,\ \forall\ k > j...N :$

$$
\begin{array}{rcll}
-\omega_k^2 t_{kij}^p - \omega_j^2 t_{jik}^p - \omega_i^2 t_{ijk}^p & = & \lambda_{ijk}^p & \qquad (R_i R_j R_k) \\
u_{kij}^p + u_{jik}^p + u_{ijk}^p & = & \mu_{ijk}^p & \qquad (S_i S_j S_k) \\
r_{ijk}^p - \omega_i^2 u_{jik}^p - \omega_j^2 u_{ijk}^p & = & \nu_{kij}^p & \qquad (R_i R_j S_k) \\
r_{ijk}^p - \omega_i^2 u_{kij}^p - \omega_k^2 u_{ijk}^p & = & \nu_{jik}^p & \qquad (R_i S_j R_k) \\
r_{ijk}^p - \omega_j^2 u_{kij}^p - \omega_k^2 u_{jik}^p & = & \nu_{ijk}^p & \qquad (S_i R_j R_k) \\
t_{jik}^p + t_{ijk}^p - \omega_k^2 s_{ijk}^p & = & \xi_{kij}^p & \qquad (S_i S_j R_k) \\
t_{kij}^p + t_{ijk}^p - \omega_j^2 s_{ijk}^p & = & \xi_{jik}^p & \qquad (S_i R_j S_k) \\
t_{kij}^p + t_{jik}^p - \omega_i^2 s_{ijk}^p & = & \xi_{ijk}^p & \qquad (R_i S_j S_k)
\end{array}
$$

C Identification of equation (15b)

The 24 following equations arise from the identification of the power-like terms (which are indicated into the brackets) from the development of eq. (15b).

$\forall\ i = 1...N:$

$$\begin{array}{rcll}
-\omega_i^2 \nu_{iii}^p &=& -\omega_p^2 r_{iii}^p - A_{iii}^p - h_{iii}^p & (R_i^3)\\
\xi_{iii}^p &=& -\omega_p^2 s_{iii}^p & (S_i^3)\\
3\lambda_{iii}^p - 2\omega_i^2 \xi_{iii}^p &=& -\omega_p^2 t_{iii}^p & (R_i^2 S_i)\\
2\nu_{iii}^p - 3\omega_i^2 \mu_{iii}^p &=& -\omega_p^2 u_{iii}^p - B_{iii}^p & (R_i S_i^2)
\end{array}$$

$\forall\ i = 1...N-1,\ \forall\ j > i...N:$

$$\begin{array}{rcll}
\lambda_{iij}^p - \omega_i^2 \xi_{iij}^p &=& -\omega_p^2 t_{jii}^p & (R_i^2 S_j)\\
\nu_{jij}^p - \omega_i^2 \mu_{ijj}^p &=& -\omega_p^2 u_{ijj}^p - B_{ijj}^p & (R_i S_j^2)\\
\nu_{iij}^p - \omega_j^2 \mu_{iij}^p &=& -\omega_p^2 u_{jii}^p - B_{jii}^p & (R_j S_i^2)\\
\lambda_{ijj}^p - \omega_j^2 \xi_{jij}^p &=& -\omega_p^2 t_{ijj}^p & (R_j^2 S_i)\\
-\omega_j^2 \nu_{jii}^p - \omega_i^2 \nu_{iij}^p &=& -\omega_p^2 r_{iij}^p - A_{iij}^p - A_{jii}^p - h_{iij}^p & (R_i^2 R_j)\\
-\omega_j^2 \nu_{jij}^p - \omega_i^2 \nu_{ijj}^p &=& -\omega_p^2 r_{ijj}^p - A_{jij}^p - A_{ijj}^p - h_{ijj}^p & (R_i R_j^2)\\
\xi_{jii}^p + \xi_{iij}^p &=& -\omega_p^2 s_{iij}^p & (S_i^2 S_j)\\
\xi_{jij}^p + \xi_{ijj}^p &=& -\omega_p^2 s_{ijj}^p & (S_i S_j^2)\\
2\lambda_{iij}^p - 2\omega_i^2 \xi_{jii}^p - \omega_j^2 \xi_{iij}^p &=& -\omega_p^2 t_{iij}^p & (R_i R_j S_i)\\
2\lambda_{ijj}^p - 2\omega_j^2 \xi_{ijj}^p - \omega_i^2 \xi_{jij}^p &=& -\omega_p^2 t_{jij}^p & (R_i R_j S_j)\\
-2\omega_i^2 \mu_{iij}^p + 2\nu_{jii}^p + \nu_{iij}^p &=& -\omega_p^2 u_{iij}^p - B_{iij}^p & (R_i S_i S_j)\\
-2\omega_j^2 \mu_{ijj}^p + \nu_{jij}^p + 2\nu_{ijj}^p &=& -\omega_p^2 u_{jij}^p - B_{jij}^p & (R_j S_i S_j)
\end{array}$$

$\forall\ i = 1...N-2,\ \forall\ j > i...N-1,\ \forall\ k > j...N:$

$$\begin{array}{rcll}
-\omega_k^2 \nu_{kij}^p - \omega_j^2 \nu_{jik}^p - \omega_i^2 \nu_{ijk}^p &=& -\omega_p^2 r_{ijk}^p - A_{ijk}^p - A_{jik}^p - A_{kij}^p - h_{ijk}^p & (R_i R_j R_k)\\
\xi_{kij}^p + \xi_{jik}^p + \xi_{ijk}^p &=& -\omega_p^2 s_{ijk}^p & (S_i S_j S_k)\\
\lambda_{ijk}^p - \omega_i^2 \xi_{jik}^p - \omega_j^2 \xi_{ijk}^p &=& -\omega_p^2 t_{kij}^p & (R_i R_j S_k)\\
\lambda_{ijk}^p - \omega_i^2 \xi_{kij}^p - \omega_k^2 \xi_{ijk}^p &=& -\omega_p^2 t_{jik}^p & (R_i S_j R_k)\\
\lambda_{ijk}^p - \omega_j^2 \xi_{kij}^p - \omega_k^2 \xi_{jik}^p &=& -\omega_p^2 t_{ijk}^p & (S_i R_j R_k)\\
\nu_{jik}^p + \nu_{ijk}^p - \omega_k^2 \mu_{ijk}^p &=& -\omega_p^2 u_{kij}^p - B_{kij}^p & (S_i S_j R_k)\\
\nu_{kij}^p + \nu_{ijk}^p - \omega_j^2 \mu_{ijk}^p &=& -\omega_p^2 u_{jik}^p - B_{jik}^p & (S_i R_j S_k)\\
\nu_{kij}^p + \nu_{jik}^p - \omega_i^2 \mu_{ijk}^p &=& -\omega_p^2 u_{ijk}^p - B_{ijk}^p & (R_i S_j S_k)
\end{array}$$

D Cubic coefficients

We give here the values of all the non-zero coefficients appearing in equation (20). Solutions are given for all $p = 1...N$.

$\forall\ i = 1...N,\ i \neq p:$

$$
\begin{aligned}
r^p_{iii} &= \tfrac{1}{D^{(1)}_{ip}}\left[(7\omega_i^2-\omega_p^2)(h^p_{iii}+A^p_{iii})+2\omega_i^4B^p_{iii}\right]\\
u^p_{iii} &= \tfrac{1}{D^{(1)}_{ip}}\left[6h^p_{iii}+6A^p_{iii}+(3\omega_i^2-\omega_p^2)B^p_{iii}\right]\\
\mu^p_{iii} &= u^p_{iii}\\
\nu^p_{iii} &= \tfrac{1}{D^{(1)}_{ip}}\left[(9\omega_i^2-3\omega_p^2)(h^p_{iii}+A^p_{iii})+2\omega_p^2\omega_i^2B^p_{iii}\right]
\end{aligned}
$$

where $D^{(1)}_{ip} = (\omega_p^2-\omega_i^2)(\omega_p^2-9\omega_i^2)$.

$\forall\ i = 1...N-1,\ i \neq p,\ \forall j > i...N:$

$$
\begin{aligned}
r^p_{ijj} =& \frac{\omega_i^2(\omega_i^2-2\omega_j^2-2\omega_p^2)+(\omega_p^2-4\omega_j^2)(\omega_p^2-2\omega_j^2)}{(\omega_i^2-\omega_p^2)D^{(1)}_{ijp}}\left[A^p_{ijj}+A^p_{jij}+h^p_{ijj}\right]\\
&-\frac{2\omega_j^4(\omega_p^2-4\omega_j^2+3\omega_i^2)}{(\omega_i^2-\omega_p^2)D^{(1)}_{ijp}}B^p_{ijj}+\frac{2\omega_i^2\omega_j^2}{D^{(1)}_{ijp}}B^p_{jij}\\
u^p_{ijj} =& \frac{\omega_i^2(\omega_i^2-2\omega_j^2-2\omega_p^2)+(\omega_p^2-4\omega_j^2)(\omega_p^2-2\omega_j^2)}{(\omega_i^2-\omega_p^2)D^{(1)}_{ijp}}B^p_{ijj}\\
&+\frac{8\omega_j^2-6\omega_i^2-2\omega_p^2}{(\omega_i^2-\omega_p^2)D^{(1)}_{ijp}}[A^p_{ijj}+A^p_{jij}+h_{ijj}]-\frac{2\omega_i^2}{D^{(1)}_{ijp}}B^p_{jij}\\
u^p_{jij} =& \frac{1}{D^{(1)}_{ijp}}\left[4\,(A^p_{jij}+\ A^p_{ijj}+\ h^p_{ijj})-4\,{w_j}^2B^p_{ijj}+(4\,{w_j}^2-{w_p}^2+{w_i}^2)B^p_{jij}\right]\\
\nu^p_{ijj} =& \frac{6\omega_i^2\omega_j^2+2\omega_j^2\omega_p^2+2\omega_p^2\omega_i^2-8\omega_j^4-\omega_p^4-\omega_i^4}{(\omega_p^2-\omega_i^2)D^{(1)}_{ijp}}\left[A^p_{ijj}+A^p_{jij}+h^p_{ijj}\right]\\
&+\frac{2\omega_j^4(3\omega_p^2+\omega_i^2-4\omega_j^2)}{(\omega_p^2-\omega_i^2)D^{(1)}_{ijp}}B^p_{ijj}-\frac{\omega_j^2(-\omega_p^2+4\omega_j^2-\omega_i^2)}{D^{(1)}_{ijp}}B^p_{jij}
\end{aligned}
$$

$$
\begin{aligned}
\nu^p_{jij} =& \frac{8\omega_j^2-2\omega_i^2-2\omega_p^2}{D^{(1)}_{ijp}}\left[A^p_{ijj}+A^p_{jij}+h^p_{ijj}\right]\\
&+\frac{2\omega_j^2\omega_p^2-8\omega_j^4+2\omega_j^2\omega_i^2}{D^{(1)}_{ijp}}B^p_{ijj}+\frac{\omega_p^2\omega_i^2+4\omega_j^2\omega_i^2-\omega_i^4}{D^{(1)}_{ijp}}B^p_{jij}\\
\mu^p_{ijj} =& \frac{6\omega_i^2\omega_j^2+2\omega_j^2\omega_p^2+2\omega_p^2\omega_i^2-8\omega_j^4-\omega_p^4-\omega_i^4}{(\omega_p^2-\omega_i^2)D^{(1)}_{ijp}}B^p_{ijj}\\
&+\frac{6\omega_p^2+2\omega_i^2-8\omega_j^2}{(\omega_p^2-\omega_i^2)D^{(1)}_{ijp}}\left[A^p_{ijj}+A^p_{jij}+h^p_{ijj}\right]+\frac{4\omega_j^2-\omega_i^2-\omega_p^2}{D^{(1)}_{ijp}}B^p_{jij}
\end{aligned}
$$

where $D^{(1)}_{ijp} = (\omega_p+\omega_i-2\omega_j)(\omega_p+\omega_i+2\omega_j)(-\omega_p+\omega_i+2\omega_j)(-\omega_p+\omega_i-2\omega_j)$.

$\forall\ i = 1 ... N-1,\ \forall j > i ... N,\ j \neq p\ :$

$$
\begin{aligned}
r^p_{iij} =& \frac{2\omega_i^2(4\omega_i^2 - 3\omega_p^2 - \omega_j^2) + (\omega_p - \omega_j)^2(\omega_p + \omega_j)^2}{(\omega_p^2 - \omega_j^2)D^{(2)}_{ijp}} \left[-A^p_{iij} - A^p_{jii} - h^p_{iij}\right] \\
& - \frac{2\omega_i^4(4\omega_i^2 - \omega_p^2 - 3\omega_j^2)}{(\omega_p^2 - \omega_j^2)D^{(2)}_{ijp}} B^p_{jii} - \frac{2\omega_i^2\omega_j^2}{D^{(2)}_{ijp}} B^p_{iij} \\
u^p_{jii} =& - \frac{2\omega_i^2(4\omega_i^2 - 3\omega_p^2 - \omega_j^2) + (\omega_p - \omega_j)^2(\omega_p + \omega_j)^2}{(\omega_p^2 - \omega_j^2)D^{(2)}_{ijp}} B^p_{jii} \\
& + \frac{8\omega_i^2 - 2\omega_p^2 - 6\omega_j^2}{(\omega_p^2 - \omega_j^2)D^{(2)}_{ijp}} \left[-A^p_{jii} - A^p_{iij} - h^p_{iij}\right] - \frac{2\omega_j^2}{D^{(2)}_{ijp}} B^p_{iij} \\
u^p_{iij} =& \frac{1}{D^{(2)}_{ijp}} \left[4(A^p_{jii} + A^p_{iij} + h^p_{iij}) - 4\omega_i^2 B^p_{jii} - (\omega_p^2 - \omega_j^2 - 4\omega_i^2) B^p_{iij}\right] \\
\nu^p_{jii} =& \frac{8\omega_i^4 + \omega_j^4 + \omega_p^4 - 2\omega_p^2\omega_i^2 - 6\omega_i^2\omega_j^2 - 2\omega_j^2\omega_p^2}{(\omega_j^2 - \omega_p^2)D^{(2)}_{ijp}} \left[A^p_{jii} + A^p_{iij} + h^p_{iij}\right] \\
& + \frac{8\omega_i^6 - 2\omega_j^2\omega_i^4 - 6\omega_p^2\omega_i^4}{(\omega_j^2 - \omega_p^2)D^{(2)}_{ijp}} B^p_{jii} + \frac{\omega_i^2(\omega_j^2 - 4\omega_i^2 + \omega_p^2)}{D^{(2)}_{ijp}} B^p_{iij} \\
\nu^p_{iij} =& \frac{8\omega_i^2 - 2\omega_p^2 - 2\omega_j^2}{D^{(2)}_{ijp}} \left[A^p_{jii} + A^p_{iij} + h^p_{iij}\right] \\
& + \frac{2\omega_i^2\omega_j^2 - 8\omega_i^4 + 2\omega_i^2\omega_p^2}{D^{(2)}_{ijp}} B^p_{jii} + \frac{-\omega_j^4 + 4\omega_i^2\omega_j^2 + \omega_j^2\omega_p^2}{D^{(2)}_{ijp}} B^p_{iij} \\
\mu^p_{iij} =& \frac{8\omega_i^2 - 2\omega_j^2 - 6\omega_p^2}{(\omega_j^2 - \omega_p^2)D^{(2)}_{ijp}} \left[A^p_{iij} + A^p_{jii} + h^p_{iij}\right] + \frac{4\omega_i^2 - \omega_j^2 - \omega_p^2}{D^{(2)}_{ijp}} B^p_{iij} \\
& - \frac{6\omega_i^2\omega_j^2 + 2\omega_j^2\omega_p^2 + 2\omega_i^2\omega_p^2 - 8\omega_i^4 - \omega_p^4 - \omega_j^4}{(\omega_j^2 - \omega_p^2)D^{(2)}_{ijp}} B^p_{jii}
\end{aligned}
$$

where $D^{(2)}_{ijp} = (\omega_p + 2\omega_i - \omega_j)(\omega_p + 2\omega_i + \omega_j)(-\omega_p + 2\omega_i + \omega_j)(-\omega_p + 2\omega_i - \omega_j)$.

$\forall\ i = 1...N-2,\ \forall j > i...N-1,\ \forall k > j...N:$

$$
\begin{aligned}
u^p_{ijk} \;=\; \frac{1}{D^{(3)}_{ijkp}}[\;\; (\;\; & 2\omega_j^4 - 6\omega_i^4 - 4\omega_k^2\omega_p^2 + 4\omega_i^2\omega_p^2 + 4\omega_i^2\omega_k^2 + 2\omega_k^4 \\
& +4\omega_i^2\omega_j^2 + 2\omega_p^4 - 4\omega_k^2\omega_j^2 - 4\omega_j^2\omega_p^2)(h^p_{ijk} + A^p_{ijk} + A^p_{kij} + A^p_{jik}) \\
+(\;\; & -\omega_k^2\omega_j^4 - \omega_i^2\omega_k^4 - 2\omega_j^2\omega_i^2\omega_p^2 + 3\omega_j^2\omega_p^4 - \omega_k^4\omega_j^2 - 2\omega_i^2\omega_k^2\omega_p^2 \\
& +3\omega_i^2\omega_p^4 - \omega_j^4\omega_i^2 - \omega_i^4\omega_k^2 + 3\omega_k^2\omega_p^4 + \omega_i^6 - \omega_p^6 - 3\omega_k^4\omega_p^2 + \omega_k^6 - 3\omega_j^4\omega_p^2 \\
& +10\omega_i^2\omega_k^2\omega_j^2 + \omega_j^6 - 2\omega_k^2\omega_j^2\omega_p^2 - 3\omega_i^4\omega_p^2 - \omega_j^2\omega_i^4)B^p_{ijk} \\
+(\;\; & -2\omega_j^4\omega_i^2 + 4\omega_i^4\omega_p^2 + 4\omega_j^2\omega_i^4 + 6\omega_i^2\omega_k^4 - 4\omega_i^4\omega_k^2 \\
& +4\omega_j^2\omega_i^2\omega_p^2 - 2\omega_i^2\omega_p^4 - 2\omega_i^6 - 4\omega_i^2\omega_k^2\omega_p^2 - 4\omega_i^2\omega_k^2\omega_j^2)B^p_{jik} \\
+(\;\; & -2\omega_i^2\omega_p^4 + 6\omega_j^4\omega_i^2 + 4\omega_i^2\omega_k^2\omega_p^2 - 4\omega_i^2\omega_k^2\omega_j^2 - 4\omega_j^2\omega_i^2\omega_p^2 \\
& -4\omega_j^2\omega_i^4 - 2\omega_i^2\omega_k^4 + 4\omega_i^4\omega_p^2 - 2\omega_i^6 + 4\omega_i^4\omega_k^2)B^p_{kij}] \\
u^p_{jik} \;=\; \frac{1}{D^{(3)}_{ijkp}}[\;\; (\;\; & -2\omega_p^4 + 6\omega_j^4 - 2\omega_i^4 - 4\omega_k^2\omega_j^2 + 4\omega_k^2\omega_p^2 \\
& -2\omega_k^4 + 4\omega_i^2\omega_k^2 - 4\omega_i^2\omega_j^2 + 4\omega_i^2\omega_p^2 - 4\omega_j^2\omega_p^2)(h^p_{ijk} + A^p_{ijk} + A^p_{kij} + A^p_{jik}) \\
+(\;\; & 2\omega_j^6 + 4\omega_k^2\omega_j^2\omega_p^2 + 4\omega_i^2\omega_k^2\omega_j^2 + 2\omega_j^2\omega_i^4 + 4\omega_k^2\omega_j^4 \\
& -4\omega_j^2\omega_i^2\omega_p^2 + 2\omega_j^2\omega_p^4 - 4\omega_j^4\omega_i^2 - 4\omega_j^4\omega_p^2 - 6\omega_k^4\omega_j^2)B^p_{ijk} \\
+(\;\; & 3\omega_j^4\omega_p^2 + \omega_k^2\omega_j^4 - 3\omega_k^2\omega_p^4 + \omega_i^2\omega_k^4 + \omega_k^4\omega_j^2 + 3\omega_i^4\omega_p^2 \\
& +2\omega_i^2\omega_k^2\omega_p^2 + \omega_j^4\omega_i^2 + \omega_i^4\omega_k^2 - \omega_i^6 + 3\omega_k^4\omega_p^2 - \omega_j^6 - \omega_k^6 + 2\omega_j^2\omega_i^2\omega_p^2 + \omega_p^6 \\
& +2\omega_k^2\omega_j^2\omega_p^2 - 3\omega_j^2\omega_p^4 - 10\omega_i^2\omega_k^2\omega_j^2 - 3\omega_i^2\omega_p^4 + \omega_j^2\omega_i^4)B^p_{jik} \\
+(\;\; & 4\omega_i^2\omega_k^2\omega_j^2 + 4\omega_j^4\omega_i^2 + 2\omega_j^2\omega_p^4 - 6\omega_j^2\omega_i^4 + 2\omega_k^4\omega_j^2 - 4\omega_j^4\omega_p^2 \\
& +2\omega_j^6 - 4\omega_k^2\omega_j^4 - 4\omega_k^2\omega_j^2\omega_p^2 + 4\omega_j^2\omega_i^2\omega_p^2)B^p_{kij}]
\end{aligned}
$$

$$
\begin{aligned}
u^p_{kij} \;=\; \frac{1}{D^{(3)}_{ijkp}}[\;\; (\;\; & 2\omega_p^4 + 2\omega_j^4 + 4\omega_i^2\omega_k^2 - 6\omega_k^4 + 4\omega_k^2\omega_p^2 - 4\omega_i^2\omega_p^2 \\
& +4\omega_k^2\omega_j^2 - 4\omega_i^2\omega_j^2 - 4\omega_j^2\omega_p^2 + 2\omega_i^4)(h^p_{ijk} + A^p_{ijk} + A^p_{kij} + A^p_{jik}) \\
+(\;\; & -4\omega_i^2\omega_k^2\omega_j^2 - 2\omega_k^6 - 2\omega_i^4\omega_k^2 - 4\omega_k^2\omega_j^2\omega_p^2 + 4\omega_i^2\omega_k^2\omega_p^2 \\
& +6\omega_k^2\omega_j^4 - 2\omega_k^2\omega_p^4 + 4\omega_i^2\omega_k^4 - 4\omega_k^4\omega_j^2 + 4\omega_k^4\omega_p^2)B^p_{ijk} \\
+(\;\; & -2\omega_k^6 - 4\omega_i^2\omega_k^4 + 4\omega_k^2\omega_j^2\omega_p^2 - 4\omega_i^2\omega_k^2\omega_p^2 - 2\omega_k^2\omega_p^4 + 6\omega_i^4\omega_k^2 \\
& -4\omega_i^2\omega_k^2\omega_j^2 + 4\omega_k^4\omega_j^2 - 2\omega_k^2\omega_j^4 + 4\omega_k^4\omega_p^2)B^p_{jik} \\
+(\;\; & -\omega_k^2\omega_j^4 - \omega_i^2\omega_k^4 - 2\omega_j^2\omega_i^2\omega_p^2 + 3\omega_j^2\omega_p^4 - \omega_k^4\omega_j^2 - 2\omega_i^2\omega_k^2\omega_p^2 \\
& +3\omega_i^2\omega_p^4 - \omega_j^4\omega_i^2 - \omega_i^4\omega_k^2 + 3\omega_k^2\omega_p^4 + \omega_i^6 - \omega_p^6 - 3\omega_k^4\omega_p^2 + \omega_k^6 \\
& -3\omega_j^4\omega_p^2 + 10\omega_i^2\omega_k^2\omega_j^2 + \omega_j^6 - 2\omega_k^2\omega_j^2\omega_p^2 - 3\omega_i^4\omega_p^2 - \omega_j^2\omega_i^4)B^p_{kij}] \\
r^p_{ijk} \;=\; \frac{1}{D^{(3)}_{ijkp}}[\;\; (\;\; & -\omega_k^2\omega_j^4 - \omega_i^2\omega_k^4 - 2\omega_j^2\omega_i^2\omega_p^2 + 3\omega_j^2\omega_p^4 - \omega_k^4\omega_j^2 \\
& -2\omega_i^2\omega_k^2\omega_p^2 + 3\omega_i^2\omega_p^4 - \omega_j^4\omega_i^2 - \omega_i^4\omega_k^2 + 3\omega_k^2\omega_p^4 + \omega_i^6 - \omega_p^6 - 3\omega_k^4\omega_p^2 + \omega_k^6 \\
& -3\omega_j^4\omega_p^2 + 10\omega_i^2\omega_k^2\omega_j^2 + \omega_j^6 - 2\omega_k^2\omega_j^2\omega_p^2 - 3\omega_i^4\omega_p^2 - \omega_j^2\omega_i^4)(h^p_{ijk} + A^p_{ijk} + A^p_{kij} + A^p_{jik}) \\
+(\;\; & 4\omega_i^2\omega_k^2\omega_j^4 - 4\omega_k^2\omega_j^4\omega_p^2 + 4\omega_i^2\omega_k^4\omega_j^2 - 4\omega_j^2\omega_k^4\omega_p^2 - 6\omega_i^4\omega_k^2\omega_j^2 + 2\omega_k^2\omega_j^2\omega_p^4 \\
& +4\omega_i^2\omega_k^2\omega_j^2\omega_p^2 - 4\omega_j^4\omega_k^4 + 2\omega_k^2\omega_j^6 + 2\omega_j^2\omega_k^6)B^p_{ijk} \\
+(\;\; & 4\omega_i^2\omega_k^4\omega_j^2 + 4\omega_i^2\omega_k^2\omega_j^2\omega_p^2 + 2\omega_i^2\omega_k^2\omega_p^4 + 2\omega_i^2\omega_k^6 + 2\omega_i^6\omega_k^2 - 4\omega_i^4\omega_k^4 \\
& -6\omega_i^2\omega_k^2\omega_j^4 - 4\omega_i^2\omega_k^4\omega_p^2 - 4\omega_i^4\omega_k^2\omega_p^2 + 4\omega_i^4\omega_k^2\omega_j^2)B^p_{jik} \\
+(\;\; & -4\omega_i^4\omega_j^4 + 2\omega_i^2\omega_j^2\omega_p^4 - 6\omega_i^2\omega_k^4\omega_j^2 + 2\omega_i^2\omega_j^6 + 2\omega_i^6\omega_j^2 - 4\omega_i^2\omega_j^4\omega_p^2 \\
& +4\omega_i^2\omega_k^2\omega_j^2\omega_p^2 + 4\omega_i^2\omega_k^2\omega_j^4 - 4\omega_i^4\omega_j^2\omega_p^2 + 4\omega_i^4\omega_k^2\omega_j^2)B^p_{kij}]
\end{aligned}
$$

$$
\begin{aligned}
\mu^p_{ijk} \;=\; \frac{1}{D^{(3)}_{ijkp}}\Big[\;(\;& -4\omega_i^2\omega_j^2 + 4\omega_i^2\omega_p^2 + 12\omega_i^2\omega_k^2 - 4\omega_k^2\omega_j^2 + 4\omega_k^2\omega_p^2 + 2\omega_p^4 \\
& +10\omega_j^4 - 6\omega_i^4 - 6\omega_k^4 - 12\omega_j^2\omega_p^2)(h^p_{ijk} + A^p_{ijk} + A^p_{kij} + A^p_{jik}) \\
+(\;& 10\omega_i^2\omega_k^2\omega_j^2 - 2\omega_k^2\omega_j^2\omega_p^2 + 2\omega_i^2\omega_k^2\omega_p^2 - 6\omega_j^2\omega_i^2\omega_p^2 + 9\omega_k^2\omega_j^4 \\
& +3\omega_j^6 - 11\omega_k^4\omega_j^2 + \omega_k^2\omega_p^4 + 3\omega_i^2\omega_k^4 + \omega_k^4\omega_p^2 - \omega_k^6 - 3\omega_i^4\omega_p^2 \\
& +3\omega_i^2\omega_p^4 + \omega_i^6 - \omega_p^6 + 5\omega_j^2\omega_p^4 - 7\omega_j^4\omega_p^2 - 5\omega_j^4\omega_i^2 + \omega_j^2\omega_i^4 - 3\omega_i^4\omega_k^2)B^p_{ijk} \\
+(\;& -18\omega_i^2\omega_k^2\omega_j^2 - 3\omega_j^2\omega_p^4 + 5\omega_k^4\omega_j^2 + 5\omega_j^2\omega_i^4 - \omega_k^2\omega_j^4 - \omega_j^4\omega_i^2 + 7\omega_i^4\omega_p^2 \\
& +3\omega_i^2\omega_k^4 + \omega_p^6 - \omega_j^6 - 3\omega_k^6 - 3\omega_i^6 + 3\omega_j^4\omega_p^2 + 7\omega_k^4\omega_p^2 + 3\omega_i^4\omega_k^2 + 6\omega_k^2\omega_j^2\omega_p^2 \\
& +6\omega_j^2\omega_i^2\omega_p^2 - 5\omega_k^2\omega_p^4 - 6\omega_i^2\omega_k^2\omega_p^2 - 5\omega_i^2\omega_p^4)B^p_{jik} \\
+(\;& \omega_k^4\omega_j^2 + \omega_k^6 - 2\omega_j^2\omega_i^2\omega_p^2 + 10\omega_i^2\omega_k^2\omega_j^2 + 3\omega_k^2\omega_p^4 - 6\omega_k^2\omega_j^2\omega_p^2 - 3\omega_i^2\omega_k^4 \\
& +2\omega_i^2\omega_k^2\omega_p^2 + 5\omega_j^2\omega_p^4 + 3\omega_i^4\omega_k^2 - 11\omega_j^2\omega_i^4 - 3\omega_k^4\omega_p^2 + \omega_i^4\omega_p^2 + 3\omega_j^6 \\
& +\omega_i^2\omega_p^4 - \omega_i^6 - \omega_p^6 + 9\omega_j^4\omega_i^2 - 7\omega_j^4\omega_p^2 - 5\omega_k^2\omega_j^4)B^p_{kij}\Big] \\
\nu^p_{kij} \;=\; \frac{1}{D^{(3)}_{ijkp}}\Big[\;(\;& -2\omega_j^2\omega_i^2\omega_p^2 + \omega_j^4\omega_p^2 - \omega_j^6 + 3\omega_k^2\omega_j^4 + \omega_k^6 - 7\omega_i^4\omega_p^2 - 11\omega_j^4\omega_i^2 + 2\omega_k^2\omega_j^2\omega_p^2 \\
& +9\omega_j^2\omega_i^4 + 3\omega_i^6 + 5\omega_i^2\omega_p^4 + \omega_j^2\omega_p^4 - 5\omega_i^4\omega_k^2 + 10\omega_i^2\omega_k^2\omega_j^2 - 6\omega_i^2\omega_k^2\omega_p^2 + 3\omega_k^2\omega_p^4 \\
& -3\omega_k^4\omega_p^2 - \omega_p^6 + \omega_i^2\omega_k^4 - 3\omega_k^4\omega_j^2)(h^p_{ijk} + A^p_{ijk} + A^p_{kij} + A^p_{jik}) \\
+(\;& 3\omega_i^4\omega_p^4 + \omega_i^8 - 6\omega_i^4\omega_k^2\omega_p^2 - 5\omega_i^4\omega_k^4 + 6\omega_i^2\omega_k^2\omega_j^2\omega_p^2 + 5\omega_i^2\omega_j^2\omega_p^4 - 7\omega_i^2\omega_j^4\omega_p^2 \\
& +18\omega_i^4\omega_k^2\omega_j^2 + 3\omega_i^2\omega_k^6 - 3\omega_i^2\omega_k^2\omega_j^4 - 7\omega_i^2\omega_k^4\omega_p^2 - 6\omega_i^4\omega_j^2\omega_p^2 + \omega_i^6\omega_k^2 - 3\omega_i^2\omega_k^4\omega_j^2 \\
& +5\omega_i^2\omega_k^2\omega_p^4 + \omega_i^6\omega_j^2 - 5\omega_i^4\omega_j^4 - 3\omega_i^6\omega_p^2 - \omega_i^2\omega_p^6 + 3\omega_i^2\omega_j^6)B^p_{jik} \\
+(\;& 3\omega_k^2\omega_j^6 + 6\omega_i^2\omega_j^4\omega_p^2 + 5\omega_i^4\omega_j^4 + 7\omega_i^4\omega_j^2\omega_p^2 - 9\omega_i^4\omega_k^2\omega_j^2 - \omega_i^2\omega_j^6 + \omega_p^6\omega_j^2 - \omega_j^2\omega_k^4\omega_p^2 \\
& -3\omega_p^4\omega_j^4 + 3\omega_j^6\omega_p^2 + 11\omega_i^2\omega_k^4\omega_j^2 - 3\omega_i^6\omega_j^2 - 5\omega_i^2\omega_j^2\omega_p^4 - 2\omega_k^2\omega_j^4\omega_p^2 - \omega_k^2\omega_j^2\omega_p^4 \\
& +2\omega_i^2\omega_k^2\omega_j^2\omega_p^2 + \omega_j^2\omega_k^6 - 3\omega_j^4\omega_k^4 - 10\omega_i^2\omega_k^2\omega_j^4 - \omega_j^8)B^p_{ijk} \\
+(\;& -4\omega_i^4\omega_k^2\omega_j^2 - 6\omega_i^2\omega_k^4\omega_j^2 - 4\omega_i^4\omega_j^4 - 12\omega_i^4\omega_j^2\omega_p^2 + 2\omega_i^2\omega_j^2\omega_p^4 + 4\omega_i^2\omega_j^4\omega_p^2 \\
& +12\omega_i^2\omega_k^2\omega_j^4 + 10\omega_i^6\omega_j^2 - 6\omega_i^2\omega_j^6 + 4\omega_i^2\omega_k^2\omega_j^2\omega_p^2)B^p_{kij}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\nu^p_{jik} \;=\; \frac{1}{D^{(3)}_{ijkp}}\Big[\;(\;& \omega_i^2\omega_k^4 + \omega_i^4\omega_p^2 + 3\omega_j^2\omega_i^4 + \omega_k^4\omega_p^2 - \omega_p^6 - 3\omega_j^4\omega_p^2 + 3\omega_j^2\omega_p^4 + 2\omega_j^2\omega_i^2\omega_p^2 + \omega_j^6 \\
& +3\omega_k^4\omega_j^2 + \omega_i^2\omega_p^4 + 2\omega_k^2\omega_j^2\omega_p^2 - \omega_k^6 + 2\omega_i^2\omega_k^2\omega_j^2 - 10\omega_i^2\omega_k^2\omega_p^2 - 3\omega_j^4\omega_i^2 \\
& +\omega_i^4\omega_k^2 - 3\omega_k^2\omega_j^4 - \omega_i^6 + \omega_k^2\omega_p^4)(h^p_{ijk} + A^p_{ijk} + A^p_{kij} + A^p_{jik}) \\
+(\;& -3\omega_i^4\omega_k^4 - \omega_i^2\omega_k^2\omega_p^4 - 2\omega_i^2\omega_k^4\omega_j^2 - 2\omega_i^2\omega_k^4\omega_p^2 - \omega_i^4\omega_k^2\omega_p^2 + \omega_i^6\omega_k^2 \\
& +\omega_k^2\omega_j^6 - 2\omega_j^2\omega_k^4\omega_p^2 - \omega_i^4\omega_k^2\omega_j^2 + 3\omega_i^2\omega_k^6 - \omega_k^2\omega_j^2\omega_p^4 - 3\omega_p^4\omega_k^4 + 3\omega_k^6\omega_p^2 - \omega_k^2\omega_j^4\omega_p^2 \\
& -\omega_k^8 + 10\omega_i^2\omega_k^2\omega_j^2\omega_p^2 + 3\omega_j^2\omega_k^6 - 3\omega_j^4\omega_k^4 - \omega_i^2\omega_k^2\omega_j^4 + \omega_k^2\omega_p^6)B^p_{ijk} \\
+(\;& -4\omega_i^2\omega_k^2\omega_j^2\omega_p^2 + 4\omega_i^2\omega_k^4\omega_j^2 + 4\omega_i^4\omega_k^2\omega_j^2 - 2\omega_i^2\omega_k^6 + 6\omega_i^2\omega_k^2\omega_p^4 - 4\omega_i^4\omega_k^2\omega_p^2 \\
& -2\omega_i^6\omega_k^2 + 4\omega_i^4\omega_k^4 - 2\omega_i^2\omega_k^2\omega_j^4 - 4\omega_i^2\omega_k^4\omega_p^2)B^p_{jik} \\
+(\;& -2\omega_i^4\omega_j^2\omega_p^2 + 3\omega_i^6\omega_k^2 - 3\omega_i^4\omega_k^4 + 3\omega_i^6\omega_p^2 - 3\omega_i^4\omega_p^4 - \omega_i^2\omega_j^2\omega_p^4 - 2\omega_i^4\omega_k^2\omega_p^2 \\
& -\omega_i^8 - \omega_i^2\omega_k^4\omega_p^2 + \omega_i^2\omega_j^6 - 2\omega_i^4\omega_k^2\omega_j^2 - \omega_i^2\omega_k^2\omega_p^4 + 3\omega_i^6\omega_j^2 + 10\omega_i^2\omega_k^2\omega_j^2\omega_p^2 \\
& +\omega_i^2\omega_k^6 + \omega_i^2\omega_p^6 - \omega_i^2\omega_k^2\omega_j^4 - 3\omega_i^4\omega_j^4 - \omega_i^2\omega_j^4\omega_p^2 - \omega_i^2\omega_k^4\omega_j^2)B^p_{kij}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\nu^p_{ijk} \;=\; \tfrac{1}{D^{(3)}_{ijkp}}[\;\; (\quad & -11\omega_k^2\omega_j^4 - 3\omega_i^4\omega_p^2 + 9\omega_k^4\omega_j^2 - \omega_p^6 + \omega_j^4\omega_p^2 - 7\omega_k^4\omega_p^2 - \omega_j^6 + \omega_i^4\omega_k^2 \\
& +2\omega_j^2\omega_i^2\omega_p^2 - 3\omega_j^2\omega_i^4 + 3\omega_k^6 - 6\omega_i^2\omega_k^2\omega_p^2 + \omega_j^2\omega_p^4 + 3\omega_i^2\omega_p^4 + 5\omega_k^2\omega_p^4 - 5\omega_i^2\omega_k^4 \\
& +\omega_i^6 + 3\omega_j^4\omega_i^2 - 2\omega_k^2\omega_j^2\omega_p^2 + 10\omega_i^2\omega_k^2\omega_j^2)(h^p_{ijk} + A^p_{ijk} + A^p_{kij} + A^p_{jik}) \\
+(\quad & 12\omega_i^2\omega_k^2\omega_j^4 - 6\omega_k^2\omega_j^6 - 4\omega_i^2\omega_k^4\omega_j^2 + 4\omega_i^2\omega_k^2\omega_j^2\omega_p^2 + 4\omega_k^2\omega_j^4\omega_p^2 \\
& +2\omega_k^2\omega_j^2\omega_p^4 - 4\omega_j^4\omega_k^4 - 12\omega_j^2\omega_k^4\omega_p^2 - 6\omega_i^4\omega_k^2\omega_j^2 + 10\omega_j^2\omega_k^6)B^p_{ijk} \\
+(\quad & 3\omega_k^2\omega_j^6 + 6\omega_i^2\omega_k^2\omega_j^2\omega_p^2 + \omega_j^2\omega_k^6 - 3\omega_i^2\omega_k^2\omega_j^4 - \omega_k^2\omega_p^6 - 3\omega_i^4\omega_k^2\omega_j^2 \\
& +18\omega_i^2\omega_k^4\omega_j^2 - 6\omega_i^2\omega_k^4\omega_p^2 - 5\omega_j^4\omega_k^4 + 3\omega_i^6\omega_k^2 + \omega_i^2\omega_k^6 + 5\omega_i^2\omega_k^2\omega_p^4 - 6\omega_j^2\omega_k^4\omega_p^2 \\
& +\omega_k^8 + 5\omega_k^2\omega_j^2\omega_p^4 - 3\omega_k^6\omega_p^2 + 3\omega_p^4\omega_k^4 - 7\omega_i^4\omega_k^2\omega_p^2 - 5\omega_i^4\omega_k^4 - 7\omega_k^2\omega_j^4\omega_p^2)B^p_{jik} \\
+(\quad & -\omega_i^4\omega_j^2\omega_p^2 - \omega_j^8 - 5\omega_k^2\omega_j^2\omega_p^4 + 7\omega_j^2\omega_k^4\omega_p^2 + 5\omega_j^4\omega_k^4 + 3\omega_j^6\omega_p^2 - \omega_k^2\omega_j^6 \\
& -3\omega_p^4\omega_j^4 + \omega_i^6\omega_j^2 + 2\omega_i^2\omega_k^2\omega_j^2\omega_p^2 + 11\omega_i^4\omega_k^2\omega_j^2 + \omega_p^6\omega_j^2 - 10\omega_i^2\omega_k^2\omega_j^4 - 3\omega_j^2\omega_k^6 \\
& -3\omega_i^4\omega_j^4 - 9\omega_i^2\omega_k^4\omega_j^2 + 3\omega_i^2\omega_j^6 + 6\omega_k^2\omega_j^4\omega_p^2 - \omega_i^2\omega_j^2\omega_p^4 - 2\omega_i^2\omega_j^4\omega_p^2)B^p_{kij}]
\end{aligned}
$$

where $D^{(3)}_{ijkp} = (\omega_k + \omega_i - \omega_p - \omega_j)(\omega_k + \omega_i - \omega_p + \omega_j)(-\omega_k + \omega_i + \omega_p + \omega_j)(-\omega_k + \omega_i + \omega_p - \omega_j)(\omega_k + \omega_i + \omega_p - \omega_j)(\omega_k + \omega_i + \omega_p + \omega_j)(-\omega_k + \omega_i - \omega_p + \omega_j)(-\omega_k + \omega_i - \omega_p - \omega_j)$.