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# Forward Feynman-Kac type representation for semilinear nonconservative Partial Differential Equations

ANTHONY LECAVIL <sup>\*</sup>, NADIA OUDJANE <sup>†</sup> AND FRANCESCO RUSSO <sup>‡</sup>

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## Abstract

We propose a nonlinear forward Feynman-Kac type equation, which represents the solution of a non-conservative semilinear parabolic Partial Differential Equations (PDE). We show in particular existence and uniqueness. The solution of that type of equation can be approached via a weighted particle system.

**Key words and phrases:** Semilinear Partial Differential Equations; Nonlinear Feynman-Kac type functional; Particle systems; Probabilistic representation of PDEs.

**2010 AMS-classification:** 60H10; 60H30; 60J60; 65C05; 65C35; 35K58.

## 1 Introduction

This paper situates in the framework of forward probabilistic representations of nonlinear PDEs of the form

$$\begin{cases} \partial_t u = \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2 ((\Phi \Phi^t)_{i,j}(t, x, u)u) - \operatorname{div}(g(t, x, u)u) + \Lambda(t, x, u, \nabla u)u, & \text{for any } t \in [0, T], \\ u(0, dx) = u_0(dx), \end{cases} \quad (1.1)$$

where  $u_0$  is a Borel probability measure.  $u : ]0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  will be the unknown function. For that PDE we propose the forward probabilistic representation

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(s, Y_s, u(s, Y_s))dW_s + \int_0^t g(s, Y_s, u(s, Y_s))ds, & \text{with } Y_0 \sim u_0, \\ u(t, \cdot) := \frac{d\nu_t}{dx} \quad \nu_t(\varphi) := \mathbb{E} \left[ \varphi(Y_t) \exp \left\{ \int_0^t \Lambda(s, Y_s, u(s, Y_s), \nabla u(s, Y_s))ds \right\} \right], & \varphi \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}), t \in [0, T], \end{cases} \quad (1.2)$$

which is a nonlinear stochastic differential equation (NLSDE) in the spirit of McKean, see e.g. [19].

The underlying idea of our approach consists in extending, to fairly general non-conservative PDEs, the probabilistic representation of nonlinear Fokker-Planck equations which appears when  $\Lambda = 0$ . An interesting aspect of this strategy is that it is potentially able to represent an extended class of second order nonlinear PDEs.

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When  $\Lambda = 0$ , several authors have studied NLSDEs of the form (1.2). Significant contributions are due to [26], [21], [20], in the case where the non linearity with respect to  $u$  are mollified in the diffusion and drift coefficients. In [15], the authors focused on the case when the coefficients depend pointwisely on  $u$ . The authors have proved strong existence and pathwise uniqueness of (1.2), when  $\Phi$  and  $g$  are smooth and Lipschitz and  $\Phi$  is non-degenerate. Other authors have more particularly studied an NLSDE of the form

$$\begin{cases} Y_t = Y_0 + \int_0^t \Phi(u(s, Y_s)) dW_s, Y_0 \sim u_0, \\ u(t, x) dx \text{ is the law density of } Y_t, t > 0 \\ u(0, \cdot) = u_0, \end{cases} \quad (1.3)$$

for which the particular case  $d = 1$ ,  $\Phi(u) = u^k$  for  $k \geq 1$  was developed in [3]. When  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  is only assumed to be bounded, measurable and monotone, existence/uniqueness results are still available in [5, 1]. A partial extension to the multidimensional case ( $d \geq 2$ ) is exposed in [2].

The solutions of (1.1), when  $\Lambda = 0$ , are probability measures dynamics which often describe the macroscopic distribution law of a microscopic particle which behaves in a diffusive way. For that reason, those time evolution PDEs are conservative in the sense that their solutions  $u(t, \cdot)$  verify the property  $\int_{\mathbb{R}^d} u(t, x) dx$  to be constant in  $t$  (equal to 1, which is the mass of a probability measure). An interesting feature of this type of representation is that the law of the solution  $Y$  of the NLSDE can be characterized as the limiting empirical distribution of a large number of interacting particles. This is a consequence of the so called *propagation of chaos* phenomenon, already observed in the literature for the case of mollified dependence, see e.g. [16, 19, 26, 21] and [15] for the case of pointwise dependence. [8] has contributed to develop stochastic particle methods in the spirit of McKean to approach a PDE related to Burgers equation providing first the rate of convergence. Comparison with classical numerical analysis techniques was provided by [7].

In this paper we will concentrate on the novelty constituted by the introduction of  $\Lambda$  depending on  $u$  and  $\nabla u$ . For this step  $\Phi, g$  will not depend on  $u$ . In this context we will focus on semilinear PDEs of the form

$$\begin{cases} \partial_t u = L_t^* u + u \Lambda(t, x, u, \nabla u) \\ u(0, \cdot) = u_0, \end{cases} \quad (1.4)$$

with  $L^*$  a partial differential operator of the type

$$(L_t^* \varphi)(x) = \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2 ((\Phi \Phi^t)_{i,j}(t, x) \varphi)(x) - \sum_{i=1}^d \partial_i (g_i(t, x) \varphi)(x), \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^d). \quad (1.5)$$

An alternative approach for representing this type of PDE is given by forward-backward stochastic differential equations. Those were initially developed in [23], see also [22] for a survey and [24] for a recent monograph on the subject. However, the extension of those equations to fully nonlinear PDEs still requires complex developments and is the subject of active research, see for instance [9]. Branching diffusion processes provide another probabilistic representation of semilinear PDEs, see e.g. [11, 13, 12]. Here again, extensions to second order nonlinear PDEs still constitutes a difficult issue.

As suggested, our method potentially allows to reach a certain significant class of PDEs with second-order non-linearity, if we allow the diffusion coefficient to also depend on  $u$ . The general framework where  $g$  and  $\Phi$  also depend non linearly on  $u$  while  $\Lambda$  depends on  $u$  and  $\nabla u$  has been partially investigated in [18], where the dependence of the coefficients with respect to  $u$  is mollified and  $\Lambda$  does not depend on  $\nabla u$ . An associated interacting particle system converging to the solution of a regularized version of the nonlinear PDE has been proposed in [17], providing encouraging numerical performances. The originality of the

present paper is to consider a pointwise dependence of  $\Lambda$  on both  $u$  and  $\nabla u$ . The pointwise dependence on  $\nabla u$  constitutes the major technical difficulty.

More specifically, we propose to associate (1.4) with a forward probabilistic representation given by a couple  $(Y, u)$  solution of (1.2) where  $\Phi$  and  $g$  are the functions intervening in (1.5). In this case, the second line equation of (1.2) will be called **Feynman-Kac equation** and a solution  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  will be called **Feynman-Kac type representation** of (1.4). When  $\Lambda$  vanishes, the functions  $(u(t, \cdot), t > 0)$  are indeed the marginal law densities of the process  $(Y_t, t > 0)$  and (1.4) coincides with the classical Fokker-Planck PDE. When  $\Lambda \neq 0$ , the proof of well-posedness of the Feynman-Kac equation is not obvious and it is one of the contributions of the paper. The strategy used relies on two steps. Under a Lipschitz condition on  $\Lambda$  in Theorem 3.5, we first prove that a function  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  (belonging to  $L^1([0, T], W^{1,1}(\mathbb{R}^d))$ ) is a solution of the Feynman-Kac equation (1.2) if and only if it is a mild solution of (1.4). The latter concept is introduced in item 2. of Definition 2.1. Then, under Lipschitz type conditions on  $\Phi$  and  $g$ , Theorem 3.6 establishes the existence and uniqueness of a mild solution of (1.4). As a second contribution, we propose and analyze a corresponding particle system. This relies on two approximation steps: a regularization procedure based on a kernel convolution and the law of large numbers. The convergence of the particle system is stated in Theorem 5.3.

## 2 Preliminaries

### 2.1 Notations

Let  $d \in \mathbb{N}^*$ . Let us consider  $\mathcal{C}^d := \mathcal{C}([0, T], \mathbb{R}^d)$  metricized by the supremum norm  $\|\cdot\|_\infty$ , equipped with its Borel  $\sigma$ -field  $\mathcal{B}(\mathcal{C}^d)$  and endowed with the topology of uniform convergence.

If  $(E, d_E)$  is a Polish space,  $\mathcal{P}(E)$  denotes the Polish space (with respect to the weak convergence topology) of Borel probability measures on  $E$  naturally equipped with its Borel  $\sigma$ -field  $\mathcal{B}(\mathcal{P}(E))$ . The reader can consult Proposition 7.20 and Proposition 7.23, Section 7.4 Chapter 7 in [4] for more exhaustive information. When  $d = 1$ , we often omit it and we simply note  $\mathcal{C} := \mathcal{C}^1$ .  $\mathcal{C}_b(E)$  denotes the space of bounded, continuous real-valued functions on  $E$ .

In this paper,  $\mathbb{R}^d$  is equipped with the Euclidean scalar product  $\cdot$  and  $|x|$  stands for the induced norm for  $x \in \mathbb{R}^d$ . The gradient operator for functions defined on  $\mathbb{R}^d$  is denoted by  $\nabla$ . If a function  $u$  depends on a variable  $x \in \mathbb{R}^d$  and other variables, we still denote by  $\nabla u$  the gradient of  $u$  with respect to  $x$ , if there is no ambiguity.  $M_{d,p}(\mathbb{R})$  denotes the space of  $\mathbb{R}^{d \times p}$  real matrices equipped with the Frobenius norm (also denoted  $|\cdot|$ ), i.e. the one induced by the scalar product  $(A, B) \in M_{d,p}(\mathbb{R}^d) \times M_{d,p}(\mathbb{R}) \mapsto Tr(A^t B)$  where  $A^t$  stands for the transpose matrix of  $A$  and  $Tr$  is the trace operator.  $\mathcal{S}_d$  is the set of symmetric, non-negative definite  $d \times d$  real matrices and  $\mathcal{S}_d^+$  the set of strictly positive definite matrices of  $\mathcal{S}_d$ .

$\mathcal{M}_f(\mathbb{R}^d)$  is the space of finite Borel measures on  $\mathbb{R}^d$ . When it is endowed with the weak convergence topology,  $\mathcal{B}(\mathcal{M}_f(\mathbb{R}^d))$  stands for its Borel  $\sigma$ -field. It is well-known that  $(\mathcal{M}_f(\mathbb{R}^d), \|\cdot\|_{TV})$  is a Banach space, where  $\|\cdot\|_{TV}$  denotes the total variation norm.

$\mathcal{S}(\mathbb{R}^d)$  is the space of Schwartz fast decreasing test functions and  $\mathcal{S}'(\mathbb{R}^d)$  is its dual.  $\mathcal{C}_b(\mathbb{R}^d)$  is the space of bounded, continuous functions on  $\mathbb{R}^d$  and  $\mathcal{C}_0^\infty(\mathbb{R}^d)$  the space of smooth functions with compact support. For any positive integers  $p, k \in \mathbb{N}$ ,  $\mathcal{C}_b^{k,p} := \mathcal{C}_b^{k,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$  denotes the set of continuously differentiable bounded functions  $[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  with uniformly bounded derivatives with respect to the time variable  $t$  (resp. with respect to space variable  $x$ ) up to order  $k$  (resp. up to order  $p$ ). In particular, for  $k = p = 0$ ,  $\mathcal{C}_b^{0,0}$  coincides with the space of bounded, continuous functions also denoted by  $\mathcal{C}_b$ .  $\mathcal{C}_b^\infty(\mathbb{R}^d)$  is the space of

bounded and smooth functions.  $C_0(\mathbb{R}^d)$  denotes the space of continuous functions with compact support in  $\mathbb{R}^d$ . For  $r \in \mathbb{N}$ ,  $W^{r,p}(\mathbb{R}^d)$  is the Sobolev space of order  $r$  in  $(L^p(\mathbb{R}^d), \|\cdot\|_p)$ , with  $1 \leq p \leq \infty$ .  $W_{loc}^{1,1}(\mathbb{R}^d)$  denotes the space of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $f$  and  $\nabla f$  (existing in the weak sense) belong to  $L_{loc}^1(\mathbb{R}^d)$ .

For convenience we introduce the following notation.

- $V : [0, T] \times \mathcal{C}^d \times \mathcal{C} \times \mathcal{C}^d$  is defined for any functions  $x \in \mathcal{C}^d, y \in \mathcal{C}$  and  $z \in \mathcal{C}^d$ , by

$$V_t(x, y, z) := \exp\left(\int_0^t \Lambda(s, x_s, y_s, z_s) ds\right) \quad \text{for any } t \in [0, T]. \quad (2.1)$$

The finite increments theorem gives, for all  $(a, b) \in \mathbb{R}^2$ , we have

$$\exp(a) - \exp(b) = (b - a) \int_0^1 \exp(\alpha a + (1 - \alpha)b) d\alpha. \quad (2.2)$$

Therefore, if  $\Lambda$  is supposed to be bounded and Lipschitz w.r.t. to its space variables  $(x, y, z)$ , uniformly w.r.t.  $t$ , we observe that (2.2) implies it follows that, for all  $t \in [0, T], x, x' \in \mathcal{C}^d, y, y' \in \mathcal{C}, z, z' \in \mathcal{C}^d$ ,

$$|V_t(x, y, z) - V_t(x', y', z')| \leq L_\Lambda e^{tM_\Lambda} \int_0^t (|x_s - x'_s| + |y_s - y'_s| + |z_s - z'_s|) ds, \quad (2.3)$$

$M_\Lambda$  (resp.  $L_\Lambda$ ) denoting an upper bound of  $|\Lambda|$  (resp. the Lipschitz constant of  $\Lambda$ ), see also Assumption 2. In the whole paper,  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  will denote a filtered probability space and  $W$  an  $\mathbb{R}^p$ -valued  $(\mathcal{F}_t)$ -Brownian motion.

## 2.2 Mild and Weak solutions

We first introduce the following assumption.

**Assumption 1.** 1.  $\Phi$  and  $g$  are functions defined on  $[0, T] \times \mathbb{R}^d$  taking values in  $M_{d,p}(\mathbb{R}^d)$  and  $\mathbb{R}^d$ .

There exist  $L_\Phi, L_g > 0$  such that for any  $t \in [0, T], (x, x') \in \mathbb{R}^d \times \mathbb{R}^d$ ,

$$\begin{aligned} |\Phi(t, x) - \Phi(t, x')| &\leq L_\Phi |x - x'|, \\ |g(t, x) - g(t, x')| &\leq L_g |x - x'|. \end{aligned}$$

2. The functions  $s \in [0, T] \mapsto |\Phi(s, 0)|$  and  $s \in [0, T] \mapsto |g(s, 0)|$  are bounded.

In the whole paper we will write  $a = \Phi\Phi^t$ ; in particular  $a : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{S}_d$ . Through some definitions, we make here precise in which sense we will consider solutions of the PDE (1.4). We are interested in different concepts of solutions  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  of that semilinear PDE where, for  $t \in [0, T], L_t$  is given by

$$(L_t \varphi)(x) = \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, x) \partial_{ij}^2 \varphi(x) + \sum_{i=1}^d g_i(t, x) \partial_i \varphi(x), \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d). \quad (2.4)$$

Its "adjoint"  $L_t^*$  defined in (1.5), verifies

$$\int_{\mathbb{R}^d} L_t \varphi(x) \psi(x) dx = \int_{\mathbb{R}^d} \varphi(x) L_t^* \psi(x) dx, \quad (\varphi, \psi) \in \mathcal{C}_0^\infty(\mathbb{R}^d), t \in [0, T], \quad (2.5)$$

and the initial condition  $u_0$  of (1.4) has to be understood in the sense that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^d} \varphi(x) u_t(dx) = \int_{\mathbb{R}^d} \varphi(x) u_0(dx), \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d),$$

since, a priori, it can be irregular and not necessarily a function.

We observe that if  $\Lambda = 0$ , (1.4) is the classical Fokker-Planck equation which can be understood in the sense of distributions, i.e., for all  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ ,  $t \in [0, T]$ ,

$$\int_{\mathbb{R}^d} u(t, dx)\varphi(x) = \int_{\mathbb{R}^d} u_0(dx)\varphi(x) + \int_0^t \int_{\mathbb{R}^d} u(s, dx)(L_s\varphi)(x)ds. \quad (2.6)$$

Under Assumption 1 it is well-known (see Introduction and Section 2.2, Chapter 2 in [25]), that there exists a *good* family of probability transition  $P(s, x_0, t, \cdot)$  (see Introduction and Section 2.2, Chapter 2 in [25]), for which the Fokker-Planck equation (understood in the sense of distributions) is verified, i.e.

$$\begin{cases} \partial_t P(s, x_0, t, \cdot) = L_t^* P(s, x_0, t, \cdot) \\ \lim_{t \downarrow s} P(s, x_0, t, \cdot) = \delta_{x_0}, \quad 0 \leq s < t \leq T, x_0 \in \mathbb{R}^d. \end{cases} \quad (2.7)$$

Given a random variable  $Y_0$ , classical theorems for SDE with Lipschitz coefficients imply strong existence and pathwise uniqueness for the SDE

$$dY_t = \Phi(t, Y_t)dW_t + g(t, Y_t)dt. \quad (2.8)$$

By the classical theory of Markov processes (see e.g. Chapter 2 in [25]), we know that the transition probability function  $P$ , satisfying (2.7), defines and characterizes uniquely the law of the process  $Y$ , provided the law  $u_0$  of  $Y_0$  is specified. In particular, we have the following.

1. For  $t \in [0, T]$ , the marginal law of  $Y_t$  is given for all  $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$  by

$$\mathbb{E}[\varphi(Y_t)] = \int_{\mathbb{R}^d} u_0(dx_0) \int_{\mathbb{R}^d} \varphi(x)P(0, x_0, t, dx). \quad (2.9)$$

2. For all  $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$  and  $0 \leq s < t \leq T$ ,

$$\mathbb{E}[\varphi(Y_t)|Y_s] = \int_{\mathbb{R}^d} \varphi(x)P(s, Y_s, t, dx). \quad (2.10)$$

Let  $\Lambda : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  be bounded, Borel measurable, we recall the notions of **weak solution** and **mild solution** associated to (1.4).

**Definition 2.1.** Let  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a Borel function such that for every  $t \in ]0, T]$ ,  $u(t, \cdot) \in W_{loc}^{1,1}(\mathbb{R}^d)$ .

1.  $u$  will be called **weak solution** of (1.4) if for all  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ ,  $t \in [0, T]$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)u(t, x)dx - \int_{\mathbb{R}^d} \varphi(x)u_0(dx) &= \int_0^t \int_{\mathbb{R}^d} u(s, x)L_s\varphi(x)dxds \\ &+ \int_0^t \int_{\mathbb{R}^d} \varphi(x)\Lambda(s, x, u(s, x), \nabla u(s, x))u(s, x)dxds. \end{aligned} \quad (2.11)$$

2.  $u$  will be called **mild solution** of (1.4) if for all  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ ,  $t \in [0, T]$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)u(t, x)dx &= \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} u_0(dx_0)P(0, x_0, t, dx) \\ &+ \int_{[0, t] \times \mathbb{R}^d} \left( \int_{\mathbb{R}^d} \varphi(x)P(s, x_0, t, dx) \right) \Lambda(s, x_0, u(s, x_0), \nabla u(s, x_0))u(s, x_0)dx_0ds. \end{aligned} \quad (2.12)$$

The object of the first lemma below is to show to what extent the concept of mild solution is equivalent to the weak one.

**Lemma 2.2.** *We assume there exists a unique weak solution  $v$  of*

$$\begin{cases} \partial_t v = L_t^* v \\ v(0, \cdot) = 0, \end{cases} \quad (2.13)$$

where  $L_t^*$  is given by (1.5). Then,  $u$  is a mild solution of (1.4) if and only if  $u$  is a weak solution of (1.4)

**Remark 2.3.** *There exist several sets of technical assumptions (see e.g. [6, 10]) leading to the uniqueness assumed in Lemma 2.2 above. In particular, under items 1., 2. and 3. of Assumption 2 stated below (which will constitute our framework in the sequel), Theorem 4.7 in Chapter 4 of [10] ensures (classical) existence and uniqueness of the solution of (2.13), see also Lemma 6.4 below.*

*Proof.* We first suppose that  $u$  is a mild solution of (1.4). Taking into account that  $P(s, x_0, t, \cdot)$  is a distributional solution of (2.7), classical computations show that  $u$  is indeed a weak solution of (1.4).

Conversely, suppose that  $u$  is a weak solution of (1.4), in the sense of Definition 2.1. We also consider

$$\begin{aligned} \bar{v}(t, dx) &:= \int_{\mathbb{R}} P(0, x_0, t, dx) u_0(dx_0) + \int_0^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) u(s, dx_0) \Lambda(s, x_0, u(s, x_0), \nabla u(s, x_0)) dx_0 \\ &= \int_{\mathbb{R}} P(0, x_0, t, dx) u_0(dx_0) + \int_0^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \bar{\Lambda}(u)(s, x_0) dx_0, \end{aligned} \quad (2.14)$$

where  $\bar{\Lambda}(u)(t, x) := u(s, x) \Lambda(s, x, u(s, x), \nabla u(s, x))$  for  $(s, x) \in [0, T] \times \mathbb{R}^d$ . We want to ensure that  $u = \bar{v}$ .

On the one hand, integrating the function  $\bar{v}(t, \cdot)$  against a test function and using again that  $P(s, x_0, t, \cdot)$  is a distributional solution of (2.7), we obtain that  $\bar{v}$  is a weak solution of

$$\begin{cases} \partial_t \bar{v} = L_t^* \bar{v} + \bar{\Lambda}(u) \\ \bar{v}(0, \cdot) = u_0, \end{cases} \quad (2.15)$$

On the other hand,  $u$  being a weak solution of (1.4), it also satisfies (2.15) (in the sense of distributions). We set  $v := \bar{v} - u$ . It follows that  $v$  and  $\hat{v} := 0$  both satisfy (2.13). Uniqueness of the solution of (2.13) implies that  $v = 0$ , which concludes the proof.  $\square$

### 3 Feynman-Kac type representation

We suppose here the validity of Assumption 1. Let  $u_0 \in \mathcal{P}(\mathbb{R}^d)$  and fix a random variable  $Y_0$  distributed according to  $u_0$  and consider the strong solution  $Y$  of (2.8).

The aim of this section is to show how a mild solution of (1.4) can be linked with a Feynman-Kac type equation, where we recall that a solution is given by a function  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the second line equation of (1.2).

Given  $\tilde{\Lambda} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  a bounded, Borel measurable function, let us consider the measure-valued map  $\mu : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$  defined by

$$\int_{\mathbb{R}^d} \varphi(x) \mu(t, dx) = \mathbb{E} \left[ \varphi(Y_t) \exp \left( \int_0^t \tilde{\Lambda}(s, Y_s) ds \right) \right], \text{ for all } \varphi \in \mathcal{C}_b(\mathbb{R}^d), t \in [0, T]. \quad (3.1)$$

The first proposition below shows how the map  $t \mapsto \mu(t, \cdot)$  can be characterized as a solution of the linear parabolic PDE

$$\begin{cases} \partial_t v = L_t^* v + \tilde{\Lambda}(t, x) v \\ v(0, \cdot) = u_0. \end{cases} \quad (3.2)$$



Before stating the corresponding proposition, we introduce the notion of *measure-mild solution*.

**Definition 3.1.** A measure-valued map  $\mu : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$  will be called **measure-mild solution** of (3.2) if for all  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ ,  $t \in [0, T]$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \mu(t, dx) &= \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} u_0(dx_0) P(0, x_0, t, dx) \\ &+ \int_{[0, t] \times \mathbb{R}^d} \left( \int_{\mathbb{R}^d} P(r, x_0, t, dx) \varphi(x) \right) \tilde{\Lambda}(r, x_0) \mu(r, dx_0) dr. \end{aligned} \quad (3.3)$$

**Remark 3.2.** 1. Since  $\mu$  is a (finite) measure valued function, by usual approximation arguments, it is not difficult to show that an equivalent formulation for Definition 2.1 can be expressed taking  $\varphi$  in  $\mathcal{C}_b(\mathbb{R}^d)$  instead of  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ .

2. Although the definition of **mild solution** (see item 2. of Definition of 2.1) and the one of **measure-mild solution** seem to be formally close, the two concepts do not make sense in the same situations. Indeed, the notion of mild-solution makes sense for PDEs with nonlinear terms of the general form  $\Lambda(t, x, u, \nabla u)$ , whereas a measure-mild solution can exist only for linear PDEs. However, in the case where a measure  $\mu$  on  $\mathbb{R}^d$ , absolutely continuous w.r.t. the Lebesgue measure  $dx$ , is a measure-mild solution of the linear PDE (3.2), its density indeed coincides with the mild solution (in the sense of item 2. of Definition 2.1) of (3.2).

**Proposition 3.3.** Under Assumption 1 the measure-valued map  $\mu$  defined by (3.1) is the unique measure-mild solution of

$$\begin{cases} \partial_t v = L_t^* v + \tilde{\Lambda}(t, x) v \\ v(0, \cdot) = u_0, \end{cases} \quad (3.4)$$

where the operator  $L_t^*$  is defined by (1.5).

*Proof.* We first prove that a function  $\mu$  defined by (3.1) is a measure-mild solution of (3.4).

Observe that for all  $t \in [0, T]$ ,

$$\exp \left( \int_0^t \tilde{\Lambda}(r, Y_r) dr \right) = 1 + \int_0^t \tilde{\Lambda}(r, Y_r) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds} dr. \quad (3.5)$$

From (3.1), it follows that for all test function  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$  and  $t \in [0, T]$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \mu(t, dx) &= \mathbb{E} \left[ \varphi(Y_t) \exp \left( \int_0^t \tilde{\Lambda}(r, Y_r) dr \right) \right] \\ &= \mathbb{E}[\varphi(Y_t)] + \int_0^t \mathbb{E} \left[ \varphi(Y_t) \tilde{\Lambda}(r, Y_r) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds} \right] dr. \end{aligned} \quad (3.6)$$

On the one hand, by (2.9), we have

$$\mathbb{E}[\varphi(Y_t)] = \int_{\mathbb{R}^d} u_0(dx_0) \int_{\mathbb{R}^d} \varphi(x) P(0, x_0, t, dx), \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \text{ and } t \in [0, T]. \quad (3.7)$$

On the other hand, using (2.10) yields, for  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ ,  $0 \leq r \leq t$ ,

$$\begin{aligned} \mathbb{E} \left[ \varphi(Y_t) \tilde{\Lambda}(r, Y_r) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds} \right] &= \mathbb{E} \left[ \tilde{\Lambda}(r, Y_r) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds} \mathbb{E} \left[ \varphi(Y_t) \middle| Y_r \right] \right] \\ &= \mathbb{E} \left[ \left( \tilde{\Lambda}(r, Y_r) \int_{\mathbb{R}^d} \varphi(x) P(r, Y_r, t, dx) \right) e^{\int_0^r \tilde{\Lambda}(s, Y_s) ds} \right] \\ &= \int_{\mathbb{R}^d} \left( \tilde{\Lambda}(r, x_0) \int_{\mathbb{R}^d} \varphi(x) P(r, x_0, t, dx) \right) \mu(r, dx_0), \end{aligned} \quad (3.8)$$



where the third equality above comes from (3.1) applied to the bounded, measurable test function  $z \mapsto \tilde{\Lambda}(r, z) \int_{\mathbb{R}^d} \varphi(x) P(r, z, t, dx)$ . Injecting (3.8) and (3.7) in the right-hand side (r.h.s.) of (3.6) gives for all  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ ,  $t \in [0, T]$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \mu(t, dx) &= \int_{\mathbb{R}^d} u_0(dx_0) \int_{\mathbb{R}^d} P(0, x_0, t, dx) \varphi(x) dx \\ &+ \int_0^t \int_{\mathbb{R}^d} \mu(r, dx_0) \tilde{\Lambda}(r, x_0) \int_{\mathbb{R}^d} P(r, x_0, t, dx) \varphi(x) dr. \end{aligned} \quad (3.9)$$

It remains now to prove uniqueness of the measure-mild solution of (3.4).

We recall that  $\mathcal{M}_f(\mathbb{R}^d)$  denotes the vector space of finite Borel measures on  $\mathbb{R}^d$ , that is here equipped with the total variation norm  $\|\cdot\|_{TV}$ . We also recall that an equivalent definition of the total variation norm is given by

$$\|\mu\|_{TV} = \sup_{\substack{\psi \in \mathcal{C}_b(\mathbb{R}^d) \\ \|\psi\|_\infty \leq 1}} \left| \int_{\mathbb{R}^d} \psi(x) \mu(dx) \right|. \quad (3.10)$$

Consider  $t \in [0, T]$  and let  $\mu_1, \mu_2$  be two measure-mild solutions of PDE (3.4). We set  $\nu := \mu_1 - \mu_2$ . Since  $\tilde{\Lambda}$  is bounded, we observe that (3.1) implies  $\|\nu(t, \cdot)\|_{TV} < +\infty$ . Moreover, taking into account item 1. of Remark 3.2, we have that  $\nu$  satisfies,

$$\forall \varphi \in \mathcal{C}_b(\mathbb{R}^d), \int_{\mathbb{R}^d} \varphi(x) \nu(t, dx) = \int_0^t \int_{\mathbb{R}^d} \tilde{\Lambda}(r, x_0) \nu(r, dx_0) \int_{\mathbb{R}^d} \varphi(x) P(r, x_0, t, dx) dr. \quad (3.11)$$

Taking the supremum over  $\varphi$  such that  $\|\varphi\|_\infty \leq 1$  in each side of (3.11), we get

$$\|\nu(t, \cdot)\|_{TV} \leq \sup_{(s,x) \in [0,t] \times \mathbb{R}^d} |\tilde{\Lambda}(s, x)| \int_0^t \|\nu(r, \cdot)\|_{TV} dr. \quad (3.12)$$

Gronwall's lemma implies that  $\nu(t, \cdot) = 0$ . Uniqueness of measure-mild solution for (3.4) follows. This ends the proof.  $\square$

The next lemma shows how a measure-mild solution of (3.4), which is a function defined on  $[0, T]$  can be built by defining it recursively on each sub-interval of the form  $[r, r + \tau]$ . In particular, it will be used in Theorem 3.6 and Proposition 4.4. Its proof is postponed in Appendix (see Section 6.3).

**Lemma 3.4.** *Let us fix  $\tau > 0$  be a real constant and  $\delta := (\alpha_0 := 0 < \dots < \alpha_k := k\tau < \dots < \alpha_N := T)$  be a finite partition of  $[0, T]$ .*

*A measure-valued map  $\mu : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$  satisfies*

$$\begin{cases} \mu(0, \cdot) = u_0 \\ \mu(t, dx) = \int_{\mathbb{R}^d} P(k\tau, x_0, t, dx) \mu(k\tau, dx_0) + \int_{k\tau}^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \tilde{\Lambda}(s, x_0) \mu(s, dx_0), \end{cases} \quad (3.13)$$

*for all  $t \in [k\tau, (k+1)\tau]$  and  $k \in \{0, \dots, N-1\}$ , if and only if  $\mu$  is a measure-mild solution (in the sense of Definition 3.1) of (3.4).*

We now come back to the case where the bounded, Borel measurable real-valued function  $\Lambda$  is defined on  $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ . Let  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  belonging to  $L^1([0, T], W^{1,1}(\mathbb{R}^d))$ . In the sequel, we set  $\tilde{\Lambda}^u(t, x) := \Lambda(t, x, u(t, x), \nabla u(t, x))$ .  $\mu^u$  will denote the measure-valued map  $\mu$  defined by (3.1) with  $\tilde{\Lambda} = \tilde{\Lambda}^u$ , i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \mu^u(t, dx) = \mathbb{E} \left[ \varphi(Y_t) \exp \left( \int_0^t \tilde{\Lambda}^u(s, Y_s) ds \right) \right], \text{ for all } \varphi \in \mathcal{C}_b(\mathbb{R}^d), t \in [0, T]. \quad (3.14)$$

By Proposition 3.3, it follows that  $\mu^u$  is the unique measure-mild solution of the linear PDE (3.4) with  $\tilde{\Lambda} = \tilde{\Lambda}^u$ . (3.14) can be interpreted as a Feynman-Kac type representation for the measure-mild solution  $\mu^u$  of the linear PDE (3.4), for the corresponding  $\tilde{\Lambda}^u$ . More generally, Theorem 3.5 below establishes such representation formula for a mild solution of the semilinear PDE (1.4).

**Theorem 3.5.** *Assume that Assumption 1 is fulfilled. We indicate by  $Y$  the unique strong solution of (2.8). Suppose that  $\Lambda : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is bounded and Borel measurable. A function  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  in  $L^1([0, T], W^{1,1}(\mathbb{R}^d))$  is a mild solution of (1.4) if and only if, for all  $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$ ,  $t \in [0, T]$ ,*

$$\int_{\mathbb{R}^d} \varphi(x) u(t, x) dx = \mathbb{E} \left[ \varphi(Y_t) \exp \left( \int_0^t \Lambda(s, Y_s, u(s, Y_s), \nabla u(s, Y_s)) ds \right) \right]. \quad (3.15)$$

A function  $u$  verifying (3.15) will be called a **Feynman-Kac type representation** of (1.4).

*Proof.* We first suppose that  $u$  is a mild solution of (1.4). The aim is then to show that  $u$  satisfies the Feynman-Kac equation (3.15).

Since  $\Lambda$  is supposed to be bounded, Borel and  $u$  is Borel, it is clear that

$$\tilde{\Lambda}^u(t, x) := \Lambda(t, x, u(t, x), \nabla u(t, x)), \quad (3.16)$$

is bounded and Borel measurable. On the one hand, by applying Proposition 3.3 with  $\tilde{\Lambda}^u$ , it follows that  $\mu^u$  (defined by (3.14)) is the unique measure-mild solution of

$$\begin{cases} \partial_t v = L_t^* v + \tilde{\Lambda}^u(t, x) v \\ v(0, \cdot) = u_0. \end{cases} \quad (3.17)$$

On the other hand, since  $u$  is supposed to be a mild solution of (1.4),  $u(t, x) dx$  is also a measure-mild solution of (3.17). By uniqueness of the solution of (3.17), for all  $t \in [0, T]$ , we have  $u(t, x) dx = \mu^u(t, dx)$ . Taking into account item 1. of Remark 3.2, it implies for all  $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$ ,  $t \in [0, T]$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) u(t, x) dx &= \int_{\mathbb{R}^d} \varphi(x) \mu^u(t, dx) \\ &= \mathbb{E} \left[ \varphi(Y_t) \exp \left( \int_0^t \tilde{\Lambda}^u(s, Y_s) ds \right) \right], \text{ by (3.14)} \\ &= \mathbb{E} \left[ \varphi(Y_t) \exp \left( \int_0^t \Lambda(s, Y_s, u(s, Y_s), \nabla u(s, Y_s)) ds \right) \right], \text{ by (3.16)}. \end{aligned}$$

Conversely, suppose that  $u$  satisfies the Feynman-Kac equation (3.15). Recalling (3.16) and setting  $\bar{\mu}(t, dx) := u(t, x) dx$ , (3.15) can be re-written

$$\int_{\mathbb{R}^d} \varphi(x) \bar{\mu}(t, dx) = \mathbb{E} \left[ \varphi(Y_t) \exp \left( \int_0^t \tilde{\Lambda}^u(s, Y_s) ds \right) \right], \quad \varphi \in \mathcal{C}_b(\mathbb{R}^d), t \in [0, T]. \quad (3.18)$$

Proposition 3.3 applied again with  $\tilde{\Lambda}^u$  shows that  $\bar{\mu}$  is the unique measure-mild solution of (3.17). In particular for every  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) u(t, x) dx &= \int_{\mathbb{R}^d} \varphi(x) \bar{\mu}(t, dx) \\ &= \int_{\mathbb{R}^d} u_0(dx_0) \int_{\mathbb{R}^d} P(0, x_0, t, dx) \varphi(x) dx \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} P(r, x_0, t, dx) \varphi(x) \right) \Lambda(r, x_0, u(r, x_0), \nabla u(r, x_0)) u(r, x_0) dx_0 dr. \end{aligned} \quad (3.19)$$

This shows that  $u$  is a mild solution of (1.4) and concludes the proof.  $\square$

We now precise more restrictive assumptions to ensure regularity properties of the transition probability function  $P(s, x_0, t, dx)$  used in the sequel.

**Assumption 2.** 1.  $\Phi$  and  $g$  are functions defined on  $[0, T] \times \mathbb{R}^d$  taking values respectively in  $M_{d,p}(\mathbb{R})$  and  $\mathbb{R}^d$ . There exist  $\alpha \in ]0, 1]$ ,  $C_\alpha, L_\Phi, L_g > 0$ , such that for any  $(t, t', x, x') \in [0, T] \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ ,

$$\begin{aligned} |\Phi(t, x) - \Phi(t', x')| &\leq C_\alpha |t - t'|^\alpha + L_\Phi |x - x'|, \\ |g(t, x) - g(t', x')| &\leq C_\alpha |t - t'|^\alpha + L_g |x - x'|. \end{aligned}$$

2.  $\Phi$  and  $g$  belong to  $C_b^{0,3}$ . In particular,  $\Phi, g$  are uniformly bounded and  $M_\Phi$  (resp.  $M_g$ ) denote the upper bound of  $|\Phi|$  (resp.  $|g|$ ).

3.  $\Phi$  is non-degenerate, i.e. there exists  $c > 0$  such that for all  $x \in \mathbb{R}^d$

$$\inf_{s \in [0, T]} \inf_{v \in \mathbb{R}^d \setminus \{0\}} \frac{\langle v, \Phi(s, x) \Phi^t(s, x) v \rangle}{|v|^2} \geq c > 0. \quad (3.20)$$

4.  $\Lambda$  is a Borel real-valued function defined on  $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$  and Lipschitz uniformly w.r.t.  $(t, x)$  i.e. there exists a finite positive real,  $L_\Lambda$ , such that for any  $(t, x, z_1, z'_1, z_2, z'_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$ , we have

$$|\Lambda(t, x, z_1, z_2) - \Lambda(t, x, z'_1, z'_2)| \leq L_\Lambda (|z_1 - z'_1| + |z_2 - z'_2|). \quad (3.21)$$

5.  $\Lambda$  is supposed to be uniformly bounded: let  $M_\Lambda$  be an upper bound for  $|\Lambda|$ .

6.  $u_0$  is a Borel probability measure on  $\mathbb{R}^d$  admitting a bounded density (still denoted by the same letter) belonging to  $W^{1,1}(\mathbb{R}^d)$ .

**Theorem 3.6.** Under Assumption 2, there exists a unique mild solution  $u$  of (1.4) in  $L^1([0, T], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$ .

The idea of the proof will be first to construct a unique "mild solution"  $u^k$  of (1.4) on each subintervals of the form  $[k\tau, (k+1)\tau]$  with  $k \in \{0, \dots, N-1\}$  and  $\tau > 0$  a constant supposed to be fixed for the moment. This will be the object of Lemma 3.7. Secondly we will show that the function  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ , defined by being equal to  $u^k$  on each  $[k\tau, (k+1)\tau]$ , is indeed a mild solution of (1.4) on  $[0, T] \times \mathbb{R}^d$ . This will be a consequence of Lemma 3.4. Finally, uniqueness will follow classically from Lipschitz property of  $\Lambda$ .

Let us fix  $\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . For  $r \in [0, T - \tau]$ , we first define a function  $\widehat{u}_0$  on  $[r, r + \tau] \times \mathbb{R}^d$  by setting

$$\widehat{u}_0(r, \phi)(t, x) := \int_{\mathbb{R}^d} p(r, x_0, t, x) \phi(x_0) dx_0, \quad (t, x) \in [r, r + \tau] \times \mathbb{R}^d. \quad (3.22)$$

Consider now the map  $\Pi : L^1([r, r + \tau], W^{1,1}(\mathbb{R}^d)) \rightarrow L^1([r, r + \tau], W^{1,1}(\mathbb{R}^d))$  given by

$$\Pi(v)(t, x) := \int_r^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x) \Lambda(s, x_0, v + \widehat{u}_0(r, \phi), \nabla(v + \widehat{u}_0(r, \phi)))(v + \widehat{u}_0(r, \phi))(s, x_0) dx_0, \quad (3.23)$$

$$\Lambda(t, z, v, \nabla v) := \Lambda(t, z, v(t, z), \nabla v(t, z)) \quad \text{with} \quad (t, z) \in [0, T] \times \mathbb{R}^d, \quad (3.24)$$

that will also be used in the sequel.

Later, the dependence on  $r, \phi$  will be omitted when it is self-explanatory. Since  $\phi$  and  $u_0$  belong to  $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , we have

$$\|\widehat{u}_0(t, \cdot)\|_1 \leq \|\phi\|_1 \quad \text{and} \quad \|\widehat{u}_0(t, \cdot)\|_\infty \leq \|\phi\|_\infty, \quad \text{if } t \in [r, r + \tau]. \quad (3.25)$$

The lemma below establishes, under a suitable choice of  $\tau > 0$ , existence and uniqueness of the mild solution on  $[r, r + \tau]$ , with initial condition  $\phi$  at time  $r$ , i.e. existence and uniqueness of the fixed-point for the application  $\Pi$ .

**Lemma 3.7.** *Assume the validity of Assumption 2. Let  $\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ .*

*Let  $M > 0$  such that  $M \geq \max(\|\phi\|_\infty; \|\phi\|_1)$ . Then, there is  $\tau > 0$  only depending on  $M_\Lambda$  and on  $C_u, c_u$  (the constants coming from inequality (6.17), only depending on  $\Phi, g$ ) such that for any  $r \in [0, T - \tau]$ ,  $\Pi$  admits a unique fixed-point in  $L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$ , where  $B(0, M)$  (resp.  $B_\infty(0, M)$ ) denotes the centered ball in  $W^{1,1}(\mathbb{R}^d)$  (resp.  $L^\infty([r, r + \tau] \times \mathbb{R}^d, \mathbb{R})$ ) of radius  $M$ .*

*Proof.* We first insist on the fact that all along the proof, the dependence of  $\widehat{u}_0$  w.r.t.  $r, \phi$  in (3.23) will be omitted to simplify notations. Let us fix  $r \in [0, T - \tau]$ .

By item 1. of Lemma 6.4, the transition probabilities are absolutely continuous and  $P(s, x_0, t, dx) = p(s, x_0, t, x)dx$  for some Borel function  $p$ . The rest of the proof relies on a fixed-point argument in the Banach space  $L^1([r, r + \tau], W^{1,1}(\mathbb{R}^d))$  equipped with the norm  $\|f\|_{1,1} := \int_r^{r+\tau} \|f(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds$  and for the map  $\Pi$  (3.23). Moreover, we emphasize that  $L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$  is complete as a closed subset of  $L^1([r, r + \tau], B(0, M))$ .

We first check that  $\Pi(L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)) \subset L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$ . Let us fix  $v \in L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$ . Indeed, for  $t \in [r, r + \tau]$ ,

$$\begin{aligned} \|\Pi(v)(t, \cdot)\|_1 &= \int_{\mathbb{R}^d} |\Pi(v)(t, x)| dx \\ &\leq M_\Lambda \int_r^t (\|v(s, \cdot)\|_1 + \|\widehat{u}_0(s, \cdot)\|_1) ds \\ &\leq 2M_\Lambda M \tau, \end{aligned} \tag{3.26}$$

where we have used the fact that  $x \mapsto p(s, x_0, t, x)$  is a probability density, the boundedness of  $\Lambda$  and the bounds  $\|v(s, \cdot)\|_1 \leq M$  and  $\|\widehat{u}_0(s, \cdot)\|_1 \leq M$  for  $s \in [r, r + \tau]$ .

Let us fix  $t \in [r, r + \tau]$ . Since the transition probability function  $x \mapsto p(s, x_0, t, x)$  is twice continuously differentiable for  $0 \leq s < t \leq T$  (see item 2. of Lemma 6.4) and taking into account inequality (6.17), we differentiate under the integral sign to obtain that  $\nabla \Pi(v)(t, \cdot)$  exists (in the sense of distributions) and is a real-valued function such that for almost all  $x \in \mathbb{R}^d$ ,

$$\nabla \Pi(v)(t, x) = \int_r^t ds \int_{\mathbb{R}^d} \nabla_x p(s, x_0, t, x) (v + \widehat{u}_0)(s, x_0) \Lambda(s, x_0, v + \widehat{u}_0, \nabla(v + \widehat{u}_0)) dx_0. \tag{3.27}$$

Integrating each side of (3.27) on  $\mathbb{R}^d$  w.r.t.  $dx$  and using inequality (6.17) (with  $(m_1, m_2) = (0, 1)$ ) yield

$$\begin{aligned} \|\nabla \Pi(v)(t, \cdot)\|_1 &= \int_{\mathbb{R}^d} |\nabla \Pi(v)(t, x)| dx \\ &\leq M_\Lambda \int_r^t \frac{ds}{\sqrt{t-s}} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} C_u \frac{e^{-c_u \frac{|x-x_0|}{t-s}}}{\sqrt{(t-s)^d}} (|v(s, x_0)| + |\widehat{u}_0(s, x_0)|) dx_0 \\ &= \hat{C} M_\Lambda \int_r^t \frac{ds}{\sqrt{t-s}} \int_{\mathbb{R}^d} (|v(s, x_0)| + |\widehat{u}_0(s, x_0)|) dx_0 \\ &\leq \hat{C} M_\Lambda \int_r^t (\|v(s, \cdot)\|_1 + \|\widehat{u}_0(s, \cdot)\|_1) \frac{ds}{\sqrt{t-s}} \\ &\leq 4\hat{C} M_\Lambda M \sqrt{\tau}, \end{aligned} \tag{3.28}$$

with  $\hat{C} := \hat{C}(C_u, c_u) > 0$  and  $C_u, c_u$  are the constants coming from inequality (6.17) and only depending on  $\Phi$  and  $g$ . Consequently, taking into account (3.26) and (3.28), we obtain,

$$\|\Pi(v)\|_{1,1} = \int_r^{r+\tau} \|\Pi(v)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} dt \leq 2MM_\Lambda(\tau^2 + 2\hat{C}\tau\sqrt{\tau}). \quad (3.29)$$

Moreover using again inequality (6.17), with  $(m_1, m_2) = (0, 0)$ , gives existence of a constant  $\bar{C} := \bar{C}(C_u, c_u)$  such that

$$\|\Pi(v)\|_\infty \leq 2\bar{C}MM_\Lambda\tau. \quad (3.30)$$

Now, setting

$$\tau := \min\left(\sqrt{\frac{1}{6M_\Lambda}}; \left(\frac{1}{12\hat{C}M_\Lambda}\right)^{\frac{2}{3}}; \frac{1}{6CM_\Lambda}\right), \quad (3.31)$$

we have

$$2MM_\Lambda(\tau^2 + 2\hat{C}\tau\sqrt{\tau}) \leq \frac{2M}{3} \quad \text{and} \quad 2\bar{C}MM_\Lambda\tau \leq \frac{M}{3},$$

which implies

$$\|\Pi(v)\|_{1,1} \leq M \quad \text{and} \quad \|\Pi(v)\|_\infty \leq M.$$

We deduce that  $\Pi(v) \in L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$ .

Let us fix  $t \in [r, r + \tau]$ ,  $v_1, v_2 \in L^1([r, r + \tau], B(0, M)) \cap B_\infty(0, M)$ .  $\Lambda$  being bounded and Lipschitz, the notation introduced in (3.24) and inequality (2.3) imply

$$\begin{aligned} \|\Pi(v_1)(t, \cdot) - \Pi(v_2)(t, \cdot)\|_1 &\leq \int_r^t ds \int_{\mathbb{R}^d} \left| v_1(s, x_0)\Lambda(s, x_0, v_1 + \widehat{u}_0, \nabla(v_1 + \widehat{u}_0)) - v_2(s, x_0)\Lambda(s, x_0, v_2 + \widehat{u}_0, \nabla(v_2 + \widehat{u}_0)) \right| dx_0 \\ &\quad + \int_r^t ds \int_{\mathbb{R}^d} |\widehat{u}_0(s, x_0)| \left| \Lambda(s, x_0, v_1 + \widehat{u}_0, \nabla(v_1 + \widehat{u}_0)) - \Lambda(s, x_0, v_2 + \widehat{u}_0, \nabla(v_2 + \widehat{u}_0)) \right| dx_0 \\ &\leq \int_r^t ds \left( \int_{\mathbb{R}^d} |v_1(s, x_0) - v_2(s, x_0)| |\Lambda(s, x_0, v_1 + \widehat{u}_0, \nabla(v_1 + \widehat{u}_0))| dx_0 \right. \\ &\quad \left. + L_\Lambda \int_r^t ds \int_{\mathbb{R}^d} (|\widehat{u}_0(s, x_0)| + |v_2(s, x_0)|) |v_1(s, x_0) - v_2(s, x_0)| dx_0 \right. \\ &\quad \left. + L_\Lambda \int_r^t ds \int_{\mathbb{R}^d} (|\widehat{u}_0(s, x_0)| + |v_2(s, x_0)|) |\nabla v_1(s, x_0) - \nabla v_2(s, x_0)| dx_0 \right) \\ &\leq (M_\Lambda + 2ML_\Lambda) \int_r^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds, \end{aligned} \quad (3.32)$$

where we have used the fact that  $\int_{\mathbb{R}^d} p(s, x_0, t, x) dx = 1$ ,  $0 \leq s < t \leq T$ .

In the same way and by using inequality (6.17) with  $(m_1, m_2) = (0, 1)$ ,

$$\begin{aligned} \left\| \nabla \left( \Pi(v_1) - \Pi(v_2) \right) (t, \cdot) \right\|_1 &\leq C_u(M_\Lambda + 2ML_\Lambda) \int_{\mathbb{R}^d} \int_r^t \int_{\mathbb{R}^d} \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{(t-s)^d}} e^{-c_u \frac{|x-x_0|^2}{t-s}} \left( |v_1(s, x_0) - v_2(s, x_0)| \right. \\ &\quad \left. + |\nabla v_1(s, x_0) - \nabla v_2(s, x_0)| \right) dx_0 ds dx. \end{aligned} \quad (3.33)$$

By Fubini's theorem we have

$$\left\| \nabla \left( \Pi(v_1) - \Pi(v_2) \right) (t, \cdot) \right\|_1 \leq \tilde{C} \int_r^t \frac{1}{\sqrt{t-s}} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds, \quad (3.34)$$

with  $\tilde{C} := \tilde{C}(C_u, c_u, M_\Lambda, L_\Lambda, M)$  some positive constant. From (3.32) and (3.34), we deduce there exists a strictly positive constant  $C = C(C_u, c_u, \Phi, g, \Lambda, M)$  (which may change from line to line) such that for all

$t \in [r, r + \tau]$ ,

$$\begin{aligned} \|\Pi(v_1)(t, \cdot) - \Pi(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} &\leq C \left\{ \int_r^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds \right. \\ &\quad \left. + \int_r^t \frac{1}{\sqrt{t-s}} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds \right\}. \end{aligned} \quad (3.35)$$

Iterating the procedure once again yields

$$\begin{aligned} \|\Pi^2(v_1)(t, \cdot) - \Pi^2(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} &\leq C \left\{ \int_r^t \int_r^s \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta ds \right. \\ &\quad \left. + \int_r^t \int_r^s \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta ds \right\}, \end{aligned} \quad (3.36)$$

for all  $t \in [r, r + \tau]$ . Interchanging the order in the second integral in the r.h.s. of (3.36), we obtain

$$\begin{aligned} \int_r^t \int_r^s \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta ds &= \int_r^t d\theta \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} \int_\theta^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s-\theta}} ds \\ &= \int_r^t d\theta \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} \int_0^\alpha \frac{1}{\sqrt{\alpha-\omega}} \frac{1}{\sqrt{\omega}} d\omega, \\ &\leq 4 \int_r^t \|v_1(\theta, \cdot) - v_2(\theta, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} d\theta, \end{aligned} \quad (3.37)$$

where the latter line above comes from the fact for all  $\theta > 0$ ,  $\int_0^\theta \frac{1}{\sqrt{\theta-\omega}} \frac{1}{\sqrt{\omega}} d\omega = \int_0^1 \frac{1}{\sqrt{1-\omega}} \frac{1}{\sqrt{\omega}} d\omega = \Gamma(\frac{1}{2})$ ,  $\Gamma$  denoting the Euler gamma function.

Injecting inequality (3.37) in (3.36), we obtain for all  $t \in [r, r + \tau]$

$$\|\Pi^2(v_1)(t, \cdot) - \Pi^2(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} \leq 5C \int_r^t \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.38)$$

Iterating previous inequality, one obtains the following. For all  $k \geq 1, t \in [r, r + \tau]$ ,

$$\|\Pi^{2k}(v_1)(t, \cdot) - \Pi^{2k}(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} \leq (5C)^k \int_r^t \frac{(t-s)^{k-1}}{(k-1)!} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.39)$$

By induction on  $k \geq 1$  (3.39) can indeed be established. Finally, by integrating each sides of (3.39) w.r.t.  $dt$  and using Fubini's theorem, for  $k \geq 1$ , we obtain

$$\int_r^{r+\tau} \|\Pi^{2k}(v_1)(t, \cdot) - \Pi^{2k}(v_2)(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} dt \leq (5C)^k \frac{T^{k-1}}{(k-1)!} \int_r^{r+\tau} \|v_1(s, \cdot) - v_2(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.40)$$

For  $k_0 \in \mathbb{N}$  large enough,  $\frac{(5CT)^{k_0}}{T^{(k_0-1)!}}$  will be strictly smaller than 1 and  $\Pi^{2k_0}$  will admit a unique fixed-point by Banach fixed-point theorem. In consequence, it implies easily that  $\Pi$  will also admit a unique fixed-point and this concludes the proof of Lemma 3.7.  $\square$

*Proof of Theorem 3.6.* Without restriction of generality, we can suppose there exists  $N \in \mathbb{N}^*$  such that  $T = N\tau$ , where we recall that  $\tau$  is given by (3.31). Similarly to the notations used in the preceding proof, in all the sequel, we agree that for  $M > 0$ ,  $B(0, M)$  (resp.  $B_\infty(0, M)$ ) denotes the centered ball of radius  $M$  in  $W^{1,1}(\mathbb{R}^d)$  (resp. in  $L^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$  or in  $L^\infty([r, r + \tau] \times \mathbb{R}^d, \mathbb{R})$  for  $r \in [0, T - \tau]$  according to the context).

The notations introduced in (3.24) will also be used in the present proof.

Indeed, for  $r = 0$ ,  $\phi = u_0$  and  $M \geq \max(\|u_0\|_\infty; \|u_0\|_1)$ , Lemma 3.7 implies there exists a unique function  $v^0 : [0, \tau] \times \mathbb{R}^d \rightarrow \mathbb{R}$  (belonging to  $L^1([0, \tau], B(0, M)) \cap B_\infty(0, M)$ ) such that for  $(t, x) \in [0, \tau] \times \mathbb{R}^d$ ,

$$v^0(t, x) = \int_0^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x) (v^0(s, x_0) + \widehat{u}_0^0(s, x_0)) \Lambda(s, x_0, v^0 + \widehat{u}_0^0, \nabla(v^0 + \widehat{u}_0^0)) dx_0, \quad (3.41)$$

where  $\widehat{u}_0^0(t, x)$  is given by (3.22) with  $\phi = u_0$ , i.e.

$$\widehat{u}_0^0(t, x) = \int_{\mathbb{R}^d} p(0, x_0, t, x) u_0(x_0) dx_0, \quad (t, x) \in [0, \tau] \times \mathbb{R}^d. \quad (3.42)$$

Setting  $u^0 := \widehat{u}_0^0 + v^0$ , i.e.

$$u^0(t, \cdot) = \int_{\mathbb{R}^d} p(0, x_0, t, \cdot) u_0(x_0) dx_0 + v^0(t, \cdot), \quad t \in [0, \tau], \quad (3.43)$$

it appears that  $u^0$  satisfies for all  $(t, x) \in [0, \tau] \times \mathbb{R}^d$

$$u^0(t, x) = \int_{\mathbb{R}^d} p(0, x_0, t, x) u_0(x_0) dx_0 + \int_0^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x) u^0(s, x_0) \Lambda(s, x_0, u^0, \nabla u^0) dx_0. \quad (3.44)$$

Let us fix  $k \in \{1, \dots, N-1\}$ . Suppose now given a family of functions  $u^1, u^2, \dots, u^{k-1}$ , where for all  $j \in \{1, \dots, k-1\}$ ,  $u^j \in L^1([j\tau, (j+1)\tau], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([j\tau, (j+1)\tau] \times \mathbb{R}^d, \mathbb{R})$  and satisfies for all  $(t, x) \in [j\tau, (j+1)\tau] \times \mathbb{R}^d$ ,

$$u^j(t, x) = \int_{\mathbb{R}^d} p(j\tau, x_0, t, x) u^{j-1}(j\tau, x_0) dx_0 + \int_{j\tau}^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x) u^j(s, x_0) \Lambda(s, x_0, u^j, \nabla u^j) dx_0. \quad (3.45)$$

Let us introduce

$$\begin{aligned} \widehat{u}_0^k(t, x) &:= \widehat{u}_0(u^{k-1})(t, x) \\ &= \int_{\mathbb{R}^d} p(k\tau, x_0, t, x) u^{k-1}(k\tau, x_0) dx_0, \quad (t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d, \end{aligned} \quad (3.46)$$

where the second inequality comes from (3.22) with  $r = k\tau$  and  $\phi = u^{k-1}(k\tau, \cdot)$ .

By choosing the real  $M$  large enough (i.e.  $M \geq \max(\|u^{k-1}(k\tau, \cdot)\|_\infty; \|u^{k-1}(k\tau, \cdot)\|_1)$ ), Lemma 3.7 applied with  $r = k\tau$ ,  $\phi = u^{k-1}(k\tau, \cdot)$  implies existence and uniqueness of a function  $v^k : [k\tau, (k+1)\tau] \times \mathbb{R}^d \rightarrow \mathbb{R}$  that belongs to  $L^1([k\tau, (k+1)\tau], B(0, M)) \cap B_\infty(0, M)$  and satisfying

$$v^k(t, x) = \int_{k\tau}^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x) (v^k(s, x_0) + \widehat{u}_0^k(s, x_0)) \Lambda(s, x_0, v^k + \widehat{u}_0^k, \nabla(v^k + \widehat{u}_0^k)) dx_0, \quad (3.47)$$

for all  $(t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d$ . Setting  $u^k := \widehat{u}_0^k + v^k$ , we have for all  $(t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d$

$$u^k(t, x) = \int_{\mathbb{R}^d} p(k\tau, x_0, t, x) u^{k-1}(k\tau, x_0) dx_0 + \int_{k\tau}^t ds \int_{\mathbb{R}^d} p(s, x_0, t, x) u^k(s, x_0) \Lambda(s, x_0, u^k, \nabla u^k) dx_0. \quad (3.48)$$

Consequently, by induction we can construct a family of functions  $(u^k : [k\tau, (k+1)\tau] \times \mathbb{R}^d \rightarrow \mathbb{R})_{k=0, \dots, N-1}$  such that for all  $k \in \{0, \dots, N-1\}$ ,  $u^k \in L^1([k\tau, (k+1)\tau], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([k\tau, (k+1)\tau] \times \mathbb{R}^d, \mathbb{R})$  and verifies (3.48).

We now consider the real-valued function  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  defined as being equal to  $u^k$  (defined by (3.48)) on each interval  $[k\tau, (k+1)\tau]$ . Then, Lemma 3.4 applied with  $\tau$  given by (3.31) and

$$\delta = (\alpha_0 := 0 < \dots < \alpha_k := k\tau < \dots < \alpha_N := T = N\tau) \quad , \quad \mu(t, dx) = u(t, x) dx, \quad (3.49)$$



shows that  $u$  is a mild solution of (1.4) on  $[0, T] \times \mathbb{R}^d$ , in the sense of Definition 2.1, item 2. It now remains to ensure that  $u$  is indeed the unique mild solution of (1.4) on  $[0, T] \times \mathbb{R}^d$  belonging to  $L^1([0, T], W^{1,1}(\mathbb{R}^d)) \cap L^\infty([0, T] \times \mathbb{R}^d, \mathbb{R})$ . This follows, in a classical way, by boundedness and Lipschitz property of  $\Lambda$ .

Indeed, if  $U, V$  are two mild solutions of (1.4), then very similar computations as the ones done in (3.32), (3.34), and (3.39) to obtain (3.40) give the following. There exists  $C := C(\Phi, g, \Lambda, U, V) > 0$  such that

$$\int_0^T \|U(t, \cdot) - V(t, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} dt \leq (5C)^j \frac{T^{j-1}}{(j-1)!} \int_0^T \|U(s, \cdot) - V(s, \cdot)\|_{W^{1,1}(\mathbb{R}^d)} ds. \quad (3.50)$$

If we choose  $j \in \mathbb{N}^*$  large enough so that  $(5C)^j \frac{T^{j-1}}{(j-1)!} < 1$ , we obtain  $U(t, x) = V(t, x)$  for almost all  $(t, x) \in [0, T] \times \mathbb{R}^d$ . This concludes the proof of Theorem 3.6.  $\square$

**Corollary 3.8.** *Under Assumption 2, there exists a unique function  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the Feynman-Kac equation (3.15). In particular, such  $u$  coincides with the mild solution of (1.4).*

In the case where the function  $\Lambda$  does not depend on  $\nabla u$ , existence and uniqueness of a solution of (1.4) in the mild sense can be proved under weaker assumptions. This is the object of the following result.

**Theorem 3.9.** *Assume that Assumption 1 is satisfied. Let  $u_0 \in \mathcal{P}(\mathbb{R}^d)$  admitting a bounded density (still denoted by the same letter). Let  $Y$  the the strong solution of (2.8) with prescribed  $Y_0$ .*

*We suppose that the transition probability function  $P$  (see (2.7)) admits a density  $p$  such that  $P(s, x_0, t, dx) = p(s, x_0, t, x)dx$ , for all  $s, t \in [0, T]$ ,  $x_0 \in \mathbb{R}^d$ .  $\Lambda$  is supposed to satisfy items 4. and 5. of Assumption 2. Then, there exists a unique mild solution  $u$  of (1.4) in  $L^1([0, T], L^1(\mathbb{R}^d))$ , i.e.  $u$  satisfies*

$$u(t, x) = \int_{\mathbb{R}^d} p(0, x_0, t, x) u_0(x_0) dx_0 + \int_0^t \int_{\mathbb{R}^d} p(s, x_0, t, x) u(s, x_0) \Lambda(s, x_0, u(s, x_0)) dx_0 ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (3.51)$$

*Proof.* Since this theorem can be proved in a very similar way as Theorem 3.6 but with simpler computations, we omit the details.  $\square$

## 4 Existence/uniqueness of the Regularized Feynman-Kac equation

In this section, we introduce a regularized version of PDE (1.4) to which we associate a regularized Feynman-Kac equation corresponding to a regularized version of (3.15). This regularization procedure constitutes a preliminary step for the construction of a particle scheme approximating (3.15). Indeed, as detailed in the next section devoted to the particle approximation, the point dependence of  $\Lambda$  on  $u$  and  $\nabla u$  will require to derive from a discrete measure (based on the particle system) estimates of densities  $u$  and their derivatives  $\nabla u$ , which can classically be achieved by kernel convolution.

Assumption 1 is in force. Let  $u_0$  be a Borel probability measure on  $\mathbb{R}^d$  and  $Y_0$  a random variable distributed according to  $u_0$ . We consider  $Y$  the strong solution of the SDE (2.8).

Let us consider  $(K_\varepsilon)_{\varepsilon>0}$ , a sequence of mollifiers such that

$$K_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \delta_0, \text{ (weakly)} \quad \text{and} \quad \forall \varepsilon > 0, K_\varepsilon \in W^{1,1}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d). \quad (4.1)$$

Let  $\Lambda : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  be bounded, Borel measurable. As announced, we introduce the following integro-PDE corresponding to a regularized version of (1.4)

$$\begin{cases} \partial_t \gamma_t = L_t^* \gamma_t + \gamma_t \Lambda(t, x, K_\varepsilon * \gamma_t, \nabla K_\varepsilon * \gamma_t) \\ \gamma_0 = u_0. \end{cases} \quad (4.2)$$

The concept of mild solution associated to this type of equation is clarified by the following definition.

**Definition 4.1.** A Borel measure-valued function  $\gamma : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$  will be called a **mild solution** of (4.2) if it satisfies, for all  $\varphi \in C_0^\infty(\mathbb{R}^d)$ ,  $t \in [0, T]$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) \gamma(t, dx) &= \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} u_0(dx_0) P(0, x_0, t, dx) \\ &+ \int_{[0, t] \times \mathbb{R}^d} \left( \int_{\mathbb{R}^d} \varphi(x) P(s, x_0, t, dx) \right) \Lambda(s, x_0, (K_\varepsilon * \gamma(s, \cdot))(x_0), (\nabla K_\varepsilon * \gamma(s, \cdot))(x_0)) \gamma(s, dx_0) ds. \end{aligned} \quad (4.3)$$

Similarly as Theorem 3.5, we straightforwardly derive the following equivalence result.

**Proposition 4.2.** Suppose that Assumption 1 and (4.1) are fulfilled. We indicate by  $Y$  the unique strong solution of (2.8). A Borel measure-valued function  $\gamma^\varepsilon : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$  is a mild solution of (4.2) if and only if, for all  $\varphi \in C_b(\mathbb{R}^d)$ ,  $t \in [0, T]$ ,

$$\int_{\mathbb{R}^d} \varphi(x) \gamma_t^\varepsilon(dx) = \mathbb{E} \left[ \varphi(Y_t) \exp \left( \int_0^t \Lambda(s, Y_s, (K_\varepsilon * \gamma_s^\varepsilon)(Y_s), (\nabla K_\varepsilon * \gamma_s^\varepsilon)(Y_s)) ds \right) \right]. \quad (4.4)$$

*Proof.* The proof follows the same lines as the proof of Theorem 3.5. First assume that  $\gamma^\varepsilon$  satisfies (4.4), we can show that  $\gamma^\varepsilon$  is a mild solution (4.2) by imitating the first step of the proof of Proposition 3.3. Secondly, the converse is proved by applying Proposition 3.3 with  $\tilde{\Lambda}(t, x) := \Lambda(t, x, (K_\varepsilon * \gamma_t^\varepsilon)(x), (\nabla K_\varepsilon * \gamma_t^\varepsilon)(x))$  and  $\mu(t, dx) := \gamma_t^\varepsilon(dx)$ .  $\square$

Let us now prove existence and uniqueness of a mild solution for the integro-PDE (4.2). To this end, we proceed similarly as for the proof of Theorem 3.6 using Lemma 3.7. Let  $\tau > 0$  be a constant supposed to be fixed for the moment and let us fix  $\varepsilon > 0$ ,  $r \in [0, T - \tau]$ .  $\mathcal{B}([r, r + \tau], \mathcal{M}_f(\mathbb{R}^d))$  denotes the space of bounded, measure-valued maps, where  $\mathcal{M}_f(\mathbb{R}^d)$  is equipped with the total variation norm  $\|\cdot\|_{TV}$ . We introduce the measure-valued application  $\Pi_\varepsilon : \beta \in \mathcal{B}([r, r + \tau], \mathcal{M}_f(\mathbb{R}^d)) \rightarrow \Pi_\varepsilon(\beta)$ , defined by

$$\begin{aligned} \Pi_\varepsilon(\beta)(t, dx) &= \int_r^t \int_{\mathbb{R}^d} P(s, x_0, t, dx) \Lambda(s, x_0, (K_\varepsilon * \hat{\beta}(s, \cdot))(x_0), (\nabla K_\varepsilon * \hat{\beta}(s, \cdot))(x_0)) \hat{\beta}(s, dx_0) ds \\ \hat{\beta}(s, \cdot) &= \beta(s, \cdot) + \hat{u}_0(s, \cdot), \end{aligned} \quad (4.5)$$

where the function  $\hat{u}_0$ , defined on  $[r, r + \tau] \times \mathcal{M}_f(\mathbb{R}^d)$ , is given by

$$\hat{u}_0(r, \pi)(t, dx) := \int_{\mathbb{R}^d} p(r, x_0, t, dx) \pi(dx_0), \quad t \in [r, r + \tau], \quad \pi \in \mathcal{M}_f(\mathbb{R}^d), \quad (4.6)$$

similarly to (3.22). In the sequel, the dependence of  $\hat{u}_0$  w.r.t.  $r, \pi$  will be omitted when it is self-explanatory.

**Lemma 4.3.** Assume the validity of items 4. and 5. of Assumption 2 and of (4.1). Let  $\pi \in B(0, M)$ .

Let us fix  $\varepsilon > 0$  and  $M > 0$  such that  $M \geq \|\pi\|_{TV}$ . Then, there is  $\tau > 0$  only depending on  $M_\Lambda$  such that for any  $r \in [0, T - \tau]$ ,  $\Pi_\varepsilon$  admits a unique fixed-point in  $\mathcal{B}([r, r + \tau], B(0, M))$ , where  $B(0, M)$  denotes here the centered ball in  $(\mathcal{M}_f(\mathbb{R}^d), \|\cdot\|_{TV})$  with radius  $M$ .

*Proof.* Let us define  $\tau := \frac{1}{2M_\Lambda}$ . For every  $\lambda \geq 0$ ,  $\mathcal{B}([r, r + \tau], \mathcal{M}_f(\mathbb{R}^d))$  will be equipped with one of the equivalent norms

$$\|\beta\|_{TV, \lambda} := \sup_{t \in [r, r + \tau]} e^{-\lambda t} \|\beta(t, \cdot)\|_{TV}. \quad (4.7)$$

Recalling (4.5), where  $\widehat{u}_0$  is defined by (4.6), it follows that for all  $\beta \in \mathcal{B}([r, r + \tau], B(0, M))$ ,  $t \in [r, r + \tau]$ ,

$$\|\Pi_\varepsilon(\beta)(t, \cdot)\|_{TV} \leq M_\Lambda \int_r^t \|\beta(s, \cdot)\|_{TV} ds + M_\Lambda M \tau \leq 2MM_\Lambda \tau \leq M, \quad (4.8)$$

where for the latter inequality of (4.8) we have used the definition of  $\tau := \frac{1}{2M_\Lambda}$ . We deduce that  $\Pi(\mathcal{B}([r, r + \tau], B(0, M))) \subset \mathcal{B}([r, r + \tau], B(0, M))$ .

Consider now  $\beta^1, \beta^2 \in \mathcal{B}([r, r + \tau], B(0, M))$ . For all  $\lambda > 0$  we have

$$\begin{aligned} \|\Pi_\varepsilon(\beta^1(t, \cdot)) - \Pi_\varepsilon(\beta^2(t, \cdot))\|_{TV} &\leq \int_0^t \|\beta^1(s, \cdot) - \beta^2(s, \cdot)\|_{TV} (L_\Lambda \|K_\varepsilon\|_\infty \|\beta^1(s, \cdot)\|_{TV} + M_\Lambda) ds \\ &\quad + L_\Lambda \|\nabla K_\varepsilon\|_\infty \int_r^t \|\beta^1(s, \cdot)\|_{TV} \|\beta^1(s, \cdot) - \beta^2(s, \cdot)\|_{TV} ds \\ &\quad + L_\Lambda (\|K_\varepsilon\|_\infty + \|\nabla K_\varepsilon\|_\infty) \int_r^t \|\widehat{u}_0(s, \cdot)\|_{TV} \|\beta^1(s, \cdot) - \beta^2(s, \cdot)\|_{TV} ds \\ &\leq C_{\varepsilon, T} \int_0^t \|\beta^1(s, \cdot) - \beta^2(s, \cdot)\|_{TV} ds \\ &\leq C_{\varepsilon, T} \int_0^t e^{s\lambda} \|\beta^1 - \beta^2\|_{TV, \lambda} ds \\ &= C_{\varepsilon, T} \|\beta^1 - \beta^2\|_{TV, \lambda} \frac{e^{\lambda t} - 1}{\lambda}, \end{aligned} \quad (4.9)$$

with  $C_{\varepsilon, T} := 2L_\Lambda M (\|K_\varepsilon\|_\infty + \|\nabla K_\varepsilon\|_\infty) + M_\Lambda$ . It follows

$$\begin{aligned} \|\Pi_\varepsilon(\beta^1) - \Pi_\varepsilon(\beta^2)\|_{TV, \lambda} &= \sup_{t \in [r, r + \tau]} e^{-\lambda t} \|\Pi(\beta^1)(t, \cdot) - \Pi(\beta^2)(t, \cdot)\|_{TV} \\ &\leq C_{\varepsilon, T} \|\beta^1 - \beta^2\|_{TV, \lambda} \sup_{t \geq 0} \left( \frac{1 - e^{-\lambda t}}{\lambda} \right) \\ &\leq \frac{C_{\varepsilon, T}}{\lambda} \|\beta^1 - \beta^2\|_{TV, \lambda}. \end{aligned} \quad (4.10)$$

Hence, taking  $\lambda > C_{\varepsilon, T}$ ,  $\Pi_\varepsilon$  is a contraction on  $\mathcal{B}([r, r + \tau], B(0, M))$ .

Since  $\mathcal{B}([r, r + \tau], (\mathcal{M}_f(\mathbb{R}^d), \|\cdot\|_{TV, \lambda}))$  is a Banach space whose  $\mathcal{B}([r, r + \tau], B(0, M))$  is a closed subset, the proof ends by a simple application of Banach fixed-point theorem.  $\square$

The next step is to show how the proposition above, with the help of Lemma 3.4, permits us to construct a mild solution of (4.2). The reasoning is similar to the one explained in the proof of Theorem 3.6. Indeed, without restriction of generality, we can suppose there exists  $N \in \mathbb{N}^*$  such that  $T = N\tau$ . Then, for all  $k = 0, \dots, N - 1$ , Lemma 4.3 applied on each interval  $[k\tau, (k + 1)\tau]$  (with  $r = k\tau$ ,  $\pi = \beta_\varepsilon^{k-1}(k\tau, \cdot)$  for  $k \geq 1$  and  $\pi = u_0$  for  $k = 0$ ) gives existence of a family of measure-valued maps  $(\beta_\varepsilon^k : [k\tau, (k + 1)\tau] \rightarrow \mathcal{M}_f(\mathbb{R}^d), k = 0, \dots, N - 1)$  defined by

$$\begin{aligned} \beta_\varepsilon^k(t, dx) &= \int_{k\tau}^t \int_{\mathbb{R}^d} P(s, x_0, t, dx) \Lambda(s, x_0, (K_\varepsilon * \widehat{\beta}_\varepsilon^k(s, \cdot))(x_0), (\nabla K_\varepsilon * \widehat{\beta}_\varepsilon^k(s, \cdot))(x_0)) \widehat{\beta}_\varepsilon^k(s, dx_0) ds. \\ \widehat{\beta}_\varepsilon^k(s, \cdot) &= \beta_\varepsilon^k(s, \cdot) + \widehat{u}_0^k(s, \cdot), \end{aligned} \quad (4.11)$$

where for  $k = 0, t \in [0, \tau]$ ,

$$\widehat{u}_0^0(t, dx) = \int_{\mathbb{R}^d} P(0, x_0, t, dx) u_0(dx_0), \quad \text{by (4.6) with } \pi = u_0, \quad (4.12)$$

and for all  $k \in \{1, \dots, N\}$ ,  $t \in [k\tau, (k+1)\tau]$ ,

$$\begin{aligned} \widehat{u}_0^k(t, dx) &:= \widehat{u}_0(\beta_\varepsilon^{k-1})(t, dx) \\ &= \int_{\mathbb{R}^d} P(k\tau, x_0, t, dx) \beta_\varepsilon^{k-1}(k\tau, dx_0), \quad \text{by (4.6) with } \pi = \beta_\varepsilon^{k-1}(k\tau, \cdot). \end{aligned} \quad (4.13)$$

We now consider the following measure-valued maps  $\widehat{U}_0 : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$  and  $\beta_\varepsilon : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$  defined by their restrictions on each interval  $[k\tau, (k+1)\tau]$ ,  $k = 0, \dots, N-1$  such that

$$\widehat{U}_0(t, x) := \widehat{u}_0^k(t, x) \quad \text{and} \quad \beta_\varepsilon(t, x) := \beta_\varepsilon^k(t, x) \quad \text{for } (t, x) \in [k\tau, (k+1)\tau] \times \mathbb{R}^d, \quad (4.14)$$

and we finally define  $\gamma^\varepsilon : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$  by

$$\gamma^\varepsilon := \widehat{U}_0 + \beta_\varepsilon \text{ on } [0, T] \times \mathbb{R}^d. \quad (4.15)$$

To ensure that  $\gamma^\varepsilon$  is indeed a mild solution on  $[0, T] \times \mathbb{R}^d$  (in the sense of Definition 4.1) of the integro-PDE (4.2), it is enough to apply Lemma 3.4 with  $\tau := \frac{1}{2M_\Lambda}$ ,  $\mu(t, dx) := \gamma^\varepsilon(t, dx)$  and  $(\alpha_k := k\tau)_{k=0, \dots, N}$ .

Previous discussion leads us to the following proposition.

**Proposition 4.4.** *Suppose the validity of Assumption 1 and items 4. and 5. of Assumption 2. Suppose also that (4.1) is fulfilled. Let us fix  $\varepsilon > 0$  and let  $\gamma^\varepsilon$  denote the map defined by (4.15). The following statements hold.*

1.  $\gamma^\varepsilon$  is the unique mild solution of the integro-PDE (4.2), see Definition 4.1.
2.  $\gamma^\varepsilon$  is the unique solution to the regularized Feynman-Kac equation (4.4).

*Proof.* The existence of a mild solution  $\gamma^\varepsilon$  of (4.2) has already been proved through the discussion just above. It remains to justify uniqueness. Consider  $\gamma^{\varepsilon,1}, \gamma^{\varepsilon,2}$  be two mild solutions of (4.3). Then, with similar computations as the ones leading to inequality (4.10), there exists a constant  $\mathfrak{C} := \mathfrak{C}(M_\Lambda, L_\Lambda, \|K_\varepsilon\|_\infty, \|\nabla K_\varepsilon\|_\infty) > 0$  such that

$$\|\gamma^{\varepsilon,1} - \gamma^{\varepsilon,2}\|_{TV,\lambda} \leq \frac{\mathfrak{C}}{\lambda} \|\gamma^{\varepsilon,1} - \gamma^{\varepsilon,2}\|_{TV,\lambda}, \quad (4.16)$$

for all  $\lambda > 0$  and where we recall that  $\|\cdot\|_{TV,\lambda}$  has been defined by (4.7). Taking  $\lambda > \mathfrak{C}$ , uniqueness follows. This shows item 1. Item 2. follows then by Proposition 4.2.  $\square$

The theorem below states the convergence of the solution of the regularized Feynman-Kac equation (4.4) to the solution to the Feynman-Kac equation (3.15). This is equivalent to the convergence of the solution of the regularized PDE (4.2) to solution of the target PDE (1.4), when the regularization parameter  $\varepsilon$  goes to zero.

**Theorem 4.5.** *Suppose the validity of Assumption 2. Suppose also that (4.1) is fulfilled. For any  $\varepsilon > 0$ , consider the real valued function  $u^\varepsilon$  such that for any  $t \in [0, T]$ ,*

$$u^\varepsilon(t, \cdot) := K_\varepsilon * \gamma_t^\varepsilon, \quad (4.17)$$

where  $\gamma^\varepsilon$  is the unique solution of both (4.4) and (4.2). Then  $u^\varepsilon$  converges to  $u$ , the unique solution of both (3.15) and (1.4), in the sense that

$$\|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 + \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1 \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \text{for any } t \in [0, T]. \quad (4.18)$$

Before proving Theorem 4.5, we state and prove a preliminary lemma.

**Lemma 4.6.** *Suppose the validity of Assumption 2 and of (4.1). Consider  $u$  the unique solution of (3.15), then for all  $t \in [0, T]$*

$$u(t, x) = F_0(t, x) + \int_0^t \mathbb{E} \left[ p(s, Y_s, t, x) \Lambda(s, Y_s, u, \nabla u) e^{\int_0^s \Lambda(r, Y_r, u, \nabla u) dr} \right] ds, \quad dx \text{ a.e.} \quad (4.19)$$

For a given  $\varepsilon > 0$ , consider  $u^\varepsilon$  defined by (4.17). Then for almost all  $x \in \mathbb{R}^d$  and all  $t \in [0, T]$ ,

$$u^\varepsilon(t, x) = (K_\varepsilon * F_0(t, \cdot))(x) + \int_0^t \mathbb{E} \left[ (K_\varepsilon * p(s, Y_s, t, \cdot))(x) \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^s \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) dr} \right] ds, \quad (4.20)$$

where  $F_0(t, x) := \int_{\mathbb{R}^d} p(0, x_0, t, x) u_0(x_0) dx_0$  for  $t > 0, x \in \mathbb{R}^d$  and  $F_0(0, \cdot) := u_0$ . We remark that we have used again the notation

$$\Lambda(s, \cdot, v, \nabla v) := \Lambda(s, \cdot, v(s, \cdot), \nabla v(s, \cdot)), \quad t \in [0, T], \quad (4.21)$$

for  $v \in L^1([0, T], W^{1,1}(\mathbb{R}^d))$ .

*Proof.* Equalities (4.19) and (4.20) are proved in a very similar way, so we only provide the proof of equation (4.20).

We observe that for all  $t \in [0, T], v \in L^1([0, T], W^{1,1}(\mathbb{R}^d))$ ,

$$e^{\int_0^t \Lambda(r, Y_r, v(r, Y_r), \nabla v(r, Y_r)) dr} = 1 + \int_0^t \Lambda(r, Y_r, v(r, Y_r), \nabla v(r, Y_r)) e^{\int_0^r \Lambda(s, Y_s, v(s, Y_s), \nabla v(s, Y_s)) ds} ds. \quad (4.22)$$

Taking into account the notation introduced in (4.21), (4.22) above implies for almost all  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} u^\varepsilon(t, x) &= \mathbb{E} \left[ K_\varepsilon(x - Y_t) \exp \left\{ \int_0^t \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds \right\} \right] \\ &= \mathbb{E} \left[ K_\varepsilon(x - Y_t) \right] + \int_0^t \mathbb{E} \left[ K_\varepsilon(x - Y_t) \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^r \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds} \right] dr \\ &= \int_{\mathbb{R}^d} K_\varepsilon(x - y) \int_{\mathbb{R}^d} p(0, x_0, t, y) u_0(x_0) dx_0 dy + \\ &\quad \int_0^t \mathbb{E} \left[ \mathbb{E} \left[ K_\varepsilon(x - Y_t) \middle| Y_r \right] \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^r \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds} \right] dr \\ &= (K_\varepsilon * F_0)(t, \cdot)(x) + \\ &\quad \int_0^t \mathbb{E} \left[ \left( \int_{\mathbb{R}^d} K_\varepsilon(x - y) p(r, Y_r, t, y) dy \right) \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^r \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds} \right] dr \\ &= (K_\varepsilon * F_0)(t, \cdot)(x) + \\ &\quad \int_0^t \mathbb{E} \left[ (K_\varepsilon * p(r, Y_r, t, \cdot))(x) \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^r \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) ds} \right] dr. \end{aligned} \quad (4.23)$$

This ends the proof.  $\square$

*Proof of Theorem 4.5.* In this proof,  $C$  denotes a real constant that may change from line to line, only depending on  $M_\Lambda, L_\Lambda, C_u, c_u$  and  $\|u_0\|_\infty$ , where we recall that the constants  $C_u, c_u$  come from inequality (6.17) (and only depend on  $\Phi, g$ ).

We first observe that for  $t = 0$ , the convergence of  $u^\varepsilon(0, \cdot)$  (resp.  $\nabla u^\varepsilon(0, \cdot)$ ) to  $u(0, \cdot)$  (resp.  $\nabla u(0, \cdot)$ ) in  $L^1(\mathbb{R}^d)$ -norm when  $\varepsilon$  goes to 0 is clear. Let us fix  $t \in (0, T]$ .

By Lemma 4.6, for almost all  $x \in \mathbb{R}^d$ , we have the decomposition

$$\begin{aligned}
u^\varepsilon(t, x) - u(t, x) &= (K_\varepsilon * F_0(t, \cdot))(x) - F_0(t, x) + \\
&\int_0^t \mathbb{E} \left[ \left\{ (K_\varepsilon * p(s, Y_s, t, \cdot))(x) - p(s, Y_s, t, x) \right\} \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^s \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) dr} \right] ds + \\
&\int_0^t \mathbb{E} \left[ p(s, Y_s, t, x) \left\{ \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^s \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) dr} - \Lambda(s, Y_s, u, \nabla u) e^{\int_0^s \Lambda(r, Y_r, u, \nabla u) dr} \right\} \right] ds.
\end{aligned} \tag{4.24}$$

By integrating the absolute value of both sides of (4.24) w.r.t.  $dx$ , it follows there exists a constant  $C > 0$  such that

$$\begin{aligned}
\|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 &\leq C \left\{ \|K_\varepsilon * F_0 - F_0\|_1 + \int_0^t \mathbb{E} \left[ \|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds \right. \\
&\quad + \int_0^t \mathbb{E} \left[ |u^\varepsilon(s, Y_s) - u(s, Y_s)| + |\nabla u^\varepsilon(s, Y_s) - \nabla u(s, Y_s)| \right] ds \\
&\quad \left. + \int_0^t \int_0^s \mathbb{E} \left[ |u^\varepsilon(r, Y_r) - u(r, Y_r)| + |\nabla u^\varepsilon(r, Y_r) - \nabla u(r, Y_r)| \right] dr ds \right\}. \tag{4.25}
\end{aligned}$$

Moreover, denoting  $p_s$  the law density of  $Y_s$ , by inequality (6.18) of Lemma 6.4 we get

$$\begin{aligned}
\mathbb{E} \left[ |u^\varepsilon(s, Y_s) - u(s, Y_s)| \right] &= \int_{\mathbb{R}^d} |u^\varepsilon(s, x) - u(s, x)| p_s(x) dx \\
&\leq C \|u_0\|_\infty \int_{\mathbb{R}^d} |u^\varepsilon(s, x) - u(s, x)| dx \\
&= C \|u^\varepsilon(s, \cdot) - u(s, \cdot)\|_1, \quad s \in [0, T], \tag{4.26}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E} \left[ |\nabla u^\varepsilon(s, Y_s) - \nabla u(s, Y_s)| \right] &= \int_{\mathbb{R}^d} |\nabla u^\varepsilon(s, x) - \nabla u(s, x)| p_s(x) dx \\
&\leq C \|u_0\|_\infty \int_{\mathbb{R}^d} |\nabla u^\varepsilon(s, x) - \nabla u(s, x)| dx \\
&= C \|\nabla u^\varepsilon(s, \cdot) - \nabla u(s, \cdot)\|_1, \quad s \in [0, T]. \tag{4.27}
\end{aligned}$$

Injecting (4.26) and (4.27) into the r.h.s. of (4.25), it comes

$$\begin{aligned}
\|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 &\leq C \left\{ \|K_\varepsilon * F_0 - F_0\|_1 + \int_0^t \mathbb{E} \left[ \|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds \right. \\
&\quad \left. + \int_0^t \|u^\varepsilon(s, \cdot) - u(s, \cdot)\|_1 + \|\nabla u^\varepsilon(s, \cdot) - \nabla u(s, \cdot)\|_1 ds \right\}. \tag{4.28}
\end{aligned}$$

Now, we need to bound  $\|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1$ . To this end, we can remark that for almost all  $x \in \mathbb{R}^d$ ,

$$\nabla u(t, x) = \nabla F_0(t, x) + \int_0^t \mathbb{E} \left[ \nabla_x p(s, Y_s, t, x) \Lambda(s, Y_s, u, \nabla u) e^{\int_0^s \Lambda(r, Y_r, u, \nabla u) dr} \right] ds, \tag{4.29}$$

and

$$\begin{aligned}
\nabla u^\varepsilon(t, x) &= (K_\varepsilon * \nabla F_0(t, \cdot))(x) \\
&\quad + \int_0^t \mathbb{E} \left[ (K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot))(x) \Lambda(s, Y_s, u^\varepsilon, \nabla u^\varepsilon) e^{\int_0^s \Lambda(r, Y_r, u^\varepsilon, \nabla u^\varepsilon) dr} \right] ds. \tag{4.30}
\end{aligned}$$

These equalities follow by computing the derivative of  $u(t, \cdot)$  and  $u^\varepsilon(t, \cdot)$  in the sense of distributions. Taking into account (4.29) and (4.30), it is easy to see that very similar arguments as those invoked above to prove (4.28), lead to

$$\begin{aligned} \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1 &\leq C \left\{ \|K_\varepsilon * \nabla F_0(t, \cdot) - \nabla F_0(t, \cdot)\|_1 + \int_0^t \mathbb{E} \left[ \|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \right] ds \right. \\ &\quad \left. + \int_0^t \|u^\varepsilon(s, \cdot) - u(s, \cdot)\|_1 + \|\nabla u^\varepsilon(s, \cdot) - \nabla u(s, \cdot)\|_1 ds \right\}. \end{aligned} \quad (4.31)$$

Gathering (4.28) together with (4.31) yields

$$\begin{aligned} \|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 + \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1 &\leq C \left\{ \|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 + \|K_\varepsilon * \nabla F_0(t, \cdot) - \nabla F_0(t, \cdot)\|_1 + \right. \\ &\quad \int_0^t \mathbb{E} \left[ \|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds + \\ &\quad \int_0^t \mathbb{E} \left[ \|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \right] ds \left. \right\} \\ &\quad + \int_0^t \|u^\varepsilon(s, \cdot) - u(s, \cdot)\|_1 + \|\nabla u^\varepsilon(s, \cdot) - \nabla u(s, \cdot)\|_1 ds. \end{aligned} \quad (4.32)$$

Applying Gronwall's lemma to the real-valued function

$$t \mapsto \|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 + \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1,$$

we obtain

$$\begin{aligned} \|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 + \|\nabla u^\varepsilon(t, \cdot) - \nabla u(t, \cdot)\|_1 &\leq C e^{CT} \left\{ \|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 + \|K_\varepsilon * \nabla F_0(t, \cdot) - \nabla F_0(t, \cdot)\|_1 + \right. \\ &\quad \int_0^t \mathbb{E} \left[ \|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds + \\ &\quad \left. \int_0^t \mathbb{E} \left[ \|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \right] ds \right\}. \end{aligned} \quad (4.33)$$

Since  $F_0(t, \cdot)$ ,  $\nabla F_0(t, \cdot)$ ,  $x \mapsto p(s, x_0, t, x)$  and  $x \mapsto \nabla_x p(s, x_0, t, x)$  belong to  $L^1(\mathbb{R}^d)$ , classical properties of convergence of the mollifiers give

$$\|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 \rightarrow 0, \quad \|K_\varepsilon * \nabla F_0(t, \cdot) - \nabla F_0(t, \cdot)\|_1 \rightarrow 0, \quad (4.34)$$

and

$$\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \rightarrow 0, \quad \|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \rightarrow 0, \quad a.s. \quad (4.35)$$

Moreover, by inequality (6.17) of Lemma 6.4, there exists a constant  $C := C(C_u, c_u) > 0$  such that for  $0 \leq s < t \leq T$ ,

$$\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 + \|K_\varepsilon * \nabla_x p(s, Y_s, t, \cdot) - \nabla_x p(s, Y_s, t, \cdot)\|_1 \leq 2C \left(1 + \frac{1}{\sqrt{t-s}}\right) a.s. \quad (4.36)$$

Lebesgue dominated convergence theorem then implies that the third and fourth terms in the r.h.s. of (4.33) converge to 0 when  $\varepsilon$  goes to 0. This ends the proof.  $\square$



**Proposition 4.7.** We assume here that  $(K_\varepsilon)_{\varepsilon>0}$  is explicitly given by

$$K_\varepsilon(x) := \frac{1}{\varepsilon^d} K\left(\frac{x}{\varepsilon}\right), \quad (4.37)$$

with  $K \geq 0$  satisfying

$$\int_{\mathbb{R}^d} K(x) dx = 1, \quad \int_{\mathbb{R}^d} x K(x) dx = 0 \quad \text{and} \quad \kappa := \frac{1}{2} \int_{\mathbb{R}^d} |x| K(x) dx < \infty. \quad (4.38)$$

Let  $u^\varepsilon$  be the real-valued function defined by (4.17), (such that for any  $t \in [0, T]$ ,  $u^\varepsilon(t, \cdot) := K_\varepsilon * \gamma_t^\varepsilon$ ) with  $K_\varepsilon$  given by (4.37). Under Assumption 2 and in the particular case where the function  $\Lambda(t, x, u)$  does not depend on the gradient  $\nabla u$ , there exists a constant  $C := C(\kappa, C_u, c_u) > 0$  such that, for all  $t \in (0, T]$ ,

$$\|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 \leq \varepsilon C \left( \frac{1}{\sqrt{t}} + 2\sqrt{t} \right), \quad (4.39)$$

with  $C_u, c_u$  denoting the constants given by (6.17) (only depending on  $\Phi, g$ ).

*Proof.* This proof is based on the same arguments as the ones used in the proof of Theorem 4.5 since in the present case,  $\Lambda$  only depends on  $(t, x, u)$  and not on  $\nabla u$ . In particular, we obtain for  $t \in ]0, T]$ ,

$$\|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_1 \leq C e^{CT} \|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 + \int_0^t \mathbb{E} \left[ \|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \right] ds, \quad (4.40)$$

that corresponds to inequality (4.33) in the proof above, without the term containing the gradient  $\nabla u$ . Invoking inequality (6.17) of Lemma 6.4 and inequality (6.9) of Lemma 6.3 with  $H = K$ , and successively with  $f = F_0(t, \cdot)$  and  $f = p(s, y, t, \cdot)$  for fixed  $y \in \mathbb{R}^d$ , imply that

$$\|K_\varepsilon * F_0(t, \cdot) - F_0(t, \cdot)\|_1 \leq \frac{\varepsilon \kappa \mathfrak{C}}{\sqrt{t}}, \quad 0 < t \leq T, \quad (4.41)$$

and

$$\|K_\varepsilon * p(s, Y_s, t, \cdot) - p(s, Y_s, t, \cdot)\|_1 \leq \frac{\varepsilon \kappa \mathfrak{C}}{\sqrt{t-s}}, \quad \text{a.e.}, \quad 0 \leq s < t < T, \quad (4.42)$$

with  $\mathfrak{C} = \mathfrak{C}(C_u, c_u)$ . This concludes the proof of (4.39).  $\square$

## 5 Particles system algorithm

To simplify notations in the rest of the paper,  $f_t$  will denote  $f(t)$  where  $f : [0, T] \rightarrow E$  is an  $E$ -valued Borel function and  $(E, d_E)$  a metric space.

In previous sections, we have studied existence, uniqueness for a semilinear PDE of the form (1.4) and we have established a Feynman-Kac type representation for the corresponding solution  $u$ , see Theorem 3.5. The regularized form of (1.4) is the integro-PDE (4.2) for which we have established well-posedness in Proposition 4.4 1. In the sequel, we denote by  $\gamma^\varepsilon$  the corresponding solution and again by  $u^\varepsilon(t, x) := (K_\varepsilon * \gamma_t^\varepsilon)(x)$  (see (4.17)). We recall that  $u^\varepsilon$  converges to  $u$ , when the regularization parameter  $\varepsilon$  vanishes to 0, see Theorem 4.5. In the present section, we propose a Monte Carlo approximation  $u^{\varepsilon, N}$  of  $u^\varepsilon$ , providing an original numerical approximation of the semilinear PDE (1.4), when both the number of particles  $N \rightarrow \infty$  and the regularization parameter  $\varepsilon \rightarrow 0$  slowly enough. Let  $u_0$  be a Borel probability measure on  $\mathcal{P}(\mathbb{R}^d)$ .

## 5.1 Convergence of the particle system

We suppose the validity of Assumption 2.

For fixed  $N \in \mathbb{N}^*$ , let  $(W^i)_{i=1, \dots, N}$  be a family of independent Brownian motions and  $(Y_0^i)_{i=1, \dots, N}$  be i.i.d. random variables distributed according to  $u_0(x)dx$ . For any  $\varepsilon > 0$ , we define the measure-valued functions  $(\gamma_t^{\varepsilon, N})_{t \in [0, T]}$  such that for any  $t \in [0, T]$

$$\begin{cases} \xi_t^i = \xi_0^i + \int_0^t \Phi(s, \xi_s^i) dW_s^i + \int_0^t g(s, \xi_s^i) ds, & \text{for } i = 1, \dots, N, \\ \xi_0^i = Y_0^i & \text{for } i = 1, \dots, N, \\ \gamma_t^{\varepsilon, N} = \frac{1}{N} \sum_{i=1}^N V_t(\xi^i, (K_\varepsilon * \gamma^{\varepsilon, N})(\xi^i), (\nabla K_\varepsilon * \gamma^{\varepsilon, N})(\xi^i)) \delta_{\xi_t^i}, & . \end{cases} \quad (5.1)$$

where  $(K_\varepsilon)_{\varepsilon > 0}$  denotes a sequence of mollifiers such that for all  $\varepsilon > 0$ ,  $K_\varepsilon \in W^{1,1}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$  and we recall that  $V_t$  is given by (2.1). The first line of (5.1) is a  $d$ -dimensional classical SDE whose strong existence and pathwise uniqueness are ensured by classical theorems for Lipschitz coefficients. Moreover  $\xi^i, i = 1, \dots, N$  are i.i.d. In the following lemma, we prove by a fixed-point argument that the third line equation of (5.1) has a unique solution.

**Lemma 5.1.** *We suppose the validity of Assumption 2. Let us fix  $\varepsilon > 0$  and  $N \in \mathbb{N}^*$ . Consider the i.i.d. system  $(\xi^i)_{i=1, \dots, N}$  of particles, solution of the two first equations of (5.1). Then, there exists a unique function  $\gamma^{\varepsilon, N} : [0, T] \rightarrow \mathcal{M}_f(\mathbb{R}^d)$  such that for all  $t \in [0, T]$ ,  $\gamma_t^{\varepsilon, N}$  is solution of (5.1).*

*Proof.* The proof relies on a fixed-point argument applied to the map  $T^{\varepsilon, N} : \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d)) \rightarrow \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d))$  given by

$$T^{\varepsilon, N}(\gamma)(t) : \gamma \mapsto \frac{1}{N} \sum_{i=1}^N V_t(\xi^i, (K_\varepsilon * \gamma)(\xi^i), (\nabla K_\varepsilon * \gamma)(\xi^i)) \delta_{\xi_t^i}. \quad (5.2)$$

In the rest of the proof, the notation  $T_t^{\varepsilon, N}(\gamma)$  will denote  $T^{\varepsilon, N}(\gamma)(t)$ .

In order to apply the Banach fixed-point theorem, we emphasize that  $\mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d))$  is equipped with one of the equivalent norms  $\|\cdot\|_{TV, \lambda}$ ,  $\lambda \geq 0$ , defined by

$$\|\gamma\|_{TV, \lambda} := \sup_{t \in [0, T]} e^{-\lambda t} \|\gamma(t, \cdot)\|_{TV}, \quad (5.3)$$

and for which  $(\mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d)), \|\cdot\|_{TV, \lambda})$  is still complete.

From now on, it remains to ensure that  $T^{\varepsilon, N}$  is indeed a contraction with respect  $\|\gamma\|_{TV, \lambda}$  for some  $\lambda$ . To simplify notations, we set for all  $i \in \{1, \dots, N\}$ ,

$$T_t^{\varepsilon, N, i}(\gamma) := V_t(\xi^i, (K_\varepsilon * \gamma)(\xi^i), (\nabla K_\varepsilon * \gamma)(\xi^i)), \quad (t, \gamma) \in [0, T] \times \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d)), \quad (5.4)$$

to re-write (5.2) in the form

$$T_t^{\varepsilon, N}(\gamma) = \frac{1}{N} \sum_{i=1}^N T_t^{\varepsilon, N, i}(\gamma) \delta_{\xi_t^i}, \quad (t, \gamma) \in [0, T] \times \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d)). \quad (5.5)$$

Let  $\lambda > 0$ . Consider now  $\gamma^1, \gamma^2 \in \mathcal{C}([0, T], \mathcal{M}_f(\mathbb{R}^d))$ . On the one hand, taking into account (2.1) and (2.3),

for all  $t \in [0, T]$ ,  $i \in \{1, \dots, N\}$ , we have

$$\begin{aligned}
|T_t^{\varepsilon, N, i}(\gamma^1) - T_t^{\varepsilon, N, i}(\gamma^2)| &\leq L_\Lambda e^{TM_\Lambda} \int_0^t \left( |(K_\varepsilon * \gamma^1)(\xi_s^i) - (K_\varepsilon * \gamma^2)(\xi_s^i)| \right. \\
&\quad \left. + |(\nabla K_\varepsilon * \gamma^1)(\xi_s^i) - (\nabla K_\varepsilon * \gamma^2)(\xi_s^i)| \right) ds \\
&\leq C \int_0^t \|\gamma_s^1 - \gamma_s^2\|_{TV} ds \\
&\leq C \int_0^t e^{s\lambda} \|\gamma^1 - \gamma^2\|_{TV, \lambda} ds \\
&= C \|\gamma^1 - \gamma^2\|_{TV, \lambda} \frac{e^{\lambda t} - 1}{\lambda}, \tag{5.6}
\end{aligned}$$

with  $C = C(T, \|K_\varepsilon\|_\infty, \|\nabla K_\varepsilon\|_\infty, L_\Lambda, M_\Lambda)$ . It follows that

$$\begin{aligned}
\|T^{\varepsilon, N}(\gamma^1) - T^{\varepsilon, N}(\gamma^2)\|_{TV, \lambda} &\leq \frac{1}{N} \sum_{i=1}^N \|T^{\varepsilon, N, i}(\gamma^1)\delta_{\xi^i} - T^{\varepsilon, N, i}(\gamma^2)\delta_{\xi^i}\|_{TV, \lambda} \\
&\leq \frac{C}{\lambda} \|\gamma^1 - \gamma^2\|_{TV, \lambda}. \tag{5.7}
\end{aligned}$$

By taking  $\lambda > C$  and invoking Banach fixed-point theorem, we end the proof.  $\square$

After the preceding preliminary considerations, we can state and prove the main result of the section.

**Proposition 5.2.** *We suppose the validity of Assumption 2. Assume that the kernel  $K$  is a probability density verifying*

$$K \in W^{1,1}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |x|^{d+1} K(x) dx < \infty, \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^{d+1} |\nabla K(x)| dx < \infty. \tag{5.8}$$

For  $\varepsilon > 0$ , we suppose that  $K_\varepsilon$  is explicitly given by (4.37) with  $K$  satisfying (5.8). Let  $u^\varepsilon$  be the real valued function defined by (4.17), and  $u^{\varepsilon, N}$  such that for any  $t \in [0, T]$ ,

$$u^{\varepsilon, N}(t, \cdot) := K_\varepsilon * \gamma_t^{\varepsilon, N}, \tag{5.9}$$

where  $\gamma^{\varepsilon, N}$  is defined by the third line of (5.1). There is a constant  $C$  (only depending on  $M_\Phi, M_g, M_\Lambda, \|K\|_\infty, \|\nabla K\|_\infty, L_\Phi, L_g, L_\Lambda, T$ ) such that the following holds. For all  $t \in [0, T]$  and  $N \in \mathbb{N}^*$ ,  $\varepsilon > 0$  verifying  $\min(N, N\varepsilon^d) > C$  we have

$$\mathbb{E} \left[ \|u_t^{\varepsilon, N} - u_t^\varepsilon\|_1 \right] + \mathbb{E} \left[ \|\nabla u_t^{\varepsilon, N} - \nabla u_t^\varepsilon\|_1 \right] \leq \frac{C}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}}. \tag{5.10}$$

*Proof.* Let us fix  $\varepsilon > 0$ ,  $N \in \mathbb{N}^*$ . For any  $\ell = 1, \dots, d$ , we introduce the real-valued function  $G_\varepsilon^\ell$  defined on  $\mathbb{R}^d$  such that

$$G_\varepsilon^\ell(x) := \frac{1}{\varepsilon^d} \frac{\partial K}{\partial x_\ell} \left( \frac{x}{\varepsilon} \right), \quad \text{for almost all } x \in \mathbb{R}^d. \tag{5.11}$$

By (5.8), there exists a finite positive constant  $C$  independent of  $\varepsilon$  such that  $\|G_\varepsilon^\ell\|_\infty \leq \frac{C}{\varepsilon^d}$  and  $\|G_\varepsilon^\ell\|_1 = \|G_1^\ell\|_1 \leq C$ . In the sequel,  $C$  will always denote a finite positive constant independent of  $(\varepsilon, N)$  that may change from line to line. For any  $t \in [0, T]$ , we introduce the random Borel measure  $\tilde{\gamma}_t^{\varepsilon, N}$  on  $\mathbb{R}^d$ , defined by

$$\tilde{\gamma}_t^{\varepsilon, N} := \frac{1}{N} \sum_{i=1}^N V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) \delta_{\xi^i}. \tag{5.12}$$

One can first decompose the error on the l.h.s of inequality (5.10) as follows

$$\begin{aligned}
\mathbb{E}\left[\|u_t^{\varepsilon,N} - u_t^\varepsilon\|_1\right] + \mathbb{E}\left[\|\nabla u_t^{\varepsilon,N} - \nabla u_t^\varepsilon\|_1\right] &= \mathbb{E}\left[\|K_\varepsilon * (\gamma_t^{\varepsilon,N} - \gamma_t^\varepsilon)\|_1\right] + \frac{1}{\varepsilon} \sum_{\ell=1}^d \mathbb{E}\left[\|G_\varepsilon^\ell * (\gamma_t^{\varepsilon,N} - \gamma_t^\varepsilon)\|_1\right] \\
&\leq \mathbb{E}\left[\|K_\varepsilon * (\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N})\|_1\right] + \frac{1}{\varepsilon} \sum_{\ell=1}^d \mathbb{E}\left[\|G_\varepsilon^\ell * (\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N})\|_1\right] \\
&\quad + \mathbb{E}\left[\|K_\varepsilon * (\tilde{\gamma}_t^{\varepsilon,N} - \gamma_t^\varepsilon)\|_1\right] + \frac{1}{\varepsilon} \sum_{\ell=1}^d \mathbb{E}\left[\|G_\varepsilon^\ell * (\tilde{\gamma}_t^{\varepsilon,N} - \gamma_t^\varepsilon)\|_1\right] \\
&= \mathbb{E}\left[\|A_t^{\varepsilon,N}\|_1\right] + \mathbb{E}\left[\|A_t'^{\varepsilon,N}\|_1\right] + \mathbb{E}\left[\|B_t^{\varepsilon,N}\|_1\right] + \mathbb{E}\left[\|B_t'^{\varepsilon,N}\|_1\right], \tag{5.13}
\end{aligned}$$

where, for all  $t \in [0, T]$ ,

$$\left\{ \begin{aligned}
A_t^{\varepsilon,N}(x) &:= \frac{1}{N} \sum_{i=1}^N K_\varepsilon(x - \xi_t^i) \left[ V_t(\xi^i, u^{\varepsilon,N}(\xi^i), \nabla u^{\varepsilon,N}(\xi^i)) - V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) \right] \\
A_t'^{\varepsilon,N}(x) &:= \frac{1}{\varepsilon} \sum_{\ell=1}^d \left| \frac{1}{N} \sum_{i=1}^N G_\varepsilon^\ell(x - \xi_t^i) \left[ V_t(\xi^i, u^{\varepsilon,N}(\xi^i), \nabla u^{\varepsilon,N}(\xi^i)) - V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) \right] \right| \\
B_t^{\varepsilon,N}(x) &:= \frac{1}{N} \sum_{i=1}^N K_\varepsilon(x - \xi_t^i) V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) - \mathbb{E}\left[ K_\varepsilon(x - \xi_t^1) V_t(\xi^1, u^\varepsilon(\xi^1), \nabla u^\varepsilon(\xi^1)) \right] \\
B_t'^{\varepsilon,N}(x) &:= \frac{1}{\varepsilon} \sum_{\ell=1}^d \left| \frac{1}{N} \sum_{i=1}^N G_\varepsilon^\ell(x - \xi_t^i) V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) - \mathbb{E}\left[ G_\varepsilon^\ell(x - \xi_t^1) V_t(\xi^1, u^\varepsilon(\xi^1), \nabla u^\varepsilon(\xi^1)) \right] \right|.
\end{aligned} \right. \tag{5.14}$$

We will proceed in two steps, first bounding  $\mathbb{E}\left[\|B_t^{\varepsilon,N}\|_1\right]$  and  $\mathbb{E}\left[\|B_t'^{\varepsilon,N}\|_1\right]$  and then  $\mathbb{E}\left[\|A_t^{\varepsilon,N}\|_1\right]$  and  $\mathbb{E}\left[\|A_t'^{\varepsilon,N}\|_1\right]$ .

**Step 1: Bounding  $\mathbb{E}\|B_t^{\varepsilon,N}\|_1$  and  $\mathbb{E}\|B_t'^{\varepsilon,N}\|_1$ .** For any  $i \in \{1, \dots, N\}$  and  $(t, x) \in [0, T] \times \mathbb{R}^d$  we set

$$P_i^\varepsilon(t, x) := K_\varepsilon(x - \xi_t^i) V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) - \mathbb{E}\left[ K_\varepsilon(x - \xi_t^i) V_t(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) \right]. \tag{5.15}$$

Notice that for fixed  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,  $(P_i^\varepsilon(t, x))_{i=1, \dots, N}$  are i.i.d. centered square integrable random variables. Hence using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}\left[\|B_t^{\varepsilon,N}\|_1\right] &= \int_{\mathbb{R}^d} \mathbb{E}\left[\left| \frac{1}{N} \sum_{i=1}^N P_i^\varepsilon(t, x) \right|\right] dx \\
&\leq \int_{\mathbb{R}^d} \sqrt{\mathbb{E}\left[\left(\frac{1}{N} \sum_{i=1}^N (P_i^\varepsilon(t, x))\right)^2\right]} dx \\
&= \frac{1}{\sqrt{N}} \int_{\mathbb{R}^d} \sqrt{\mathbb{E}\left[(P_1^\varepsilon(t, x))^2\right]} dx. \tag{5.16}
\end{aligned}$$

By the boundedness assumption on  $\Lambda$  (item 5. of Assumption 2.,

$$\mathbb{E}[(P_1^\varepsilon(t, x))^2] \leq 4e^{2M\Lambda T} \mathbb{E}[(K_\varepsilon(x - \xi_t^1))^2],$$

which implies

$$\mathbb{E}\left[\|B_t^{\varepsilon,N}\|_1\right] \leq \frac{C}{\sqrt{N}} \int_{\mathbb{R}^d} \sqrt{\mathbb{E}\left[(K_\varepsilon(x - \xi_t^1))^2\right]} dx = \frac{C}{\sqrt{N}} \frac{\sqrt{\int_{\mathbb{R}^d} K^2(x) dx}}{\sqrt{\varepsilon^d}} \int_{\mathbb{R}^d} \sqrt{H_\varepsilon * p_t(x)} dx, \tag{5.17}$$

where  $p_t$  is the law density of  $\xi_t^1$  and  $H_\varepsilon$  is the probability density on  $\mathbb{R}^d$  such that for almost all  $x \in \mathbb{R}^d$ ,  $H_\varepsilon(x) := \frac{1}{\int_{\mathbb{R}^d} K^2(x) dx} \frac{1}{\varepsilon^d} K^2\left(\frac{x}{\varepsilon}\right)$ , which is well-defined thanks to assumption (5.8). Finally, applying Lemma 6.2 with  $G = \frac{K^2}{\|K\|_2^2}$  and  $f = p_t$  we obtain

$$\mathbb{E}\left[\|B_t^{\varepsilon,N}\|_1\right] \leq \frac{C}{\sqrt{N\varepsilon^d}}, \quad \text{for } \varepsilon \text{ small enough.} \quad (5.18)$$

Proceeding similarly for the term  $B_t^{\prime\varepsilon,N}$  leads to

$$\mathbb{E}\left[\|B_t^{\prime\varepsilon,N}\|_1\right] \leq \frac{C}{\sqrt{N\varepsilon^2}} \sum_{\ell=1}^d \int_{\mathbb{R}^d} \sqrt{\mathbb{E}\left[(G_\varepsilon^\ell(x - \xi_t^1))^2\right]} dx = \frac{C}{\sqrt{N\varepsilon^2}} \sum_{\ell=1}^d \frac{\sqrt{\int_{\mathbb{R}^d} |\frac{\partial K}{\partial x_\ell}(x)|^2 dx}}{\sqrt{\varepsilon^d}} \int_{\mathbb{R}^d} \sqrt{H_\varepsilon^\ell * p_t(x)} dx, \quad (5.19)$$

where  $H_\varepsilon^\ell, \ell = 1, \dots, d$  denotes the probability densities on  $\mathbb{R}^d$  such that for almost all  $x \in \mathbb{R}^d$ ,  $H_\varepsilon^\ell(x) := \frac{1}{\int_{\mathbb{R}^d} |\frac{\partial K}{\partial x_\ell}(x)|^2 dx} \frac{1}{\varepsilon^d} |\frac{\partial K}{\partial x_\ell}\left(\frac{x}{\varepsilon}\right)|^2$ . Applying again Lemma 6.2 with  $G = \frac{|\frac{\partial K}{\partial x_\ell}|^2}{\|\frac{\partial K}{\partial x_\ell}\|_2^2}, \ell = 1, \dots, d$  and  $f = p_t$  we obtain

$$\mathbb{E}\left[\|B_t^{\prime\varepsilon,N}\|_1\right] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}}, \quad \text{for } \varepsilon \text{ small enough.} \quad (5.20)$$

**Step 2: Bounding  $\mathbb{E}\|A_t^{\varepsilon,N}\|_1$  and  $\mathbb{E}\|A_t^{\prime\varepsilon,N}\|_1$ .** Recall that  $A_t^{\varepsilon,N}(x) = K_\varepsilon * (\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N})(x)$  and  $A_t^{\prime\varepsilon,N}(x) = \frac{1}{\varepsilon} \sum_{\ell=1}^d |G_\varepsilon^\ell * (\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N})|(x)$ , which yields

$$\mathbb{E}\left[\|A_t^{\varepsilon,N}\|_1\right] + \mathbb{E}\left[\|A_t^{\prime\varepsilon,N}\|_1\right] \leq \frac{C}{\varepsilon} \mathbb{E}\left[\|\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N}\|_{TV}\right]. \quad (5.21)$$

We are now interested in bounding the r.h.s. of (5.21).

Recalling (5.1), (5.12) and inequality (2.3), we have

$$\begin{aligned} \mathbb{E}\left[\|\gamma_t^{\varepsilon,N} - \tilde{\gamma}_t^{\varepsilon,N}\|_{TV}\right] &= \frac{1}{N} \sum_{i=1}^N \mathbb{E}\left[|V_t(\xi^i, u^{\varepsilon,N}, \nabla u^{\varepsilon,N}) - V_t(\xi^i, u^\varepsilon, \nabla u^\varepsilon)|\right] \\ &\leq C \mathbb{E}\left[\int_0^t (|u_s^{\varepsilon,N} - u_s^\varepsilon|(\xi_s^1) + |\nabla u_s^{\varepsilon,N} - \nabla u_s^\varepsilon|(\xi_s^1)) ds\right] \\ &\leq C \int_0^t \left(\mathbb{E}\left[|K_\varepsilon * (\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N})(\xi_s^1)|\right] + \mathbb{E}\left[|K_\varepsilon * (\tilde{\gamma}_s^{\varepsilon,N} - \gamma_s^\varepsilon)(\xi_s^1)|\right] ds\right), \\ &\quad + \frac{C}{\varepsilon} \sum_{\ell=1}^d \int_0^t \left(\mathbb{E}\left[|G_\varepsilon^\ell * (\gamma_s^{\varepsilon,N} - \tilde{\gamma}_s^{\varepsilon,N})(\xi_s^1)|\right] + \mathbb{E}\left[|G_\varepsilon^\ell * (\tilde{\gamma}_s^{\varepsilon,N} - \gamma_s^\varepsilon)(\xi_s^1)|\right] ds\right). \end{aligned} \quad (5.22)$$

By inequality (6.18) in Lemma 6.4, there exists a finite constant  $C > 0$  such that  $\|p_s\|_\infty \leq C\|u_0\|_\infty$  for all  $s \in [0, T]$ . Thus using inequality (5.18), we obtain

$$\begin{aligned} &\mathbb{E}\left[|K_\varepsilon * (\tilde{\gamma}_s^{\varepsilon,N} - \gamma_s^\varepsilon)(\xi_s^1)|\right] \\ &\leq \frac{1}{N} \mathbb{E}\left[|K_\varepsilon(0)V_s(\xi^1, u^\varepsilon(\xi^1), \nabla u^\varepsilon(\xi^1)) - \mathbb{E}[K_\varepsilon(\xi_s^1 - \xi_s^2)V_s(\xi^2, u^\varepsilon(\xi^2), \nabla u^\varepsilon(\xi^2)) | \xi^2]| \right] \\ &+ \frac{N-1}{N} \frac{1}{N-1} \int_{\mathbb{R}^d} \left| \sum_{i=2}^N \left[ K_\varepsilon(x - \xi_s^i)V_s(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i)) - \mathbb{E}[K(x - \xi_s^i)V_s(\xi^i, u^\varepsilon(\xi^i), \nabla u^\varepsilon(\xi^i))] \right] \right| p_s(x) dx \\ &\leq \frac{C}{N\varepsilon^d} + \frac{N-1}{N} \frac{C}{\sqrt{(N-1)\varepsilon^d}} \\ &\leq \frac{C}{\sqrt{N\varepsilon^d}} \quad (\text{for } (N \text{ and } N\varepsilon^d) \text{ sufficiently large}), s \in [0, T]. \end{aligned} \quad (5.23)$$

Similarly we get

$$\sum_{\ell=1}^d \mathbb{E} \left[ \left| \frac{1}{\varepsilon} G_\varepsilon^\ell * (\tilde{\gamma}_s^{\varepsilon, N} - \gamma_s^\varepsilon)(\xi_s^1) \right| \right] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}}, \quad s \in [0, T]. \quad (5.24)$$

Moreover, for all  $s \in [0, T]$ , the boundedness of  $|K|$  and  $|\nabla K|$  implies

$$\mathbb{E} \left[ |K_\varepsilon * (\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N})(\xi_s^1)| \right] + \sum_{\ell=1}^d \mathbb{E} \left[ \left| \frac{1}{\varepsilon} G_\varepsilon^\ell * (\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N})(\xi_s^1) \right| \right] \leq \frac{C}{\varepsilon^{d+1}} \left[ \|\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N}\|_{TV} \right]. \quad (5.25)$$

Injecting inequalities (5.23) (5.24) and (5.25) into (5.22) gives

$$\mathbb{E} \left[ \|\gamma_t^{\varepsilon, N} - \tilde{\gamma}_t^{\varepsilon, N}\|_{TV} \right] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}} + \frac{C}{\varepsilon^{d+1}} \int_0^t \mathbb{E} \left[ \|\gamma_s^{\varepsilon, N} - \tilde{\gamma}_s^{\varepsilon, N}\|_{TV} \right] ds.$$

By Gronwall's lemma we obtain  $\mathbb{E} \left[ \|\gamma_t^{\varepsilon, N} - \tilde{\gamma}_t^{\varepsilon, N}\|_{TV} \right] \leq \frac{C}{\sqrt{N\varepsilon^{d+2}}} e^{\frac{C}{\varepsilon^{d+1}}}$ , which together with (5.21) completes the proof by implying the inequality

$$\mathbb{E}[\|A_t^{\varepsilon, N}\|_1] + \mathbb{E}[\|A_t^{\prime\varepsilon, N}\|_1] \leq \frac{C}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}}. \quad (5.26)$$

□

**Corollary 5.3.** *Assume that the same assumptions as in Proposition 5.2 are fulfilled.*

*If  $\varepsilon \rightarrow 0$ ,  $N \rightarrow +\infty$  such that  $\frac{1}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \rightarrow 0$ , (where  $C$  is the constant coming from Proposition 5.2) then*

$$\mathbb{E} \left[ \|u_t^{\varepsilon, N} - u_t\|_1 \right] + \mathbb{E} \left[ \|\nabla u_t^{\varepsilon, N} - \nabla u_t\|_1 \right] \rightarrow 0. \quad (5.27)$$

*Proof.* Let us fix  $\varepsilon > 0$ ,  $N \in \mathbb{N}^*$ ,  $t \in [0, T]$ . The proof is based on the bound

$$\begin{aligned} \mathbb{E} \left[ \|u_t^{\varepsilon, N} - u_t\|_1 \right] + \mathbb{E} \left[ \|\nabla u_t^{\varepsilon, N} - \nabla u_t\|_1 \right] &\leq \mathbb{E} \left[ \|u_t^{\varepsilon, N} - u_t^\varepsilon\|_1 \right] + \mathbb{E} \left[ \|\nabla u_t^{\varepsilon, N} - \nabla u_t^\varepsilon\|_1 \right] \\ &\quad + \|u_t^\varepsilon - u_t\|_1 + \|\nabla u_t^\varepsilon - \nabla u_t\|_1, \\ &\leq \frac{C}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} + \|u_t^\varepsilon - u_t\|_1 + \|\nabla u_t^\varepsilon - \nabla u_t\|_1, \end{aligned} \quad (5.28)$$

where we have used Proposition 5.2 for the second inequality above.

Taking into account Theorem 4.5 above, it appears clearly that the convergence of  $u^{\varepsilon, N}$  (resp.  $\nabla u^{\varepsilon, N}$ ) to  $u$  (resp.  $\nabla u$ ) will hold as soon as  $\frac{1}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \rightarrow 0$  when  $\varepsilon \rightarrow 0$ ,  $N \rightarrow +\infty$ . This concludes the proof of the corollary. □

**Remark 5.4.** *In the statement of Corollary 5.3 appears the condition  $\frac{1}{\sqrt{N\varepsilon^{d+4}}} e^{\frac{C}{\varepsilon^{d+1}}} \rightarrow 0$  when  $\varepsilon \rightarrow 0$ ,  $N \rightarrow +\infty$ . This requires a "trade-off" between the speed of convergence of  $N$  and  $\varepsilon$ . Setting  $\Phi(\varepsilon) := \varepsilon^{-(d+4)} e^{\frac{2C}{\varepsilon^{d+1}}}$ , the trade-off condition can be formulated as*

$$\frac{\Phi(\varepsilon)}{N} \rightarrow 0 \quad \text{when} \quad \varepsilon \rightarrow 0, \quad N \rightarrow +\infty. \quad (5.29)$$

*An example of such trade-off between  $N$  and  $\varepsilon$  can be given by the relation  $\varepsilon(N) = \left(\frac{1}{\log(N)}\right)^{\frac{1}{d+4}}$ .*

## 6 Appendix

### 6.1 General inequalities

If  $f$  is a probability density on  $\mathbb{R}^d$ ,  $I(f)$  denotes the quantity  $I(f) := \int_{\mathbb{R}^d} |x|^{d+1} f(x) dx$ .

**Lemma 6.1** (Multidimensional Carlson's inequality). *Let  $f$  be a probability density on  $\mathbb{R}^d$  such that  $I(f) < \infty$ , then*

$$\int_{\mathbb{R}^d} \sqrt{f(x)} dx \leq A_d I(f)^{\frac{d}{2(d+1)}} \quad \text{where} \quad A_d = \left( \frac{(2\pi)^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \right)^{1/2}. \quad (6.1)$$

We refer to [14], where a more precise estimate is proved. From Carlson's inequality, we deduce the following lemma.

**Lemma 6.2.** *Let  $G$  and  $f$  be two probability densities defined on  $\mathbb{R}^d$  such that*

$$I(G) < \infty, \quad \text{and} \quad I(f) < \infty. \quad (6.2)$$

*Then for any strictly positive real  $\varepsilon \leq (1/I(G))^{\frac{1}{d+1}}$ ,*

$$\int_{\mathbb{R}^d} \sqrt{(G_\varepsilon * f)(x)} dx \leq 2^{\frac{d}{2}} A_d [1 + I(f)] \quad \text{where} \quad A_d = \left( \frac{(2\pi)^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \right)^{1/2}, \quad (6.3)$$

*where  $G_\varepsilon(\cdot) := \frac{1}{\varepsilon^d} G(\frac{\cdot}{\varepsilon})$ .*

*Proof.* By Carlson's inequality (6.1) we have

$$\int_{\mathbb{R}^d} \sqrt{(G_\varepsilon * f)(x)} dx \leq A_d [I(G_\varepsilon * f)]^{\frac{d}{2(d+1)}}. \quad (6.4)$$

Then, by Minkowski's inequality,

$$\begin{aligned} [I(G_\varepsilon * f)]^{\frac{1}{d+1}} &= \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^{d+1} G_\varepsilon(x-y) f(y) dy dx \right]^{\frac{1}{d+1}} \\ &= \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} |y+u|^{d+1} G(u) f(y) dy du \right]^{\frac{1}{d+1}} \\ &\leq I(f)^{\frac{1}{d+1}} + \varepsilon I(G)^{\frac{1}{d+1}}. \end{aligned}$$

Since  $x \in \mathbb{R}^+ \mapsto x^d$  is convex, it follows

$$I(G_\varepsilon * f)^{\frac{d}{2(d+1)}} \leq 2^{\frac{d-1}{2}} \left[ [I(f)]^{\frac{d}{d+1}} + \varepsilon^d [I(G)]^{\frac{d}{d+1}} \right]^{\frac{1}{2}}.$$

Hence, as soon as  $\varepsilon \leq (1/I(G))^{\frac{1}{d+1}}$ , we have

$$[I(G_\varepsilon * f)]^{\frac{d}{2(d+1)}} \leq 2^{\frac{d}{2}} [1 + I(f)], \quad (6.5)$$

which, owing to (6.4), concludes the proof.  $\square$

**Lemma 6.3.** *Let  $H$  be a density kernel on  $\mathbb{R}^d$  satisfying*

$$H \geq 0, \quad \int_{\mathbb{R}^d} H(x) dx = 1, \quad \int_{\mathbb{R}^d} x H(x) dx = 0. \quad (6.6)$$

*Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a real-valued function. For any  $\varepsilon > 0$ , we consider the function  $H_\varepsilon$  given by*

$$H_\varepsilon(\cdot) := \frac{1}{\varepsilon^d} H\left(\frac{\cdot}{\varepsilon}\right). \quad (6.7)$$

*If  $a := \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 H(x) dx < \infty$  (resp.  $\tilde{a} := \int_{\mathbb{R}^d} |x| H(x) dx < \infty$ ) and  $f \in W^{2,p}$  (resp.  $f \in W^{1,p}$ ) for some integer  $p \geq 1$ , then for any  $\varepsilon > 0$ ,*

$$\|H_\varepsilon * f - f\|_p \leq \varepsilon^2 a \sum_{i,j=1}^d \|\partial_i \partial_j f\|_p. \quad (6.8)$$

$$\left( \text{resp.} \quad \|H_\varepsilon * f - f\|_p \leq \varepsilon \tilde{a} \sum_{i=1}^d \|\partial_i f\|_p \right). \quad (6.9)$$



*Proof.* The proof is modeled on [14] and it is only written in the case  $f \in W^{2,p}$ . The case  $f \in W^{1,p}$  follows exactly the same reasoning. We omit the corresponding details.

For  $\varepsilon > 0$  and any integer  $1 \leq i < j \leq d$  let us introduce the real-valued function  $L_\varepsilon^{i,j}$  defined on  $\mathbb{R}^d$  with values in  $\bar{\mathbb{R}}_+$ , associated with  $H$  such that for almost all  $x \in \mathbb{R}^d$ ,

$$L_\varepsilon^{i,j}(x) = \frac{x_i x_j}{\varepsilon^2} \int_0^1 \frac{1-t}{t^2} H_{\varepsilon t}(x) dt, \quad (6.10)$$

where  $x_i$  is the  $i$ -th coordinate of  $x$  and  $H_t$  given by (6.7). Observe that, for any  $\varepsilon > 0$ ,  $1 \leq i < j \leq d$ ,

$$\|L_\varepsilon^{i,j}\|_1 = \int_{\mathbb{R}^d} |L_\varepsilon^{i,j}(x)| dx \leq a, \quad (6.11)$$

which implies that  $L_\varepsilon^{i,j} < \infty$  a.e.

Developing  $f$  according to the Lagrange expansion up to order two, yields, for almost all  $(x, y) \in (\mathbb{R}^d)^2$ ,

$$f(x-y) = f(x) - \sum_{i=1}^d (\partial_i f)(x) y_i + \sum_{i,j=1}^d \int_0^1 (1-t) \partial_i \partial_j f(x-ty) y_i y_j dt.$$

Integrating this expression against  $H_\varepsilon$  w.r.t.  $y$  and using the symmetry of  $H$ , yields for almost all  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} (H_\varepsilon * f)(x) - f(x) &= \int_{\mathbb{R}^d} [f(x-y) - f(x)] H_\varepsilon(y) dy \\ &= \sum_{i,j=1}^d \int_{\mathbb{R}^d} \int_0^1 (1-t) \partial_i \partial_j f(x-ty) y_i y_j dt H_\varepsilon(y) dy \\ &= \varepsilon^2 \sum_{i,j=1}^d \int_{\mathbb{R}^d} \partial_i \partial_j f(x-u) \frac{u_i u_j}{\varepsilon^2} \int_0^1 \frac{1-t}{t^2} H_{\varepsilon t}(u) dt du \\ &= \varepsilon^2 \sum_{i,j=1}^d (L_\varepsilon^{i,j} * (\partial_i \partial_j f))(x), \end{aligned} \quad (6.12)$$

Taking the  $L^p$  norm in equality (6.12), Young's inequality yields

$$\|H_\varepsilon * f - f\|_p \leq \varepsilon^2 \sum_{i,j=1}^d \|\partial_i \partial_j f\|_p \|L_\varepsilon^{i,j}\|_1,$$

which gives the result by recalling (6.11).  $\square$

## 6.2 About transition kernels

In the following lemma, we state well-known technical properties about the transition probability function of a diffusion process. All the statements below are established in [10].

**Lemma 6.4.** *We assume here the validity of Assumption 2. Consider a stochastic process  $Z$ , solution of the SDE*

$$Z_t = Z_0 + \int_0^t \Phi(s, Z_s) dW_s + \int_0^t g(s, Z_s) ds, \quad (6.13)$$

with  $Z_0$  a random variable admitting a bounded density  $u_0$ .  $P(s, x_0, t, \Gamma)$  denotes its transition probability function, for all  $(s, x_0, t, \Gamma) \in [0, T] \times \mathbb{R}^d \times [0, T] \times \mathcal{B}(\mathbb{R}^d)$ . The following statements hold.

1. The transition probability function  $P$  admits a density, i.e. there exists a Borel function  $p : (s, x_0, t, x) \mapsto p(s, x_0, t, x)$  such that for all  $(s, x_0, t) \in [0, T] \times \mathbb{R}^d \times [0, T]$ ,

$$P(s, x_0, t, \Gamma) = \int_{\Gamma} p(s, x_0, t, x) dx \quad , \quad \Gamma \in \mathcal{B}(\mathbb{R}^d) . \quad (6.14)$$

2. The function  $p$  satisfies (in the classical sense) Kolmogorov backward (B) and forward (F) equations: i.e.  $(s, x_0) \mapsto p(s, x_0, t, x)$  belongs to  $C^{1,2}([0, t] \times \mathbb{R}^d, \mathbb{R})$  and satisfies

$$(B) \quad \begin{cases} \partial_s p + L_s p = 0 , & 0 \leq s < t \leq T \\ p(s, x_0, t, x) \xrightarrow{s \uparrow t} \delta_x , & \text{weakly} , \end{cases} \quad (6.15)$$

where for  $s \in [0, T]$ ,  $\varphi \in C_0^\infty(\mathbb{R}^d)$ ,  $L_s \varphi$  has been defined in (2.4);

$(t, x) \mapsto p(s, x_0, t, x)$  is in  $C^{1,2}([s, T] \times \mathbb{R}^d, \mathbb{R})$  and satisfies

$$(F) \quad \begin{cases} \partial_t p = L_t^* p , & 0 \leq s < t \leq T \\ p(s, x_0, t, x) \xrightarrow{t \downarrow s} \delta_{x_0} , & \text{weakly} , \end{cases} \quad (6.16)$$

where for  $t \in [0, T]$ , for  $\varphi \in C_0^\infty(\mathbb{R}^d)$ ,  $L_t^* \varphi$  has been defined by (2.5).

In particular,  $p$  is twice continuously differentiable w.r.t.  $x_0$  and  $x$ .

3. There exist real constants  $C_u, c_u > 0$  such that, for  $0 \leq s < t \leq T$ ,  $(x_0, x) \in \mathbb{R}^d \times \mathbb{R}^d$  and for all multi-index  $m := (m_1, m_2)$  whose length  $|m| := m_1 + m_2$  is less or equal to 2, we have

$$\left| \frac{\partial^{m_1}}{\partial z_i} \frac{\partial^{m_2}}{\partial x_j} p(s, z, t, x) \right| \leq \frac{C_u}{(t-s)^{\frac{d+|m_1|}{2}}} e^{-c_u \frac{|x-z|^2}{t-s}} , \quad 0 \leq s < t \leq T, (z, x) \in (\mathbb{R}^d)^2 . \quad (6.17)$$

In particular, there exists a constant  $C > 0$  (only depending on  $C_u, c_u$ ) such that for all  $t \in [0, T]$ , the law density  $p_t$  of  $Y_t$  satisfies

$$\|p_t\|_\infty \leq C \|u_0\|_\infty , \quad (6.18)$$

where  $p_t$  is given by  $\int_{\mathbb{R}^d} p(0, x_0, t, x) u_0(x_0) dx_0$ .

*Proof.* Under Assumption 2, the results are a result of Theorems 4.6, 4.7, Section 4 and Theorem 5.4, Section 5 in Chapter 6 in [10].  $\square$

### 6.3 Proof of technicalities of Section 3

We give in this section the proof of Lemma 3.4.

*Proof of Lemma 3.4.* We only prove the direct implication since the converse follows easier with similar arguments. Without restriction of generality, we can assume that  $T = N\tau$  for some integer  $N \in \mathbb{N}$ . The aim is to prove, for all  $n \in \{1, \dots, N\}$ ,

$$(H_n) \quad \begin{cases} \mu(t, dx) = \int_{\mathbb{R}^d} P(0, x_0, t, dx) u_0(dx_0) + \int_0^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \tilde{\Lambda}(s, x_0) \mu(s, dx_0) , \\ \text{for all } t \in [0, n\tau] . \end{cases} \quad (6.19)$$

We are going to proceed by induction on  $n$ . For  $n = 1$ , formula (6.19) follows from (3.13) by taking  $k = 0$ .

We suppose now that  $(H_{n-1})$  holds for some integer  $n \geq 1$ . Then, by taking  $t = (n-1)\tau$  in the first line

equation of (6.19), it follows immediately that

$$\mu((n-1)\tau, dx_0) = \int_{\mathbb{R}^d} P(0, \widetilde{x}_0, (n-1)\tau, dx_0) u_0(d\widetilde{x}_0) + \int_0^{(n-1)\tau} ds \int_{\mathbb{R}^d} P(s, \widetilde{x}_0, (n-1)\tau, dx_0) \widetilde{\Lambda}(s, \widetilde{x}_0) \mu(s, d\widetilde{x}_0). \quad (6.20)$$

On the other hand, since (3.13) is valid for all  $t \in [(n-1)\tau, n\tau]$  by plugging  $k = n-1$ , we obtain

$$\mu(t, dx) = \int_{\mathbb{R}^d} P((n-1)\tau, x_0, t, dx) \mu((n-1)\tau, dx_0) + \int_{(n-1)\tau}^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \widetilde{\Lambda}(s, x_0) \mu(s, dx_0), \quad (6.21)$$

for all  $t \in [(n-1)\tau, n\tau]$ . Inserting (6.20) in (6.21) yields

$$\begin{aligned} \mu(t, dx) &= \int_{\mathbb{R}^d} u_0(d\widetilde{x}_0) \int_{\mathbb{R}^d} P(0, \widetilde{x}_0, (n-1)\tau, dx_0) P((n-1)\tau, x_0, t, dx) \\ &\quad + \int_0^{(n-1)\tau} ds \int_{\mathbb{R}^d} \mu(s, d\widetilde{x}_0) \widetilde{\Lambda}(s, \widetilde{x}_0) \int_{\mathbb{R}^d} P(s, \widetilde{x}_0, (n-1)\tau, dx_0) P((n-1)\tau, x_0, t, dx) \\ &\quad + \int_{(n-1)\tau}^t ds \int_{\mathbb{R}^d} P(s, x_0, t, dx) \widetilde{\Lambda}(s, x_0) \mu(s, dx_0), \quad t \in [(n-1)\tau, n\tau]. \end{aligned} \quad (6.22)$$

Invoking the Chapman-Kolmogorov equation satisfied by the transition probability function  $P(s, x_0, t, dx)$  (see e.g. expression (2.1) in Section 2.2, Chapter 2 in [25]), we have

$$P(s, \widetilde{x}_0, t, dx) = \int_{\mathbb{R}^d} P(s, \widetilde{x}_0, \theta, dz) P(\theta, z, t, dx), \quad s < \theta < t, (\widetilde{x}_0, z) \in \mathbb{R}^d \times \mathbb{R}^d. \quad (6.23)$$

Applying (6.23) with  $\theta = (n-1)\tau$ , it follows that for all  $t \in [0, n\tau]$ ,

$$\begin{aligned} \mu(t, dx) &= \int_{\mathbb{R}^d} u_0(d\widetilde{x}_0) P(0, \widetilde{x}_0, t, dx) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} P(s, \widetilde{x}_0, t, dx) \widetilde{\Lambda}(s, \widetilde{x}_0) \mu(s, d\widetilde{x}_0). \end{aligned} \quad (6.24)$$

This shows that  $(H_n)$  holds. □

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