On the Hybrid Cramér-Rao Bound and Its Application to Dynamical Phase Estimation

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Abstract

This letter deals with the Cramér-Rao bound for the estimation of a hybrid vector with both random and deterministic parameters. We point out the specificity of the case when the deterministic and the random vectors of parameters are statistically dependent. The relevance of this expression is illustrated by studying a practical phase estimation problem in a non data-aided communication context.

I. INTRODUCTION

A natural problematic when designing an estimator is the evaluation of its performance. Lower bounds on the Mean Square Error (MSE) mainly answer this question and the well known Cramér-Rao Bound (CRB) is widely used by the signal processing community. Depending on assumptions on the parameters, the CRB has different expressions. When the vector of parameters is assumed to be deterministic, we obtain the standard CRB [1] and when the vector of parameters is assumed to be random with an *a priori* probability density function (pdf), we obtain the so-called Bayesian CRB [2].

At the end of the eighties, an extension combining both the standard and the Bayesian CRBs has been proposed [3]. Indeed, in some practical scenarios, it is natural to represent the parameter vector by a deterministic part and by a random part. This bound has thus been called the Hybrid CRB (HCRB) and a nice tutorial can be found in [4]. Until now, results available in the literature essentially focussed on the case where the deterministic part and the random part of the parameter vector are assumed to be statistically independent (see, *e.g.*, Eqn. (5) in [3], Eqn. (13)

in [4] and Eqn. (13) in [5]). To the best of our knowledge, a closed-form expression of the HCRB with a statistical dependence between the deterministic and the random parameters has never been reported in the literature. The goal of this paper is then twofold. First, in Section II, we remind the structure of the HCRB and we point out the specificity of the case when the deterministic part and the random part of the parameter vector are statistically dependent. Second, in Section III, motivated by this analysis we give a closed-form expression of the proposed bound in the practical case of a dynamical phase subject to a linear drift in a non data-aided communication context.

II. THE HYBRID CRAMÉR-RAO BOUND

A. Background

Let $\boldsymbol{\mu} = \left(\boldsymbol{\mu}_r^T\ \boldsymbol{\mu}_d^T\right)^T \in \mathbb{R}^n$ be the parameter vector that we have to estimate. This vector is split into two sub-vectors $\boldsymbol{\mu}_d$ and $\boldsymbol{\mu}_r$ where $\boldsymbol{\mu}_d$ is assumed to be a $(n-m) \times 1$ deterministic vector and $\boldsymbol{\mu}_r$ is assumed to be a $m \times 1$ random vector with an *a priori* pdf $p\left(\boldsymbol{\mu}_r\right)$. The true value of $\boldsymbol{\mu}_d$ will be denoted $\boldsymbol{\mu}_d^\star$. We consider $\hat{\boldsymbol{\mu}}(\mathbf{y})$ as an estimator of $\boldsymbol{\mu}$ where $\mathbf{y}$ is the observation vector. The HCRB satisfies the following inequality on the MSE

$$\mathbb{E}_{\mathbf{y},\boldsymbol{\mu}_r|\boldsymbol{\mu}_d^\star} \left[\left(\hat{\boldsymbol{\mu}}(\mathbf{y}) - \boldsymbol{\mu}\right)\left(\hat{\boldsymbol{\mu}}(\mathbf{y}) - \boldsymbol{\mu}\right)^T \Big|_{\boldsymbol{\mu}_d^\star} \right] \geq \mathbf{H}^{-1}\left(\boldsymbol{\mu}_d^\star\right), \tag{1}$$

where $\mathbf{H}\left(\boldsymbol{\mu}_d^\star\right) \in \mathbb{R}^{n \times n}$ is the so-called Hybrid Information Matrix (HIM) defined as [3]

$$\mathbf{H}\left(\boldsymbol{\mu}_d^\star\right) = \mathbb{E}_{\mathbf{y},\boldsymbol{\mu}_r|\boldsymbol{\mu}_d^\star} \left[-\Delta_{\boldsymbol{\mu}}^{\boldsymbol{\mu}} \log p(\mathbf{y},\boldsymbol{\mu}_r|\boldsymbol{\mu}_d)\big|_{\boldsymbol{\mu}_d^\star} \right], \tag{2}$$

where $\left[\Delta_{\boldsymbol{\eta}}^{\boldsymbol{\nu}}\right]_{k,l} = \frac{\partial^2}{\partial[\boldsymbol{\eta}]_k \partial[\boldsymbol{\nu}]_l}$.

When the deterministic and the random parts of the parameter vector are assumed to be independent, and after some algebraic manipulations, the HIM can be rewritten as (see [4], Eqn. (18))

$$\mathbf{H}\left(\boldsymbol{\mu}_d^\star\right) = \mathbb{E}_{\boldsymbol{\mu}_r}\left[\mathbf{F}(\boldsymbol{\mu}_d^\star, \boldsymbol{\mu}_r)\right] + \begin{pmatrix} \mathbb{E}_{\boldsymbol{\mu}_r}\left[-\Delta_{\boldsymbol{\mu}_r}^{\boldsymbol{\mu}_r} \log p\left(\boldsymbol{\mu}_r\right)\right] & \mathbf{0}_{m \times (n-m)} \\ \mathbf{0}_{(n-m) \times m} & \mathbf{0}_{(n-m) \times (n-m)} \end{pmatrix}, \tag{3}$$

where

$$\mathbf{F}(\boldsymbol{\mu}_d^\star, \boldsymbol{\mu}_r) = \mathbb{E}_{\mathbf{y}|\boldsymbol{\mu}_d^\star, \boldsymbol{\mu}_r} \left[-\Delta_{\boldsymbol{\mu}}^{\boldsymbol{\mu}} \log p(\mathbf{y}|\boldsymbol{\mu}_d, \boldsymbol{\mu}_r)\big|_{\boldsymbol{\mu}_d^\star} \right]. \tag{4}$$

With this aforementioned structure, it is straightforward to reobtain the standard and the Bayesian CRBs. Indeed, if $\boldsymbol{\mu} = \boldsymbol{\mu}_d$, we have

$$\mathbf{H}^{-1}\left(\boldsymbol{\mu}_d^\star\right) = \left(\mathbb{E}_{\mathbf{y}|\boldsymbol{\mu}_d^\star} \left[-\Delta_{\boldsymbol{\mu}_d}^{\boldsymbol{\mu}_d} \log p(\mathbf{y}|\boldsymbol{\mu}_d)\big|_{\boldsymbol{\mu}_d^\star} \right] \right)^{-1}, \tag{5}$$

which is the standard CRB, and, if $\boldsymbol{\mu} = \boldsymbol{\mu}_r$, we have

$$\mathbf{H}^{-1} = \left(\mathbb{E}_{\mathbf{y},\boldsymbol{\mu}_r} \left[-\Delta_{\boldsymbol{\mu}_r}^{\boldsymbol{\mu}_r} \log p(\mathbf{y}|\boldsymbol{\mu}_r) \right] + \mathbb{E}_{\boldsymbol{\mu}_r} \left[-\Delta_{\boldsymbol{\mu}_r}^{\boldsymbol{\mu}_r} \log p\left(\boldsymbol{\mu}_r\right) \right] \right)^{-1}, \tag{6}$$

which is the Bayesian CRB.

B. Extension when $\boldsymbol{\mu}_r$ and $\boldsymbol{\mu}_d$ are statistically dependent

We now assume a possible statistical dependence between $\boldsymbol{\mu}_r$ and $\boldsymbol{\mu}_d$. In other words, $\boldsymbol{\mu}_r$ is now assumed to be a $m \times 1$ random vector with an *a priori* pdf $p\left(\boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star\right) \neq p\left(\boldsymbol{\mu}_r\right)$.

Based on the HIM definition given by Eqn. (2) and expending the log-likelihood as $\log p\left(\mathbf{y}, \boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star\right) = \log p\left(\mathbf{y} | \boldsymbol{\mu}_d^\star, \boldsymbol{\mu}_r\right) + \log p\left(\boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star\right)$, we obtain the following HIM

$$\mathbf{H}\left(\boldsymbol{\mu}_d^\star\right) = \mathbb{E}_{\boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star}\left[\mathbf{F}(\boldsymbol{\mu}_d^\star, \boldsymbol{\mu}_r)\right] + \mathbb{E}_{\boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star}\left[-\Delta_{\boldsymbol{\mu}}^{\boldsymbol{\mu}} \log p\left(\boldsymbol{\mu}_r | \boldsymbol{\mu}_d\right)\big|_{\boldsymbol{\mu}_d^\star}\right], \tag{7}$$

where $\mathbf{F}(\boldsymbol{\mu}_d^\star, \boldsymbol{\mu}_r)$ is given by Eqn. (4).

In order to explicitly show the modification in comparison with the HIM given by Eqn. (3), $\mathbf{H}\left(\boldsymbol{\mu}_d^\star\right)$ can be rewritten as follows

$$\mathbf{H}\left(\boldsymbol{\mu}_d^\star\right) = \mathbb{E}_{\boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star}\left[\mathbf{F}(\boldsymbol{\mu}_d^\star, \boldsymbol{\mu}_r)\right] + \mathbb{E}_{\boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star}\left[\begin{pmatrix} -\Delta_{\boldsymbol{\mu}_r}^{\boldsymbol{\mu}_r} \log p\left(\boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star\right) & -\Delta_{\boldsymbol{\mu}_d}^{\boldsymbol{\mu}_r} \log p\left(\boldsymbol{\mu}_r | \boldsymbol{\mu}_d\right)\big|_{\boldsymbol{\mu}_d^\star} \\ \left(-\Delta_{\boldsymbol{\mu}_d}^{\boldsymbol{\mu}_r} \log p\left(\boldsymbol{\mu}_r | \boldsymbol{\mu}_d\right)\big|_{\boldsymbol{\mu}_d^\star}\right)^T & -\Delta_{\boldsymbol{\mu}_d}^{\boldsymbol{\mu}_d} \log p\left(\boldsymbol{\mu}_r | \boldsymbol{\mu}_d\right)\big|_{\boldsymbol{\mu}_d^\star} \end{pmatrix}\right]. \tag{8}$$

Obviously, if we assume $p\left(\boldsymbol{\mu}_r | \boldsymbol{\mu}_d\right) = p\left(\boldsymbol{\mu}_r\right)$ in this expression, we straightforwardly reobtain Eqn. (3).

Based on this structure, one now has to prove that there is still an inequality, *i.e.*, a lower bound on the MSE,

$$\mathbb{E}_{\mathbf{y}, \boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star}\left[\left(\hat{\boldsymbol{\mu}}(\mathbf{y}) - \boldsymbol{\mu}\right)\left(\hat{\boldsymbol{\mu}}(\mathbf{y}) - \boldsymbol{\mu}\right)^T\big|_{\boldsymbol{\mu}_d^\star}\right] \geq \mathbf{H}^{-1}\left(\boldsymbol{\mu}_d^\star\right), \tag{9}$$

when $\mathbf{H}\left(\boldsymbol{\mu}_d^\star\right)$ is given by Eqn. (8).

Proof: Following the idea of [4] to prove the inequality (1), one defines a vector $\mathbf{h}$ such that

$$\mathbf{h} = \begin{pmatrix} \nabla_{\boldsymbol{\mu}} \log p\left(\mathbf{y}, \boldsymbol{\mu}_r | \boldsymbol{\mu}_d\right)\big|_{\boldsymbol{\mu}_d^\star} \\ \hat{\boldsymbol{\mu}}(\mathbf{y}) - \boldsymbol{\mu}\big|_{\boldsymbol{\mu}_d^\star} \end{pmatrix}, \tag{10}$$

where $\nabla_{\boldsymbol{\mu}} = \begin{pmatrix} \frac{\partial}{\partial[\boldsymbol{\mu}]_1} & \cdots & \frac{\partial}{\partial[\boldsymbol{\mu}]_n} \end{pmatrix}^T$.

Consequently, the non-negative definite matrix $\mathbf{G}\left(\boldsymbol{\mu}_d^\star\right) = \mathbb{E}_{\mathbf{y}, \boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star}\left[\mathbf{h}\,\mathbf{h}^T\right]$ can be decomposed as the following block matrix

$$\mathbf{G}\left(\boldsymbol{\mu}_d^\star\right) = \begin{pmatrix} \mathbf{H}\left(\boldsymbol{\mu}_d^\star\right) & \mathbf{L}\left(\boldsymbol{\mu}_d^\star\right) \\ \mathbf{L}^T\left(\boldsymbol{\mu}_d^\star\right) & \mathbf{R}\left(\boldsymbol{\mu}_d^\star\right) \end{pmatrix}, \tag{11}$$

where $\mathbf{R}\left(\boldsymbol{\mu}_d^\star\right)$ is the covariance matrix of $\hat{\boldsymbol{\mu}}(\mathbf{y})$, *i.e.*,

$$\mathbf{R}\left(\boldsymbol{\mu}_d^\star\right) = \mathbb{E}_{\mathbf{y}, \boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star}\left[\left(\hat{\boldsymbol{\mu}}(\mathbf{y}) - \boldsymbol{\mu}\right)\left(\hat{\boldsymbol{\mu}}(\mathbf{y}) - \boldsymbol{\mu}\right)^T\big|_{\boldsymbol{\mu}_d^\star}\right], \tag{12}$$

and, where $\mathbf{L}\left(\boldsymbol{\mu}_d^\star\right)$ is given by

$$\mathbf{L}\left(\boldsymbol{\mu}_d^\star\right) = \mathbb{E}_{\mathbf{y}, \boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star}\left[\nabla_{\boldsymbol{\mu}} \log p\left(\mathbf{y}, \boldsymbol{\mu}_r | \boldsymbol{\mu}_d\right)\big|_{\boldsymbol{\mu}_d^\star}\left(\hat{\boldsymbol{\mu}}(\mathbf{y}) - \boldsymbol{\mu}\big|_{\boldsymbol{\mu}_d^\star}\right)^T\right]. \tag{13}$$

Since $\mathbf{G}\left(\boldsymbol{\mu}_d^\star\right) \geq 0$, its Schur complement satisfies

$$\mathbf{R}\left(\boldsymbol{\mu}_d^\star\right) \geq \mathbf{L}^T\left(\boldsymbol{\mu}_d^\star\right) \mathbf{H}^{-1}\left(\boldsymbol{\mu}_d^\star\right) \mathbf{L}\left(\boldsymbol{\mu}_d^\star\right). \tag{14}$$

It is straightforward to show that, for an unbiased estimator w.r.t. the pdf $p\left(\mathbf{y}, \boldsymbol{\mu}_r | \boldsymbol{\mu}_d^\star\right)$, $\mathbf{L}\left(\boldsymbol{\mu}_d^\star\right) = \mathbf{I}_{n \times n}$. ∎

Consequently, the inequality (9) is proved and $\mathbf{H}^{-1}\left(\boldsymbol{\mu}_d^\star\right)$ is a lower bound on the MSE.

III. HCRB FOR A DYNAMICAL PHASE ESTIMATION PROBLEM

In [6], we have proposed a closed-form expression of the Bayesian CRB for the estimation of the phase offset for a BPSK transmission in a non data-aided context. In this section, we extend these previous results by providing a closed-form expression of the HCRB for the estimation of the phase offset and also of the linear drift. In this more realistic scenario, we show that we have to take into account the statistical dependence between the parameters and, consequently, the HCRB given by Eqn. (3) is not adapted to this problem.

A. Observation and state models

We consider a linearly modulated signal, obtained by applying to a square-root Nyquist transmit filter an unknown symbol sequence $\mathbf{a} = (a_1 \cdots a_K)^T$ taken from a unit energy BPSK constellation. The signal is transmitted over an additive white Gaussian noise channel. The output signal is sampled at the symbol rate which yields to the observations

$$y_k = a_k e^{j\theta_k} + n_k \quad \text{with } k = 1 \ldots K, \tag{15}$$

where $\{n_k\}$ is a sequence of i.i.d., circular, zero mean complex Gaussian noise variables with variance σ_n^2. We consider that the system operates in a non Data-Aided synchronization mode, *i.e.*, the transmitted symbols are i.i.d. with $P_r(a_k = \pm 1) = \frac{1}{2}$.

In practice, several sources of distortions affect the phase. An efficient model representing these effects is the so-called Brownian phase with a linear drift widely studied in the literature (see, *e.g.*, [7] [8] [9]). This model takes into account a constant frequency shift between the oscillators of the transmitter and of the receiver, the uncertainities due to clocks, and, the jitters of oscillators. The Brownian phase model with a linear drift is given as follows

$$\theta_k = \theta_{k-1} + \xi + w_k \quad \text{with } k = 2 \ldots K, \tag{16}$$

where, for any index k, $\{\theta_k\}$ is the sequence of phases to be estimated, ξ represents the deterministic unknown linear drift with true value $\xi^\star$, and where $\{w_k\}$ is an i.i.d. sequence of centered Gaussian random variables with known variance σ_w^2.

The parameter vector of interest is then made up of both random and deterministic parameters $\boldsymbol{\mu} = \left(\boldsymbol{\mu}_r^T \ \mu_d \right)^T$ where $\boldsymbol{\mu}_r = \boldsymbol{\theta} = (\theta_1 \cdots \theta_K)^T$ and $\mu_d = \xi$. Moreover, from Eqn. (16), it is clear that $p(\boldsymbol{\theta} | \xi^\star) \neq p(\boldsymbol{\theta})$.

B. Derivation of the HCRB

For notational convenience, we drop the dependence of the different matrices on $\mu_d^\star = \xi^\star$ in the remainder of this paper. From Eqn. (8), the HIM $\mathbf{H}$ can be rewritten into a block matrix $\mathbf{H} = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{h}_{12} \\ \mathbf{h}_{21} & H_{22} \end{pmatrix}$, where,

$$\begin{cases} \mathbf{H}_{11} = \mathbb{E}_{\mathbf{y},\boldsymbol{\theta}|\xi^\star} \left[-\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}} \log p(\mathbf{y}|\boldsymbol{\theta},\xi) \big|_{\xi^\star} \right] + \mathbb{E}_{\boldsymbol{\theta}|\xi^\star} \left[-\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}} \log p(\boldsymbol{\theta}|\xi^\star) \right], \\ \mathbf{h}_{12} = \mathbf{h}_{21}^T = \mathbb{E}_{\mathbf{y},\boldsymbol{\theta}|\xi^\star} \left[-\Delta_{\boldsymbol{\theta}}^{\xi} \log p(\mathbf{y}|\boldsymbol{\theta},\xi) \big|_{\xi^\star} \right] + \mathbb{E}_{\boldsymbol{\theta}|\xi^\star} \left[-\Delta_{\boldsymbol{\theta}}^{\xi} \log p(\boldsymbol{\theta}|\xi) \big|_{\xi^\star} \right], \\ H_{22} = \mathbb{E}_{\mathbf{y},\boldsymbol{\theta}|\xi^\star} \left[-\Delta_{\xi}^{\xi} \log p(\mathbf{y}|\boldsymbol{\theta},\xi) \big|_{\xi^\star} \right] + \mathbb{E}_{\boldsymbol{\theta}|\xi^\star} \left[-\Delta_{\xi}^{\xi} \log p(\boldsymbol{\theta}|\xi) \big|_{\xi^\star} \right]. \end{cases} \tag{17}$$

These blocks only depend on the log-likelihoods $\log p\left(\mathbf{y}|\boldsymbol{\theta},\xi^\star\right)$ and $\log p\left(\boldsymbol{\theta}|\xi^\star\right)$. Let us set $\mathbf{y}=\left(y_1\cdots y_K\right)^T$ and assume that the initial phase θ_1 does not depend on ξ, *i.e.*, $p\left(\theta_1|\xi^\star\right)=p\left(\theta_1\right)$. Using Eqn. (15) and (16), *i.e.*, the Gaussian nature of the noise and the equiprobability of the symbols, one has

$$
\begin{cases}
\log p\left(\mathbf{y}|\boldsymbol{\theta},\xi^\star\right) = \sum_{k=1}^K\left(-\log\left(\pi\sigma_n^2\right)-\frac{1+\|y_k\|^2}{\sigma_n^2}+\log\left(\cosh\left(\frac{2}{\sigma_n^2}\Re\{y_k e^{-j\theta_k}\}\right)\right)\right), \\
\log p\left(\boldsymbol{\theta}|\xi^\star\right) = \log p\left(\theta_1\right)+(K-1)\log\left(\frac{1}{\sqrt{2\pi}\sigma_w}\right)-\sum_{k=2}^K\frac{(\theta_k-\theta_{k-1}-\xi^\star)^2}{2\sigma_w^2}.
\end{cases}
\tag{18}
$$

- *Expression of* $\mathbf{H}_{11}$: assuming that we have no prior knowledge, *i.e.*, $\mathbb{E}_{\theta_1}\left[\Delta_{\theta_1}^{\theta_1}\log p(\theta_1)\right]=0$, it is shown in [6] (due to the order one Markov structure exhibited by Eqn. (16)) that $\mathbf{H}_{11}$ takes the following tridiagonal structure

$$
\mathbf{H}_{11}=b\begin{pmatrix}
A+1 & 1 & 0 & \cdots & & 0 \\
1 & A & 1 & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & & 0 \\
\vdots & & \ddots & 1 & A & 1 \\
0 & & \cdots & 0 & 1 & A+1
\end{pmatrix},
\tag{19}
$$

where $b=-1/\sigma_w^2$, and, where $A=-\sigma_w^2 J_D-2$ with $J_D=\mathbb{E}_{\mathbf{y},\boldsymbol{\theta}|\xi^\star}\left[-\Delta_{\theta_k}^{\theta_k}\log p\left(y_k|\theta_k,\xi^\star\right)\right]$.

- *Expression of* $\mathbf{h}_{12}$: since, from Eqn. (18), $\log p\left(\mathbf{y}|\boldsymbol{\theta},\xi^\star\right)$ is independent of $\xi^\star$, $\left.\Delta_{\boldsymbol{\theta}}^\xi\log p\left(\mathbf{y}|\boldsymbol{\theta},\xi\right)\right|_{\xi^\star}=0$. Consequently,

$$
\mathbf{h}_{12}=\mathbb{E}_{\boldsymbol{\theta}|\xi^\star}\left[-\left.\Delta_{\boldsymbol{\theta}}^\xi\log p\left(\boldsymbol{\theta}|\xi\right)\right|_{\xi^\star}\right].
\tag{20}
$$

Using the state model, we have

$$
\begin{cases}
\left.\Delta_\xi^{\theta_1}\log p\left(\boldsymbol{\theta}|\xi\right)\right|_{\xi^\star} = -\frac{1}{\sigma_w^2}, \\
\left.\Delta_\xi^{\theta_K}\log p\left(\boldsymbol{\theta}|\xi\right)\right|_{\xi^\star} = \frac{1}{\sigma_w^2}, \\
\left.\Delta_\xi^{\theta_k}\log p\left(\boldsymbol{\theta}|\xi\right)\right|_{\xi^\star} = 0\text{ for }k\in\{2,\ldots,K-1\}.
\end{cases}
\tag{21}
$$

Applying the expectation operator $\mathbb{E}_{\boldsymbol{\theta}|\xi^\star}\left[.\right]$, we obtain

$$
\mathbf{h}_{12}=\left(\begin{array}{ccc} \frac{1}{\sigma_w^2} & \mathbf{0}_{1\times K-2} & -\frac{1}{\sigma_w^2} \end{array}\right)^T.
\tag{22}
$$

- *Expression of* H_{22}: since, from Eqn. (18), $\log p\left(\mathbf{y}|\boldsymbol{\theta},\xi^\star\right)$ is independent of $\xi^\star$, $\left.\Delta_\xi^\xi\log p\left(\mathbf{y}|\boldsymbol{\theta},\xi\right)\right|_{\xi^\star}=0$. Consequently,

$$
H_{22}=\mathbb{E}_{\boldsymbol{\theta}|\xi^\star}\left[-\left.\Delta_\xi^\xi\log p\left(\boldsymbol{\theta}|\xi\right)\right|_{\xi^\star}\right]=\frac{K-1}{\sigma_w^2}.
\tag{23}
$$

- *Expression of the HCRB*: we now give the expression of $\mathbf{H}^{-1}$ which bounds the MSE. Thanks to the block-matrix inversion formula, we have

$$
\mathbf{H}^{-1}=\begin{pmatrix}
\mathbf{H}_{11}^{-1}+\mathbf{V}_K & -\frac{1}{\lambda}\mathbf{H}_{11}^{-1}\mathbf{h}_{12} \\
-\frac{1}{\lambda}\mathbf{h}_{12}^T\mathbf{H}_{11}^{-1} & \frac{1}{\lambda}
\end{pmatrix},
\tag{24}
$$

where $\lambda=\frac{K-1}{\sigma_w^2}-\mathbf{h}_{12}^T\mathbf{H}_{11}^{-1}\mathbf{h}_{12}$ and $\mathbf{V}_K=\frac{1}{\lambda}\mathbf{H}_{11}^{-1}\mathbf{h}_{12}\mathbf{h}_{12}^T\mathbf{H}_{11}^{-1}$.

We start to compute λ corresponding to the inverse of the minimal bound on the MSE of ξ. Due to the particular structure of matrices $\mathbf{H}_{11}$ and $\mathbf{h}_{12}$ (Eqn. (19) and (22)), we obtain

$$\lambda = \frac{K-1}{\sigma_w^2} - \frac{2}{\sigma_w^4} \left(\left[\mathbf{H}_{11}^{-1} \right]_{1,1} - \left[\mathbf{H}_{11}^{-1} \right]_{1,K} \right). \tag{25}$$

From Eqn. (19), thanks to the cofactor expression in the matrix inversion formula we have for any index k, $\left[\mathbf{H}_{11}^{-1} \right]_{1,k} = \frac{b^{k-1}}{|\mathbf{H}_{11}|} \left(d_{K-k} + b\, d_{K-k-1} \right)$ where d_k is the determinant of the following $k \times k$ matrix $\mathbf{D}_k$

$$\mathbf{D}_k = b \begin{pmatrix} A & 1 & 0 & \cdots & 0 \\ 1 & A & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & A & 1 \\ 0 & \cdots & 0 & 1 & A \end{pmatrix}. \tag{26}$$

The sequence $\{d_k\}$ satisfies the following recursive equation $d_k = A\, b\, d_{k-1} - b^2 \, d_{k-2}$ with $d_0 = 1$ and $d_1 = bA$. d_k can thus be written as $d_k = \rho_1 \left(r_1 \right)^k + \rho_2 \left(r_2 \right)^k$ where r_1, r_2, ρ_1 and ρ_2 are given by

$$\begin{cases} r_1 = \frac{b}{2} \left(A + \sqrt{A^2 - 4} \right), & r_2 = \frac{b}{2} \left(A - \sqrt{A^2 - 4} \right), \\ \rho_1 = \frac{\sqrt{A^2 - 4} + A}{2\sqrt{A^2 - 4}}, & \rho_2 = \frac{\sqrt{A^2 - 4} - A}{2\sqrt{A^2 - 4}}. \end{cases} \tag{27}$$

Consequently,

$$\left[\mathbf{H}_{11}^{-1} \right]_{1,k} = \frac{b^{k-1}}{|\mathbf{H}_{11}|} \left(\rho_1 r_1^{K-k-1} (r_1 + b) + \rho_2 r_2^{K-k-1} (r_2 + b) \right), \tag{28}$$

and

$$\lambda = \frac{K-1}{\sigma_w^2} - \frac{2}{\sigma_w^4 |\mathbf{H}_{11}|} \left(\rho_1 r_1^{K-2} (r_1 + b) + \rho_2 r_2^{K-2} (r_2 + b) - b^{K-1} \right). \tag{29}$$

From the definition of $\mathbf{V}_K$, we have

$$[\mathbf{V}_K]_{k,k} = \frac{1}{\lambda \sigma_w^4} \left(\left[\mathbf{H}_{11}^{-1} \right]_{1,k} - \left[\mathbf{H}_{11}^{-1} \right]_{1,K+1-k} \right)^2. \tag{30}$$

Using Eqn. (24), (28), and (30), we obtain, for any index k, the analytical expression of the HCRB diagonal elements

$$\begin{aligned} \left[\mathbf{H}^{-1} \right]_{k,k} = & \frac{1}{|\mathbf{H}_{11}|} \left[\rho_1^2 (b + r_1)^2 \, r_1^{K-3} + \rho_2^2 (b + r_2)^2 r_2^{K-3} - \frac{b^2}{A-2} (r_1^{k-2} r_2^{K-k-1} + r_1^{K-k-1} r_2^{k-2}) \right] \\ & + \frac{1}{\lambda \sigma_w^4 |\mathbf{H}_{11}|^2} \left[b^{k-1} \left(\rho_1 \left(r_1 \right)^{K-k-1} (b + r_1) + \rho_2 \left(r_2 \right)^{K-k-1} (b + r_2) \right) \right. \\ & \left. + b^{K-k} \left(\rho_1 \left(r_1 \right)^{k-2} (b + r_1) + \rho_2 \left(r_2 \right)^{k-2} (b + r_2) \right) \right]^2. \end{aligned} \tag{31}$$

Remark: Note that, if Eqn. (3) was used instead of Eqn. (8), the HIM would not be invertible.

C. Simulation results

We now illustrate the behavior of the HCRB versus the Signal-to-Noise Ratio (SNR) defined by $\frac{1}{\sigma_n^2}$. We consider a block of $K = 40$ BPSK transmitted symbols. For two distinct phase-noise variances ($\sigma_w^2 = 0.1\,\text{rad}^2$ and $\sigma_w^2 \rightarrow 0\,\text{rad}^2$), Figure 1 superimposes on one side the HCRB (see Eqn. (31)), the Data-Aided HCRB $\left(J_D = \frac{2}{\sigma_n^2} \right)$, and

the BCRB (see Eqn. (21) in [6]) on θ_K. For the same scenario, Figure 2 superimposes on one side the HCRB (see Eqn. (25)) and the Data-Aided HCRB on ξ.

- At high SNR, we first notice that HCRB_ξ converges to its horizontal asymptote given by $\frac{\sigma_w^2}{K-1}$ which is the standard CRB when $\boldsymbol{\theta}$ is assumed to be known. The observation noise compared to the phase noise is then not significant enough to disturb the estimation of ξ; consequently HCRB_ξ depends only on the phase noise and on the number of observations. Concerning the bounds on θ_K, HCRB_{θ_k} and BCRB_{θ_k} both have the same asymptote given by $\frac{\sigma_n^2}{2}$ which is the Modified CRB (MCRB) for one observation (see [10]). It means that, at high SNR, the observation y_K is self-sufficient to estimate θ_K and the error on ξ does not disturb the performance on θ_K. Moreover, the HCRB logically tends to the Data-Aided HCRB.

- For median SNR, HCRB_{θ_K} and HCRB_ξ leave their respective asymptote. HCRB_{θ_K} is still lower bounded by the BCRB and upper bounded by the high-SNR asymptote. This stems from the fact that taking into account a block of observations instead of one observation necessarily improves the performance. However, for large σ_w^2 values (*e.g.*, $\sigma_w^2 = 0.1\,\text{rad}^2$), HCRB_{θ_K} stays close to the MCRB because the correlation between the phase offsets θ_k is less significant than the information brought by the observation y_K. Moreover, when σ_w^2 tends to 0, HCRB_{θ_K} is above the BCRB because performance is now limited by the accuracy on the parameter ξ.

- At low SNR, n_k is preponderant compared to w_k. Both HCRB_ξ and HCRB_{θ_K} do not depend on σ_w^2: the lack of knowledge on ξ directly affects the estimation on θ_K. As expected, the knowledge of the symbols (Data-Aided HCRB) leads to a better estimation of $\boldsymbol{\theta}$ and ξ.

IV. CONCLUSION

In this paper, we have studied the hybrid Cramér-Rao bound when the random and the deterministic parts of the parameter vector are statistically dependent. We have applied this bound in order to evaluate the performance of a dynamical phase estimator where the linear drift is unknown in a non data-aided context. In particular, we have illustrated the effect of this unknown linear drift on the phase estimation performance.

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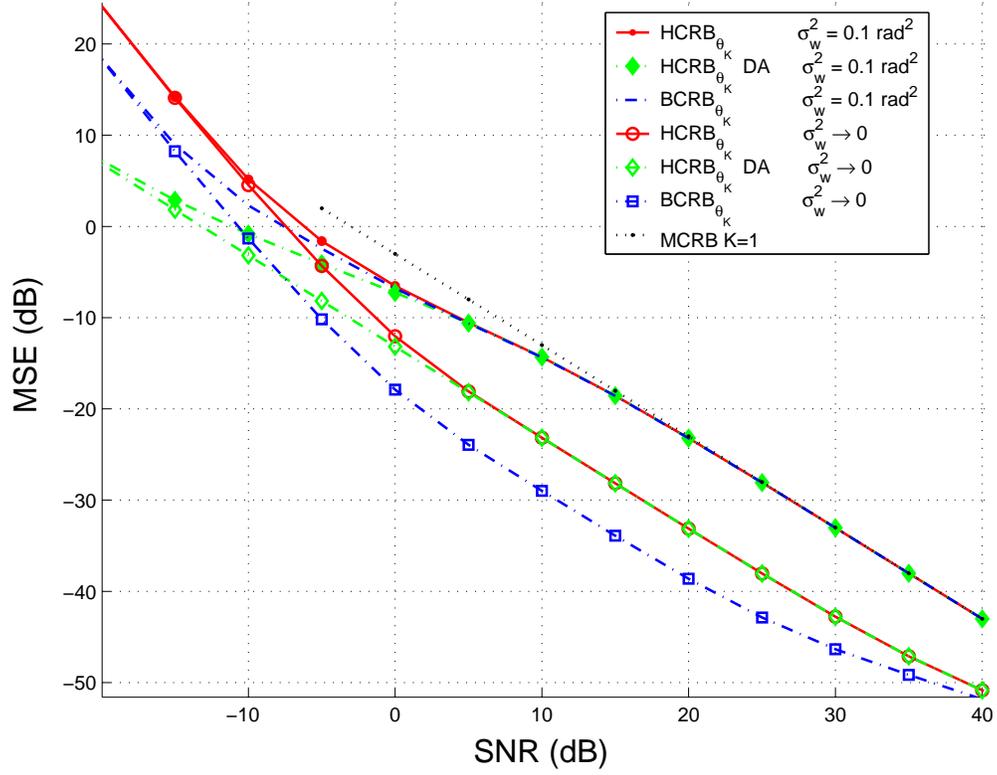[8] J. A. McNeill, "Jitter in ring oscillators," Ph.D. dissertation, Boston University, 1994.

Fig. 1. Bounds on θ_K versus the SNR ($K = 40$ observations, $\sigma_w^2 = 0.1\,\mathrm{rad}^2$, and $\sigma_w^2 \rightarrow 0\ \mathrm{rad}^2$, J_D evaluated over 10^8 Monte-Carlo trials).

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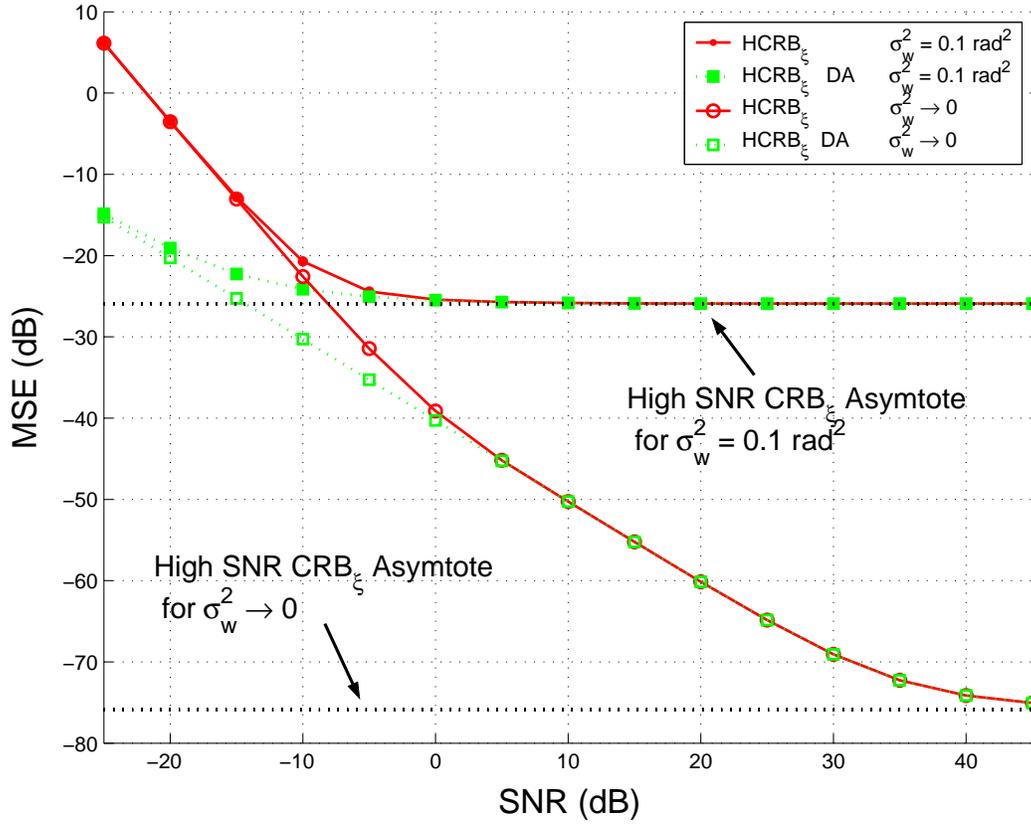
Fig. 2.   Bounds on ξ versus the SNR ($K = 40$ observations, $\sigma_w^2 = 0.1\,\mathrm{rad}^2$ and $\sigma_w^2 \to 0\,\mathrm{rad}^2$, J_D evaluated over 10^8 Monte-Carlo trials).