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# AN INVERSE OBSTACLE PROBLEM FOR THE WAVE EQUATION IN A FINITE TIME DOMAIN

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Abstract. We consider an inverse obstacle problem for the acoustic transient wave equation. More precisely, we wish to reconstruct an obstacle characterized by a Dirichlet boundary condition from lateral Cauchy data given on a subpart of the boundary of the domain and over a finite interval of time. We first give a proof of uniqueness for that problem and then propose an "exterior approach" based on a mixed formulation of quasi-reversibility and a level set method in order to actually solve the problem. Some 2D numerical experiments are provided to show that our approach is effective.

1. **Introduction.** We address an inverse obstacle problem for the wave equation, defined as follows. Let $G$ be an open, bounded and connected domain of $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary. Throughout the paper, we define an obstacle as an open domain $O \Subset G$ which is formed by a collection of a finite number of disjoint simply connected Lipschitz domains ($O$ is not necessarily connected), and such that $\Omega := G \setminus \overline{O}$ is connected. A generic point in $Q := \Omega \times (0,T)$, for $T > 0$, will be denoted $(x,t)$. Let $\Gamma$ be a non-empty open subset of $\partial G$. Given a pair of data $(g_0, g_1)$ on $\Gamma \times (0,T)$, the inverse obstacle problem consists in finding a domain $O$ (independent of time $t$) and some function $u(x,t)$ in the space

$$H^{1,1}(Q) := \{u \in L^2(0,T;H^1(\Omega)),\ u \in H^1(0,T;L^2(\Omega))\}, \tag{1}$$

following the notation of [19], and such that

$$\left\{\begin{array}{rcll} \partial_t^2 u - \Delta u & = & 0 & \text{in } \Omega \times (0,T) \\ u & = & g_0 & \text{on } \Gamma \times (0,T) \\ \partial_\nu u & = & g_1 & \text{on } \Gamma \times (0,T) \\ u & = & 0 & \text{on } \partial O \times (0,T) \\ u,\ \partial_t u & = & 0 & \text{on } \Omega \times \{0\}, \end{array}\right. \tag{2}$$

where $\nu$ is the outward unit normal to $\Omega$. The inverse problem that we address is a geometric inverse problem, the obstacle to retrieve being characterized by a Dirichlet boundary condition. We also emphasize that the data are restricted to

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a single pair of Cauchy data on a subpart $\Gamma$ of $\partial G$ only: there are no data at all on the complementary part $\partial G \setminus \overline{\Gamma}$. Lastly, those Cauchy data are known only in the finite time domain $(0,T)$, which prevents us from addressing the problem in the frequency domain. Problem (2) arises in the following practical situation. The medium $G$ is at rest for negative time $t$. At initial time $t = 0$, an experimenter generates a known pulse $g_1$ on the subpart $\Gamma$ of the surface of the medium $G$ and measures the response $g_0$ of the medium on $\Gamma$ during the time interval $(0,T)$, the complementary part of the surface being inaccessible. The goal is to retrieve the unknown obstacle $O$ from those measurements $(g_0, g_1)$ on $\Gamma \times (0,T)$.

To the best of our knowledge, there are very few papers dealing with the effective reconstruction of the obstacle: the only one we have found is [1], in which an optimization technique is applied. However, the obstacles which are seeked in the numerical experiments of [1] are a priori known to be circles in 2D, which are characterized by only three real parameters. We mention that in [2] the authors also want to retrieve a surface defined by a Dirichlet boundary condition with the help of a single incident wave, however with measurements in an infinite time interval. Needless to say that there are many contributions in the case of several incident waves, in particular in the case of infinitely many. In this vein, let us mention [3], which is based on the topological gradient, [4], which is based on the boundary control method, or [5], which relies on the Linear Sampling Method.

A crucial issue induced by considering measurements in a finite time domain $(0,T)$ is uniqueness. More precisely, the natural question is: what is the minimal value of $T$ so that the obstacle $O$ is uniquely determined by the Cauchy data $(g_0, g_1)$ on $\Gamma \times (0,T)$? From our understanding this question is still open. In [6], however, the author provides some minimal values of $T$ which guarantee uniqueness, as soon as a bound on a certain length characterizing the obstacle is assumed. Considering that our geometric setting is slightly different from that studied in [6] and also for the sake of self-containment, we give a detailed uniqueness proof in $2D$. However the main concern of this paper is the effective reconstruction of the obstacle in $2D$ from the data when uniqueness holds. Our strategy is to apply the so-called "exterior approach", first introduced in [7] in the case of the Laplace equation and then extended in [8, 9, 10] to the case of the Stokes system, heat equation and Helmholtz equation, respectively. We remark that an inverse problem such as (2) is both ill-posed and non-linear. The idea consists in addressing these two issues separately by coupling a quasi-reversibility method and a level set method. An initial guess of the obstacle being given, the exterior solution is updated from the lateral Cauchy data by solving a quasi-reversibility problem. An initial guess of the solution being given, the obstacle is updated by solving a level set problem. Our approach is hence iterative, with one step of quasi-reversibility and one step of level set at each iteration, until some stopping criterion is reached. Quasi-reversibility goes back to [11]: it is a Tikhonov regularization of some linear ill-posed PDE problem which is directly in the form of a weak formulation. This weak formulation can hence be discretized with the help of a Finite Element Method. The main drawback of the original method presented in [11] and later in [12] is the fact that the weak formulation associated with an ill-posed second-order problem corresponds to a fourth-order well-posed problem. As a consequence, the justification of such quasi-reversibility method requires some additional regularity for the exact data and the discretization requires some additional regularity for the approximation space. This is why we developed in [13] an alternative mixed formulation of quasi-reversibility

which preserves the order of the original problem. Such mixed formulation has been recently generalized in [10] and we propose to apply it once again in the case of the wave equation with lateral Cauchy data. Concerning the level set method, we reuse the method introduced in [7], which is based on a simple Poisson equation and has shown its efficiency in many situations ever since. It should be noted that, due to the specific form of the uniqueness results in the case of the wave equation in a finite time domain, both the quasi-reversibility method and the level set method are more difficult to justify than in the previous situations of elliptic or parabolic equations.

This paper is organized as follows. The next section shows in which sense all the boundary conditions in problem (2) have to be understood. Section 3 is dedicated to some uniqueness results. Firstly it concerns a well-known unique continuation result which is crucial in what follows and the proof of which is postponed in an Appendix, for sake of self-containment. Secondly it establishes a uniqueness result for the inverse problem (2) in dimension 2. Section 4 describes the mixed quasi-reversibility method while section 5 describes the level set method. By merging these two methods we derive the "exterior approach" algorithm, which is detailed in Section 6. Lastly, some numerical results which illustrate the feasibility of this approach to solve problem (2) are presented in Section 7.

## 2. **Some comments on boundary conditions.**

It is important to note that, due to the absence of boundary conditions on the boundary $\partial G \setminus \overline{\Gamma}$, the standard regularity results for the wave equation are helpless. This explains why, in order to justify that the boundary conditions in problem (2) are meaningful, we assumed that the solution $u$ belongs to the unsual space $H^{1,1}(Q)$ defined by (1). Let us give a meaning to the traces $(u, \partial_\nu u)$ of $u$ on $\Sigma := \Gamma \times (0,T)$ and the traces $(u, \partial_t u)$ of $u$ on $S_0 = \Omega \times \{0\}$. It is readily seen that $H^{1,1}(Q)$ coincides with $H^1(Q)$, since $Q$ can be viewed as a bounded Lipschitz domain of $\mathbb{R}^{d+1}$. As a result, the trace of $u \in V$ on $\partial Q$ is well-defined in $H^{\frac{1}{2}}(\partial Q)$, and in particular the traces $u|_\Sigma$ and $u|_{S_0}$ are well defined in $H^{\frac{1}{2}}(\Sigma)$ and $H^{\frac{1}{2}}(S_0)$ as the sets of restrictions of functions in $H^{\frac{1}{2}}(\partial Q)$ to $\Gamma \times (0,T)$ and $\Omega \times \{0\}$, respectively. Furthermore, let us denote, for $u \in H^1(Q)$, $\mathbf{v} := (\nabla u, -\partial_t u) \in (L^2(Q))^{d+1}$. If $u$ solves the wave equation $\partial_t^2 u - \Delta u = 0$, then

$$\mathrm{div}_{d+1}(\mathbf{v}) = \Delta u - \partial_t^2 u = 0.$$

We conclude that $\mathbf{v} \in H_{\mathrm{div}}(Q)$: this enables one to define $\mathbf{v} \cdot \nu_{d+1}|_{\partial Q} \in H^{-\frac{1}{2}}(\partial Q)$, which is the dual space of $H^{\frac{1}{2}}(\partial Q)$. Here $\nu_{d+1}$ denotes the outward unit normal to the $(d+1)$-dimensional domain $Q$. As a result, the normal derivatives $\partial_\nu u|_\Sigma$ and $\partial_t u|_{S_0}$ are well defined in $H^{-\frac{1}{2}}(\Sigma)$ and $H^{-\frac{1}{2}}(S_0)$ as the restrictions of distributions in $H^{-\frac{1}{2}}(\partial Q)$ to $\Gamma \times (0,T)$ and $\Omega \times \{0\}$, respectively. Let us recall that $H^{-\frac{1}{2}}(\Sigma)$ and $H^{-\frac{1}{2}}(S_0)$ are the dual spaces of $\tilde{H}^{\frac{1}{2}}(\Sigma)$ and $\tilde{H}^{\frac{1}{2}}(S_0)$, the latter being defined as the subsets of functions in $H^{\frac{1}{2}}(\Sigma)$ and $H^{\frac{1}{2}}(S_0)$ which once extended by 0 on the whole boundary $\partial Q$ are still in $H^{\frac{1}{2}}(\partial Q)$.

## 3. **Some uniqueness results.**

### 3.1. **A basic unique continuation result.**

In this section, we recall a useful unique continuation result for the wave equation. To this aim, we need to define the geodesic distance in $\Omega$.

**Definition 3.1.** If $\Omega \subset \mathbb{R}^d$ is a connected open domain, for $x, y \in \Omega$, the geodesic distance between $x$ and $y$ is defined by

$$d_\Omega(x,y) = \inf\{\ell(g),\ g : [0,1] \to \Omega,\ g(0) = x,\ g(1) = y\},$$

where $g$ is a continuous path in $\Omega$ of length $\ell(g)$. Here, the length of $g$ is defined as

$$\ell(g) = \sup\left\{\sum_{i=0}^{n-1} |g(t_i) - g(t_{i+1})|,\ n \in \mathbb{N},\ 0 = t_0 \le t_1 \le \cdots \le t_n = 1\right\},$$

where the sup is taken over all decompositions of $[0,1]$ into an arbitrary (finite) number of intervals.

Now, let us introduce the constant $D(\Omega,\Gamma)$ denoting the largest geodesic distance between some point $x$ in $\Omega$ and $\Gamma$, which will play a crucial role in the sequel.

**Definition 3.2.** For a connected open domain $\Omega$, we define

$$D(\Omega,\Gamma) = \sup_{x\in\Omega} d(x,\Gamma), \quad \text{where} \quad d(x,\Gamma) = \inf_{x_0\in\Gamma} d_\Omega(x,x_0).$$

A natural question is: do we have $D(\Omega,\Gamma) < +\infty$? For some fixed $x_0 \in \Omega$, we first define

$$D(\Omega,x_0) = \sup_{x\in\Omega} d_\Omega(x,x_0).$$

It is not difficult to see that we may have $D(\Omega,x_0) = +\infty$, even if $\Omega$ is a bounded domain. However we have $D(\Omega,x_0) < +\infty$ as soon as $\Omega$ is a bounded Lipschitz domain, which is a consequence of the fact that for $x_0 \in \Omega$, the application $x \mapsto d_\Omega(x,x_0)$ has a continuous extension in $\overline{\Omega}$ for the Euclidian topology [21]. In this Lipschitz case, the definition of $D(\Omega,x_0)$ can hence be extended to the case when $x_0 \in \partial\Omega$. We clearly have

$$D(\Omega,\Gamma) = \sup_{x\in\Omega}\inf_{x_0\in\Gamma} d_\Omega(x,x_0) \le \inf_{x_0\in\Gamma}\sup_{x\in\Omega} d_\Omega(x,x_0) = \inf_{x_0\in\Gamma} D(\Omega,x_0).$$

We conclude that for a bounded Lipschitz domain, $D(\Omega,\Gamma) < +\infty$.
Now let us state the following classical theorem, which concerns unique continuation for the wave equation in the presence of lateral Cauchy data, and which is proved in Appendix for the sake of self-containment.

**Theorem 3.3.** *Let us consider a Lipschitz connected open domain $\Omega$ of $\mathbb{R}^d$, with $d \ge 2$, and $\Gamma$ a non-empty open subset of $\partial\Omega$. For $T > 0$, let us denote $Q = \Omega \times (0,T)$.*
*Assume that $u \in H^1(Q)$ satisfies the system*

$$\left\{\begin{array}{rcll} \partial_t^2 u - \Delta u & = & 0 & \text{in } \Omega\times(0,T) \\ u & = & 0 & \text{on } \Gamma\times(0,T) \\ \partial_\nu u & = & 0 & \text{on } \Gamma\times(0,T) \\ u,\ \partial_t u & = & 0 & \text{on } \Omega\times\{0\}. \end{array}\right. \tag{3}$$

*Then $u$ vanishes in the subdomain $Q_0$ of $Q$ defined by*

$$Q_0 = \{(x,t) \in \Omega\times(0,T),\ d_\Omega(x,\Gamma) < T - t\}. \tag{4}$$

*In particular, if $T > D(\Omega,\Gamma)$, then $u$ vanishes in the subdomain $\Omega\times(0,T-D(\Omega,\Gamma))$.*

3.2. **Uniqueness in the inverse obstacle problem for $d = 2$.** For any $d \geq 2$, let us introduce the following definitions. For a simply connected bounded Lipschitz domain $O$ of $\mathbb{R}^d$, we introduce the diameter of the boundary $\partial O$ as

$$D(\partial O) = \sup_{x,y \in \partial O} d_{\partial O}(x, y),$$

where $d_{\partial O}(x, y)$ is the geodesic distance from $x$ to $y$ in $\partial O$. If $O$ is the collection of $I$ disjoint simply connected Lipschitz domains $O_i$, $i = 1, \cdots, I$, we denote

$$D(\partial O) = \sum_{i=1}^{I} D(\partial O_i).$$

For any $d$, we can also define the perimeter of the domain $O$, that is $P(O) = \int_{\partial O} ds$, where $s$ is the measure on $\partial O$. As above, if $O$ is the collection of $I$ disjoint simply connected Lipschitz domains $O_i$, $i = 1, \cdots, I$, we denote $P(O) = \sum_{i=1}^{I} P(O_i)$. It happens that for $d = 2$, the quantities $P(O)$ and $D(\partial O)$ coincide up to a factor 2.

**Lemma 3.4.** *For a simply connected bounded Lipschitz domain $O$ in $\mathbb{R}^2$,*

$$P(O) = 2D(\partial O).$$

*Proof.* Given $x \in \partial O$, let us consider some $y \in \partial O$ such that

$$d_{\partial O}(x, y) = \sup_{\tilde{y} \in \partial O} d_{\partial O}(x, \tilde{y}).$$

Such point $y$ exists because the function $\tilde{y} \mapsto d_{\partial O}(x, \tilde{y})$ is continuous on the compact set $\partial O$. In dimension 2, there are only two paths $g_1$ and $g_2$ joining $x$ and $y$ on $\partial O$. It happens that $\ell(g_1) = \ell(g_2)$. Indeed, if we had for example $\ell(g_1) > \ell(g_2)$, then $d_{\partial O}(x, y) = \ell(g_2)$. There would exist some $\tilde{y} \in \partial O$ such that the two paths $\tilde{g_1}$ and $\tilde{g_2}$ joining $x$ to $\tilde{y}$ satisfy

$$\ell(g_1) > \ell(\tilde{g_1}) \geq \ell(\tilde{g_2}) > \ell(g_2) = d_{\partial O}(x, y),$$

and this would contradict the fact that $y$ maximizes $\tilde{y} \mapsto d_{\partial O}(x, \tilde{y})$ on $\partial O$. Since $\ell(g_1) = \ell(g_2)$, we have

$$P(O) = \ell(g_1) + \ell(g_2) = 2\ell(g_2) = 2 \sup_{\tilde{y} \in \partial O} d_{\partial O}(x, \tilde{y}).$$

We see that $\sup_{\tilde{y} \in \partial O} d_{\partial O}(x, \tilde{y})$ is independent of $x$ and then $P(O) = 2D(\partial O)$. □

We will also need the following lemma, which is proved in [18] (see theorem IX.17 and remark 20).

**Lemma 3.5.** *Let $\Omega$ denote an open subset of $\mathbb{R}^d$, and $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$. Then $u \in H_0^1(\Omega)$, where $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.*

Let us now state our uniqueness result for problem (2) in two dimensions: it establishes a minimal time $T$ which guaranties uniqueness given an imposed bound $P$ on the perimeter of the unknown obstacle.

**Theorem 3.6.** *For $d = 2$, let us consider two domains $O_1$, $O_2$ and corresponding functions $u_1$, $u_2$ which satisfy (2) with data $(g_0, g_1)$ on $\Gamma$. We assume in addition that $u_i \in L^2(0, T; C^0(\overline{\Omega_i}))$, $i = 1, 2$, and that for all $x_0 \in \Gamma$ and all sufficiently small $t_0 > 0$, the function $g_0(x_0, \cdot)$ is not identically zero in the interval $(0, t_0)$. Let us*

*denote by $P$ an upper bound of $P(O_1)$ and $P(O_2)$ and $D = D(G,\Gamma)$. If we assume that*

$$T > P + 2D,$$

*we have $O_1 = O_2$.*

*Proof.* Let us define $\Omega_{12}$ as the connected component of $G \setminus (\overline{O_1} \cup \overline{O_2})$ which is in contact with $\Gamma$ and let also denote $O_{12} = G \setminus \overline{\Omega_{12}}$. The function $u := u_1 - u_2$ satisfies in $\Omega_{12} \times (0,T)$ the system

$$\left\{ \begin{array}{rcll} \partial_t^2 u - \Delta u & = & 0 & \text{in } \Omega_{12} \times (0,T) \\ u & = & 0 & \text{on } \Gamma \times (0,T) \\ \partial_\nu u & = & 0 & \text{on } \Gamma \times (0,T) \\ u,\ \partial_t u & = & 0 & \text{on } \Omega_{12} \times \{0\}. \end{array} \right.$$

Let us consider some point $x \in O_{12}$. For any $\varepsilon > 0$ there exists some $x_0 \in \Gamma$ such that

$$d_G(x,x_0) \leq d_G(x,\Gamma) + \varepsilon.$$

Let $g$ be a path joining $x_0$ to $x$ in $G$ such that $\ell(g) \leq d_G(x,x_0) + \varepsilon$, hence $\ell(g) \leq d_G(x,\Gamma) + 2\varepsilon \leq D + 2\varepsilon$. The path $g$ cuts the boundary of $O_{12}$ at some point $x_{12}$ such that the restriction $\tilde{g}$ of $g$ joining $x_{12}$ to $x_0$ belongs to $\Omega_{12}$. Without loss of generality, the point $x_{12}$ belongs to the boundary of $O_1$, so that $x_{12}$ is renamed $x_1$. Obviously, $d_{\Omega_{12}}(x_1,x_0) \leq \ell(\tilde{g}) \leq \ell(g)$, so that

$$D \geq d_{\Omega_{12}}(x_1,x_0) - 2\varepsilon. \tag{5}$$

Now let us consider the set $R = O_{12} \setminus \overline{O_2}$ and assume that $R$ is not empty. The boundary of $R$ is partitioned into a subpart of $\partial O_2$ and $\Gamma_{12} = \partial\Omega_{12} \cap \partial O_1$. For any point $\tilde{x}_1$ of $\Gamma_{12}$, we have

$$P(O_{12})/2 \geq d_{\Omega_{12}}(\tilde{x}_1,x_1) - \varepsilon, \tag{6}$$

where $P(O_{12})$ is the perimeter of $O_{12}$ (see Remark 1). That $P(O_{12}) \leq 2P$ together with (5) and (6) imply

$$D + P \geq d_{\Omega_{12}}(x_1,x_0) + d_{\Omega_{12}}(\tilde{x}_1,x_1) - 3\varepsilon \geq d_{\Omega_{12}}(\tilde{x}_1,x_0) - 3\varepsilon.$$

We first propagate the vanishing property of function $u$ from point $x_0$ to point $\tilde{x}_1$. By using the second part of Proposition 5 in $\Omega_{12}$, we obtain from the above estimate of $d_{\Omega_{12}}(\tilde{x}_1,x_0)$ that for $T > P + 2D$, the function $u(\tilde{x}_1,\cdot)$ vanishes in the interval $(0, T - D - P - 3\varepsilon)$. Since $\tilde{x}_1$ was an arbitrarily chosen point of $\Gamma_{12}$, we obtain that $u$ vanishes in $\Gamma_{12} \times (0, T - D - P - 3\varepsilon)$. We remark that any point of the boundary of $R$ is either on $\partial O_2$ or on $\partial O_{12} \cap \partial O_1$, we hence have $u_2 = 0$ on $\partial R \times (0, T - D - P - 3\varepsilon)$. Because of the assumption on the continuity of $u_2$, we obtain from Lemma 3.5 that $u_2 \in L^2(0, T - D - P - 3\varepsilon; H_0^1(R))$. Using the initial condition for $u_2$ we conclude that $u_2$ vanishes in $R \times (0, T - D - P - 3\varepsilon)$.
We now propagate the vanishing property of function $u_2$ from point $x_1$ to point $x_0$. If we reuse the second part of Proposition 5 in $\Omega_{12}$ as well as the estimate of $d_{\Omega_{12}}(x_1,x_0)$ given by (5), we obtain that for $T > P + 2D$, $u_2$ vanishes at point $x_0$ in the time interval $(0, T - 2D - P - 5\varepsilon)$. Since $\varepsilon$ is arbitrarily small we obtain a contradiction to the assumption on $g_0$. We conclude that $R$ is empty. Since $O_1 \cup O_2 \subset O_{12} \subset O_2$, we conclude that $O_1 \subset O_2$. The same reasoning is then applied with some point $x \in O_2$ to obtain the same contradiction, hence $O_1 = O_2$. □

**Remark 1.** Here we have used the fact that although $O_{12}$ is not a Lipschitz domain, $P(O_{12})$ is still defined since its boundary is still rectifiable and $P(O_{12}) \leq P(O_1 \cup O_2) \leq P(O_1) + P(O_2)$.

It is natural to compare the minimal time $T(O) := P(O) + 2D(G,\Gamma)$ in Theorem 3.6 with $D(\Omega,\Gamma)$. We have the following result, which shows that $T(O) \geq 2D(\Omega,\Gamma)$.

**Lemma 3.7.** *Let us consider two open domains $G$ and $O$ of $\mathbb{R}^2$, such that $G$ is a bounded Lipschitz domain and $O$ is formed by a collection of a finite number of simply connected Lipschitz domains with $O \Subset G$, $\Omega = G \setminus \overline{O}$ is connected and $\Gamma$ a non-empty open set of $\partial G$. We have*

$$2D(\Omega,\Gamma) \leq P(O) + 2D(G,\Gamma).$$

*Proof.* Let us consider some $x \in \Omega$. For arbitrarily small $\varepsilon > 0$, one may find some $x_0 \in \Gamma$ such that $d_G(x,x_0) \leq d_G(x,\Gamma) + \varepsilon$. If $d_\Omega(x,x_0) = d_G(x,x_0)$, then $d_\Omega(x,\Gamma) \leq d_\Omega(x,x_0) \leq d_G(x,\Gamma) + \varepsilon$, and since the inequality is valid for all $\varepsilon$ and all $x \in \Omega$, we obtain

$$D(\Omega,\Gamma) \leq D(G,\Gamma).$$

Otherwise, let us consider some path $g$ joining $x$ to $x_0$ in $G$ such that $\ell(g) \leq d_G(x,x_0) + \varepsilon$. Clearly, there exists a curve $h$ in $\overline{\Omega}$ joining $x_0$ to $x$ consisting of two parts: one part coincides with a sub-path of $g$, the other part coincides with a subpart of the boundary of $O$. The length of the first part is lower than $\ell(g)$ while the length of the second part is lower than $D(\partial O)$, so that $\ell(h) \leq D(\partial O) + \ell(g)$. Hence,

$$\begin{aligned} d_\Omega(x,\Gamma) \leq d_\Omega(x,x_0) \leq \ell(h) + \varepsilon &\leq D(\partial O) + d_G(x,\Gamma) + 3\varepsilon \\ &\leq D(\partial O) + D(G,\Gamma) + 3\varepsilon, \end{aligned}$$

and lastly

$$D(\Omega,\Gamma) \leq D(\partial O) + D(G,\Gamma),$$

which is the result, in view of Lemma 3.4. □

4. **The method of quasi-reversibility.** Before addressing the inverse obstacle problem, let us focus on a linear ill-posed Cauchy problem for the wave equation in the domain $\Omega \times (0,T)$, namely

$$\left\{ \begin{array}{rcll} \partial_t^2 u - \Delta u & = & 0 & \text{in } \Omega \times (0,T) \\ u & = & g_0 & \text{on } \Gamma \times (0,T) \\ \partial_\nu u & = & g_1 & \text{on } \Gamma \times (0,T) \\ u,\ \partial_t u & = & 0 & \text{on } \Omega \times \{0\}. \end{array} \right. \tag{7}$$

Problem (7) is a simplified version of problem (2) provided the obstacle $O$, and hence the complementary domain $\Omega$, is known. However, it is crucial in view of the"exterior approach" introduced hereafter, to regularize such problem without taking into account the boundary condition on $\partial O$ in the problem (7). If we assume existence of a solution $u \in H^1(Q)$ to problem (7), which means in some sense that $(g_0, g_1)$ are exact data, from Theorem 3.3 $u$ is uniquely defined in $Q_0 = \{(x,t) \in \Omega \times (0,T),\ d_\Omega(x,\Gamma) < T - t\}$, and in the particular case $T > D(\Omega,\Gamma)$, uniquely defined in $\Omega \times (0, T - D(\Omega,\Gamma))$.

We now describe our mixed formulation of quasi-reversibility. Let us introduce the sets $\tilde{\Gamma} := \partial\Omega \setminus \overline{\Gamma}$, $\tilde{\Sigma} = \tilde{\Gamma} \times (0,T)$ and $S_T = \Omega \times \{T\}$. Let us also introduce

$$V = \{u \in H^1(Q),\ u|_{S_0} = 0\}, \tag{8}$$

$$\tilde{V}_0 = \{\lambda \in H^1(Q),\ \lambda|_{\tilde{\Sigma}} = 0,\ \lambda|_{S_T} = 0\}, \tag{9}$$

$$H^{1/2}_{S_0}(\Sigma) = \{g \in L^2(\Sigma),\ g = u|_\Sigma,\ u \in H^1(Q),\ u|_{S_0} = 0\} \tag{10}$$

and for $g_0 \in H^{1/2}_{S_0}(\Sigma)$,

$$V_g = \{u \in H^1(Q),\ u|_\Sigma = g_0,\ u|_{S_0} = 0\}, \quad V_0 = \{u \in H^1(Q),\ u|_\Sigma = 0,\ u|_{S_0} = 0\}.$$

The spaces $V$, $V_0$ and $\tilde{V}_0$ are endowed with the same norm $\|\cdot\|$ given by

$$\|u\|^2 = \int_0^T \int_\Omega (\partial_t u)^2 \, dxdt + \int_0^T \int_\Omega |\nabla u|^2 \, dxdt.$$

That $\|\cdot\|$ is actually a norm in these three spaces is a consequence of Poincaré's inequality. We define $H^{1/2}_{\tilde{\Sigma},S_T}(\Sigma)$ as the set of traces on $\Sigma$ of functions in $H^1(Q)$ that vanish on $\tilde{\Sigma}$ and on $S_T$, that is in other words the restrictions on $\Sigma$ of functions in $\tilde{V}_0$. Its dual space $H^{-\frac{1}{2}}_{S_0}(\Sigma)$ coincides with the set of restrictions on $\Sigma$ of distributions in $H^{-\frac{1}{2}}(\partial Q)$ the support of which is contained in $\overline{\Sigma \cup \tilde{\Sigma} \cup S_T}$. For what follows we need the following weak characterization of the solutions to problem (7).

**Lemma 4.1.** *For $(g_0, g_1) \in H^{1/2}_{S_0}(\Sigma) \times H^{-1/2}_{S_0}(\Sigma)$, the function $u \in H^1(Q)$ is a solution to problem (7) if and only if $u \in V_g$ and for all $\mu \in \tilde{V}_0$,*

$$-\int_Q \partial_t u \, \partial_t \mu \, dxdt + \int_Q \nabla u \cdot \nabla \mu \, dxdt = \int_\Sigma g_1 \, \mu \, dsdt, \tag{11}$$

*where the last integral means duality pairing between $H^{-1/2}_{S_0}(\Sigma)$ and $H^{1/2}_{\tilde{\Sigma},S_T}(\Sigma)$.*

*Proof.* We first consider some $u \in H^1(Q)$ which solves problem (7). Then $u \in V_g$. For $\mu \in H^1(Q)$, we use the following integration by parts formula, with $\mathbf{v} = (\nabla u, -\partial_t u) \in (L^2(Q))^{d+1}$, $\mathrm{div}_{d+1}\mathbf{v} \in L^2(Q)$ and $\nabla_{d+1}\mu = (\nabla\mu, \partial_t\mu) \in (L^2(Q))^{d+1}$,

$$\int_Q \mathbf{v} \cdot \nabla_{d+1}\mu \, dxdt = -\int_Q \mu(\mathrm{div}_{d+1}\mathbf{v}) \, dxdt + \langle \mathbf{v} \cdot \nu_{d+1}, \mu\rangle_{\partial Q}\,, \tag{12}$$

where the bracket $\langle\cdot,\cdot\rangle_{\partial Q}$ has the meaning of duality between $H^{-1/2}(\partial Q)$ and $H^{1/2}(\partial Q)$. By considering now $\mu \in \tilde{V}_0$, we use the fact that $\mathrm{div}_{d+1}\mathbf{v} = 0$ in $Q$, that $\partial_t u = 0$ on $S_0$, $\mu = 0$ on $S_T$, $\partial_\nu u = g_1$ on $\Sigma$ and $\mu = 0$ on $\tilde{\Sigma}$ to obtain

$$\int_Q \mathbf{v} \cdot \nabla_{d+1}\mu \, dxdt = \langle g_1, \mu\rangle_\Sigma\,,$$

where the bracket $\langle\cdot,\cdot\rangle_\Sigma$ has the meaning of duality between $H^{-1/2}_{S_0}(\Sigma)$ and $H^{1/2}_{\tilde{\Sigma},S_T}(\Sigma)$. The weak formulation (11) is achieved. Conversely, let us consider some $u \in V_g$ which satisfies (11). By taking $\mu \in C^\infty_0(Q)$, it follows that $u$ solves the wave equation in the distributional sense in $Q$. Hence formula (12) is valid, and we obtain that for all $\mu \in \tilde{V}_0$

$$\langle \mathbf{v} \cdot \nu_{d+1}, \mu\rangle_{\partial Q} = \langle g_1, \mu\rangle_\Sigma\,,$$

so that by using the fact that $\mu = 0$ on $\tilde{\Sigma}$ and $\mu = 0$ on $S_T$,

$$\langle \mathbf{v} \cdot \nu_{d+1}, \mu\rangle_{\Sigma_0} = \langle g_1, \mu\rangle_\Sigma\,,$$

where $\Sigma_0 = \Sigma \cup S_0$ and the brackets $\langle \cdot, \cdot \rangle_{\Sigma_0}$ have the meaning of duality between $H^{-1/2}(\Sigma_0)$ and $\tilde{H}^{1/2}(\Sigma_0)$. By using the fact that $g_1 \in H_{S_0}^{-1/2}(\Sigma)$, we obtain that its extension $\tilde{g}_1$ by 0 on $\Sigma_0 = \Sigma \cup S_0$ satisfies for all $\mu \in \tilde{V}_0$,

$$\langle \mathbf{v} \cdot \nu_{d+1}, \mu \rangle_{\Sigma_0} = \langle \tilde{g}_1, \mu \rangle_{\Sigma_0} .$$

We conclude that $\mathbf{v} \cdot \nu_2 = \tilde{g}_1$ on $\Sigma_0$, then $\mathbf{v} \cdot \nu_{d+1} = g_1$ on $\Sigma$ and $\mathbf{v} \cdot \nu_{d+1} = 0$ on $S_0$, that is $\partial_\nu u = g_1$ on $\Sigma$ and $\partial_t u = 0$ on $S_0$. As a conclusion, problem (7) is satisfied by $u$. □

Due to Lemma 4.1, problem (7) is a particular instance of the abstract framework described in [10], which we recall here. We consider three Hilbert spaces $V$, $M$ and $H$, endowed with the scalar products $(\cdot, \cdot)_V$, $(\cdot, \cdot)_M$ and $(\cdot, \cdot)_H$ and corresponding norms $\| \cdot \|_V$, $\| \cdot \|_M$ and $\| \cdot \|_H$. We denote $A : V \to H$ a continuous onto operator while for some $g \in H$, we denote $V_g = \{u \in V,\ Au = g\}$, which is an affine space. The corresponding vector space is denoted $V_0$, equipped with the norm $\| \cdot \|_V$. For a continuous bilinear form $b$ on $V \times M$ and a linear form $m$ on $M$, let us consider the abstract weak formulation: find $u \in V_g$ such that for all $\mu \in M$,

$$b(u, \mu) = m(\mu). \tag{13}$$

The bilinear form $b$ is said to satisfy the $\inf - \sup$ property on $V_0 \times M$ if

**Assumption 4.2.** There exists $\alpha > 0$ such that

$$\inf_{\substack{u \in V_0 \\ u \neq 0}} \sup_{\substack{\mu \in M \\ \mu \neq 0}} \frac{b(u, \mu)}{\|u\|_V \|\mu\|_M} \geq \alpha.$$

The bilinear form $b$ is said to satisfy the solvability property on $V_0 \times M$ if

**Assumption 4.3.** For all $\mu \in M$,

$$\forall u \in V_0, \quad b(u, \mu) = 0 \Longrightarrow \mu = 0.$$

According to the Brezzi-Nečas-Babuška theorem (see, for example, [14]), the problem (13) is well-posed if and only if both conditions 4.2 and 4.3 are satisfied. Conversely, if either 4.2 or 4.3 fail to hold, then problem (13) is ill-posed. Regardless of assumptions 4.2 or 4.3, a regularized formulation of ill-posed problem (13) is the following: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in V_g \times M$ such that for all $(v, \mu) \in V_0 \times M$,

$$\left\{ \begin{array}{c} \varepsilon (u_\varepsilon, v)_V + b(v, \lambda_\varepsilon) = 0 \\ b(u_\varepsilon, \mu) - (\lambda_\varepsilon, \mu)_M = m(\mu). \end{array} \right. \tag{14}$$

The following theorem is proved in [10].

**Theorem 4.4.** *For any $g \in H$ and $m \in M'$, the problem (14) has a unique solution. In addition, if $g \in H$ and $m \in M'$ are such that (13) has at least one solution, then the solution $(u_\varepsilon, \lambda_\varepsilon) \in V_g \times M$ satisfies $(u_\varepsilon, \lambda_\varepsilon) \to (u_m, 0)$ in $V \times M$ when $\varepsilon \to 0$, where $u_m$ is the unique solution to the minimization problem*

$$\inf_{v \in K} \|v\|_V, \quad K := \{v \in V_g,\ b(v, \mu) = m(\mu),\ \forall \mu \in M\}. \tag{15}$$

Our problem (11) coincides with the general problem (13) by choosing spaces $V$ as specified by (8), $M = \tilde{V}_0$ and $H = H^{1/2}_{S_0}(\Sigma)$ as specified by (9) and (10), respectively. The operator $A : \{u \in H^1(Q),\ u|_{S_0} = 0\} \mapsto H^{1/2}_{S_0}(\Sigma)$ is the trace mapping on $\Sigma$, which is onto by definition of $H^{1/2}_{S_0}(\Sigma)$. Lastly, the bilinear form $b$ and linear form $m$ are given, for $(u, \mu) \in V \times M$, by

$$b(u,\mu) = -\int_Q \partial_t u\, \partial_t \mu\, dxdt + \int_Q \nabla u \cdot \nabla \mu\, dxdt$$

and

$$m(\mu) = \int_\Sigma g_1\, \mu\, dsdt,$$

where the integral means duality pairing between $H^{-1/2}_{S_0}(\Sigma)$ and $H^{1/2}_{\tilde{\Sigma},S_T}(\Sigma)$. The regularized formulation (14), which can be directly applied to our problem, consists in the following problem for some real $\varepsilon > 0$: for $(g_0, g_1) \in H^{1/2}_{S_0}(\Sigma) \times H^{-1/2}_{S_0}(\Sigma)$, find $(u_\varepsilon, \lambda_\varepsilon) \in V_g \times \tilde{V}_0$ such that for all $(v, \mu) \in V_0 \times \tilde{V}_0$,

$$\left\{\begin{array}{l} -\displaystyle\int_Q \partial_t v\, \partial_t \lambda_\varepsilon\, dxdt + \varepsilon \int_Q \partial_t u_\varepsilon\, \partial_t v\, dxdt \\ +\displaystyle\int_Q \nabla v \cdot \nabla \lambda_\varepsilon\, dxdt + \varepsilon \int_Q \nabla u_\varepsilon \cdot \nabla v\, dxdt = 0, \\ -\displaystyle\int_Q \partial_t u_\varepsilon\, \partial_t \mu\, dxdt - \int_Q \partial_t \lambda_\varepsilon\, \partial_t \mu\, dxdt + \int_Q \nabla u_\varepsilon \cdot \nabla \mu\, dxdt \\ -\displaystyle\int_Q \nabla \lambda_\varepsilon \cdot \nabla \mu\, dxdt = \int_\Sigma g_1 \mu\, dsdt, \end{array}\right. \tag{16}$$

where the last integral has the meaning of duality pairing between $H^{-1/2}_{S_0}(\Sigma)$ and $H^{1/2}_{\tilde{\Sigma},S_T}(\Sigma)$. Due to Theorem 4.4, which we apply in our particular case, recalling Theorem 3.3, we obtain the following result.

**Theorem 4.5.** *For any $(g_0, g_1) \in H^{1/2}_{S_0}(\Sigma) \times H^{-1/2}_{S_0}(\Sigma)$, problem (16) has a unique solution $(u_\varepsilon, \lambda_\varepsilon) \in V_g \times \tilde{V}_0$. If in addition we assume there exists $u \in H^1(Q)$ satisfying problem (7) for $(g_0, g_1) \in H^{1/2}_{S_0}(\Sigma) \times H^{-1/2}_{S_0}(\Sigma)$, the solution $(u_\varepsilon, \lambda_\varepsilon)$ of problem (16) associated with the same data $(g_0, g_1)$ satisfies*

$$\lim_{\varepsilon \to 0} u_\varepsilon = u_m \text{ in } H^1(Q), \quad \lim_{\varepsilon \to 0} \lambda_\varepsilon = 0 \text{ in } H^1(Q),$$

*where $u_m$ is the unique solution of minimal norm $\|\cdot\|$ to problem (7). In particular, $u_m$ coincides with $u$ in $Q_0$ defined by (4) and if $T > D(\Omega, \Gamma)$, $u_m$ coincides with $u$ in subdomain $\Omega \times (0, T - D(\Omega, \Gamma))$.*

## 5. The level set method.

We present a simple level set method and show it enables us to identify the obstacle $O$ assuming that the function $u$ which solves (2) is known.

### 5.1. Preliminaries.

This section presents, in a slightly more general case, some results already given in [7]. Let us consider a function $U \in H^1(G)$ such that, for some obstacle $O \Subset G$ formed by a collection of a finite number of disjoint simply

connected open Lipschitz domains and $\Omega = G \setminus \overline{O}$, we have

$$\left\{ \begin{array}{rcll} U & \geq & 0 & \text{in} \quad \Omega \\ U & = & 0 & \text{on} \quad \partial\Omega = \partial O \\ U & \leq & 0 & \text{in} \quad O, \end{array} \right. \tag{17}$$

where the second line is satisfied in the sense of trace. We now define a sequence of open domains $(\omega_n)$ by following induction. Let us choose $f \in H^{-1}(G)$ such that

$$f \geq \Delta U \tag{18}$$

in the sense of $H^{-1}(G)$ and an open domain $\omega_0$ such that $O \subset \omega_0 \Subset G$ . The open domain $\omega_n$ being given, we define

$$\omega_{n+1} = \omega_n \setminus \text{supp}(\sup(v_{\omega_n}, 0)), \tag{19}$$

where $v_\omega := w_\omega + U$ and $w_\omega$ is the unique solution $w$ in $H_0^1(\omega)$ of problem $\Delta w = f - \Delta U$. Here supp denotes the support of a function.

**Remark 2.** With additional regularity assumptions, we can give a simpler definition of $v_{\omega_n}$ and $\omega_{n+1}$. Indeed, in the case when $\omega_n$ is a Lipschitz domain, the function $v_{\omega_n}$ is the unique solution to the problem

$$\left\{ \begin{array}{rcll} \Delta v & = & f & \text{in} \quad \omega_n \\ v & = & U & \text{on} \quad \partial\omega_n, \end{array} \right.$$

while when $v_{\omega_n}$ is a continuous function, the domain $\omega_{n+1}$ is defined by

$$\omega_{n+1} = \{x \in \omega_n, \quad v_{\omega_n}(x) < 0\}.$$

Our level set method essentially relies on the weak maximum principle (see, for example, [17]).

**Lemma 5.1.** *Let $\Omega$ be an open domain of $\mathbb{R}^d$, and $u \in H^1(\Omega)$ such that $\Delta u \geq 0$ in the sense of $H^{-1}(\Omega)$, and $\sup(u, 0) \in H_0^1(\Omega)$. Then $\sup(u, 0) = 0$ in $\Omega$.*

In what follows, for a functional space $V$, $V^+$ (resp. $V^-$) denotes the set of functions in $V$ such that $V \geq 0$ (resp. $V \leq 0$) almost everywhere. For example, the function $u$ of Lemma 5.1 satisfies $u \in H_0^1(\Omega)^-$ We begin with the following proposition.

**Proposition 1.** *For all $n \in \mathbb{N}$, we have $O \subset \omega_{n+1} \subset \omega_n \Subset G$.*

*Proof.* Let us assume that $O \subset \omega_n \Subset G$. We have to prove that $O \subset \omega_{n+1}$. By using Lemma 5.1 and the fact that $f - \Delta U \geq 0$, we obtain $w_{\omega_n} \leq 0$ in $\omega_n$, that is $v_{\omega_n} = w_{\omega_n} + U \leq U$ in $\omega_n$. Since $U \leq 0$ in $O \subset \omega_n$, we obtain $v_{\omega_n} \leq 0$ in $O$. Hence $O \subset \omega_n \setminus \text{supp}(\sup(v_{\omega_n}, 0)) = \omega_{n+1}$. The proof is complete. □

Proposition (1) immediately implies the following proposition by using [15].

**Proposition 2.** *The sequence of open domains $\omega_n$ converges, in the sense of the Hausdorff distance for open domains, to the set*

$$\omega_\infty = \text{Int}\left(\bigcap_n \omega_n\right),$$

*such that $O \subset \omega_\infty \Subset G$.*

The definition of the Hausdorff distance for open domains and various relative properties are detailed in [15]. We need the following lemma, which is proved in [15] (corollary 3.1.12).

**Lemma 5.2.** *Let $\Omega$ be an open domain of $\mathbb{R}^d$, and $u, v$ two functions in $H^1(\Omega)$ (resp. $H_0^1(\Omega)$). Then $\inf(u,v)$, $\sup(u,v) \in H^1(\Omega)$ (resp $\in H_0^1(\Omega)$). Furthermore, the mappings $(u,v) \mapsto \inf(u,v)$ and $(u,v) \mapsto \sup(u,v)$ ($H^1(\Omega) \times H^1(\Omega) \to H^1(\Omega)$) are continuous.*

We now consider the following assumption, which concerns the continuity of the Dirichlet solution to the Laplace equation with respect to the domain and is extensively analyzed in [15].

**Assumption 5.3.** For $f \in H^{-1}(G)$, if $w_\omega$ denotes the solution in $H_0^1(\omega)$ of problem $\Delta w = f$ in the domain $\omega$, the sequence $(w_{\omega_n})$ tends to $w_{\omega_\infty}$ in $H_0^1(G)$ when $n \to +\infty$.

**Remark 3.** Some sufficient conditions on the sequence $(\omega_n)$ that enable us to fulfill 5.3 are given in [15]. In particular, for $d = 2$, a sufficient condition to have 5.3 is that all the sets $G \setminus \overline{\omega_n}$, $n \in \mathbb{N}$, are connected, which is a consequence of Šverák's Theorem (see theorem 3.4.14 in [15]).

**Proposition 3.** *Let us consider the open domain $\omega_\infty$ defined by Proposition 2 and assume that 5.3 is satisfied. If $R := \omega_\infty \setminus \overline{O} \neq \emptyset$, we have $U \in H_0^1(R)$.*

*Proof.* The assumption 5.3 implies that $w_{\omega_n} \to w_{\omega_\infty}$ in $H_0^1(G)$, and hence $v_{\omega_n} \to v_{\omega_\infty}$ in $H^1(G)$. Since $\omega_\infty \subset \omega_n$ for all $n$, we have $v_{\omega_n} \leq 0$ a.e. in $\omega_\infty$, hence $v_{\omega_\infty} \leq 0$ a.e. in $\omega_\infty$.
Now let us prove that $U \in H_0^1(R)$. We recall that $U - v_{\omega_\infty} \geq 0$ in $\omega_\infty$, that is $U - v_{\omega_\infty} \in H_0^1(\omega_\infty)^+$. By using corollary 3.1.13 in [15], there exists a sequence $\psi_n \in C_0^\infty(\omega_\infty)^+$ such that $\psi_n \to U - v_{\omega_\infty}$ in $H^1(\omega_\infty)$. Let $\varphi \in C_0^\infty(G)^+$, $\varphi \equiv 1$ on $\overline{\omega_\infty}$. Since $U|_{\partial O} = 0$, $U|_\Omega \geq 0$ and $\varphi|_{\partial G} = 0$, we have $\varphi U \in H_0^1(G \setminus \overline{O})^+$. As a consequence there exists $\varphi_n \in C_0^\infty(G \setminus \overline{O})^+$ such that $\varphi_n \to \varphi U$ in $H^1(G \setminus \overline{O})$. Now, by using Lemma 5.2, the functions $\theta_n := \inf(\varphi_n|_R, \psi_n|_R)$ converge to $\inf(\varphi U|_R, (U - v_{\omega_\infty})|_R) = U|_R$ in $H^1(R)$, since $v_{\omega_\infty} \leq 0$ a.e. in $\omega_\infty$ and $\varphi \equiv 1$ on $\omega_\infty$.
Let us denote $K_n$ and $L_n$ the supports of $\varphi_n$ and $\psi_n$ respectively. We have $K_n \subset G \setminus \overline{O}$ and $L_n \subset \omega_\infty$, hence $K_n \cap L_n \subset R$. For $x \in R \setminus (K_n \cap L_n)$, either $\varphi_n(x) = 0$ and $\psi_n(x) \geq 0$, or $\varphi_n \geq 0$ and $\psi_n = 0$, hence $\theta_n(x) = 0$. This implies that $\mathrm{supp}(\theta_n) \subset K_n \cap L_n$, that is $\theta_n$ in compactly supported in $R$. Since $\theta_n$ converges to $U$ in $H^1(R)$, $U \in H_0^1(R)$. □

5.2. **Convergence of the level set method.** Let us now set $d = 2$ and apply the previous results to some particular function $U$. Let us assume that some obstacle $O$ and some solution $u$ satisfy problem (2) with $T > T(O) = P(O) + 2D(G, \Gamma)$. From Theorem 3.3 and Theorem 3.6, at most one single pair $(O, u)$ is compatible with data $(g_0, g_1)$. We define a function $U$ in the whole domain $G$ such that

$$\begin{cases} U & = & \left( \displaystyle\int_0^{T - T(O)/2} (u(\cdot, t))^2 \, dt \right)^{\frac{1}{2}} \quad \text{in } \Omega \\ U|_O & \in & H_0^1(O)^-. \end{cases} \tag{20}$$

Note that the integral in (20) is uniquely defined from $(g_0, g_1)$ since $T(O) \geq 2D(\Omega, \Gamma)$. Indeed, $T - T(O)/2 \leq T - D(\Omega, \Gamma)$ and the uniqueness domain of the function $u$ contains $\Omega \times (0, T - D(\Omega, \Gamma))$ from Theorem 3.3. Besides, by taking simply $U = 0$ in $O$, we see that we can find at least one function $U$ satisfying (20). Let us prove the following result.

**Lemma 5.4.** *If the function $U$ satisfies (20), it belongs to $H^1(G)$ and satisfies the assumptions (17).*

*Proof.* First, that $u \in H^1(Q)$ implies

$$\int_\Omega U^2\,dx = \int_0^{T-T(O)/2} \int_\Omega (u(x,t))^2\,dxdt < +\infty.$$

Moreover, for $i = 1, 2$,

$$\frac{\partial U}{\partial x_i} = \frac{1}{\left(\int_0^{T-T(O)/2}(u(\cdot,t))^2\,dt\right)^{\frac{1}{2}}} \int_0^{T-T(O)/2} \frac{\partial u(\cdot,t)}{\partial x_i} u(\cdot,t)\,dt$$

Hence

$$\left|\frac{\partial U}{\partial x_i}\right| \leq \left(\int_0^{T-T(O)/2} \left(\frac{\partial u(\cdot,t)}{\partial x_i}\right)^2 dt\right)^{\frac{1}{2}}.$$

Then

$$\int_\Omega \left(\frac{\partial U}{\partial x_i}\right)^2 dx \leq \int_0^{T-T(O)/2} \int_\Omega \left(\frac{\partial u(x,t)}{\partial x_i}\right)^2 dxdt < +\infty.$$

This proves that $U|_\Omega \in H^1(\Omega)$. The homogeneous Dirichlet boundary condition satisfied by $u$ on $\partial O$ (see problem (2)) implies that $U|_{\partial\Omega} = 0$, which together with $U|_O \in H_0^1(O)$ implies that $U \in H^1(G)$. In addition, since $U \geq 0$ on $\Omega$, $U = 0$ on $\partial O = \partial\Omega$ and $U \leq 0$ on $O$, the assumptions (17) are satisfied. □

The following theorem shows that under assumption 5.3 the sequence of domains $(\omega_n)$ converges to the true obstacle $O$.

**Theorem 5.5.** *We consider some obstacle $O$ and some function $u$ which satisfy problem (2) with $T > T(O)$ ($O$ and $u$ are both uniquely defined). We assume in addition that $u \in L^2(0,T;C^0(\overline{\Omega}))$, that for all $x_0 \in \Gamma$ and all sufficiently small $t_0 > 0$, the function $g_0(x_0,\cdot)$ is not identically zero in the interval $(0,t_0)$. Let $U \in H^1(G)$ and $f \in H^{-1}(G)$ satisfy (20) and (18).*
*Let $\omega_0$ denote an open domain such that $O \subset \omega_0 \Subset G$ and $(\omega_n)$ denotes the decreasing sequence of open domains defined by (19).*
*With additional assumption 5.3, we have*

$$\mathrm{Int}\left(\bigcap_n \omega_n\right) = O,$$

*with convergence in the sense of Hausdorff distance for open domains.*

*Proof.* We already know from Proposition 2 that $O \subset \omega_\infty$. Let us denote $R = \omega_\infty \backslash \overline{O}$ and let us assume that $R \neq \emptyset$. We shall find a contradiction. From Proposition 3 we have $U \in H_0^1(R)$.
We remark that $u = 0$ in $\partial R \times (0, T - T(O)/2)$, because

$$U^2 = \int_0^{T-T(O)/2} (u(\cdot,t))^2\,dt \quad \text{in} \quad R$$

and $u(\cdot,t) \in C^0(\overline{\Omega})$ for almost all $t$. By using the initial condition in (2), we get $u = 0$ in $R \times (0, T - T(O)/2)$. By the same unique continuation argument that was used in the proof of Theorem 3.6 and since $T > T(O)$, we obtain that there exists some $x_0 \in \Gamma$ such that $u(x_0,\cdot)$ vanishes in the interval $(0, T - T(O)/2 - T(O)/2)$, that is $(0, T - T(O))$, which contradicts our assumption on $g_0$. We conclude that

$R = \omega_\infty \setminus \overline{O} = \emptyset$. As a conclusion, $O \subset \omega_\infty \subset \overline{O}$. Since $O$ has a continuous boundary, the interior of the set $\overline{O}$ is $O$, and lastly $O = \omega_\infty$. □

**Remark 4.** It is not clear for us that $T > 2D(\Omega, \Gamma)$ suffices to have uniqueness in problem (2). Note however that if uniqueness holds for $T > 2D(\Omega, \Gamma)$, then it can be seen from the proof that Theorem 5.5 is valid by replacing $T(O) = P(O) + 2D(G, \Gamma)$ by the smaller distance $2D(\Omega, \Gamma)$.

6. **Description of the "exterior approach".** In this section, we deduce from the results of sections 4 and 5 an algorithm to approximately solve the problem (2) presented in the introduction, that is to retrieve the obstacle $O$ from the lateral Cauchy data $(g_0, g_1)$ on $\Gamma \times (0, T)$. In Theorem 5.5 we have shown convergence of the level set method to the true obstacle $O$ if the field defined by (20) is based on the true solution $u$. In practice such function is unknown but following Theorem 4.5, it can be approximated in the subdomain $\Omega \times (0, T - D(\Omega, \Gamma))$ with the help of the solution $u_\varepsilon$ of the quasi-reversibility problem (16). This is why we propose the following algorithm in the continuous framework.

**Algorithm** :

1. Choose an initial guess $O_0$ such that $O \subset O_0 \Subset G$ and $G \setminus \overline{O_0}$ is connected.
2. Step 2: the domain $O_n$ being given, solve the quasi-reversibility problem (16) in $\Omega_n \times (0, T)$ with $\Omega_n := G \setminus \overline{O_n}$ for some selected parameter $\varepsilon > 0$. The solution is denoted $(u_n, \lambda_n)$.
3. Step 3: the function $u_n$ being given, solve the non-homogeneous Dirichlet problem
$$\left\{ \begin{array}{rcl} \Delta\varphi_n & = & f \quad \text{in} \quad O_n \\ \varphi_n & = & V_n := \left( \displaystyle\int_0^{T - T(O_n)/2} (u_n(\cdot, t))^2 \, dt \right)^{\frac{1}{2}} \quad \text{on} \quad \partial O_n \end{array} \right. \tag{21}$$
for some selected $f \in H^{-1}(G)$. Define
$$O_{n+1} = \{x \in O_n, \quad \varphi_n(x) < 0\}.$$
4. Go back to Step 2 until the stopping criteria is reached.

Alternatively, it can be tempting, in view of Remark 4, to replace $T(O_n)$ by $2D(\Omega_n, \Gamma)$ in the above algorithm.

**Remark 5.** Since the sequence of domain $(O_n)$ is non increasing, it is natural to investigate the monotonicity of the functions $O \mapsto T(O) = P(O) + D(G, \Gamma)$ and $O \mapsto D(\Omega, \Gamma)$ (recall that $\Omega = G \setminus \overline{O}$) with respect to inclusion. The first function is clearly neither increasing nor decreasing. We now show that this property is shared by the second function on a simple example illustrated by figure 1. The domain $G$ is an equilateral triangle $ABC$ of centroid $O$, while $\Gamma$ is the whole boundary of $G$. It is easy to build some $O_1$ and $O_2$ such that $O_1 \subset O_2$ and $D(\Omega_1, \Gamma) > D(\Omega_2, \Gamma)$. It suffices to consider two disks centered at $O$: the obstacle $O_1$ is the small green disk while $O_2$ is the big red disk on the right figure of 1. Building some $O_1$ and $O_2$ such that $O_1 \subset O_2$ and $D(\Omega_1, \Gamma) < D(\Omega_2, \Gamma)$ is more involved. The obstacle $O_2$ is chosen as the domain delimited by the isosceles trapezoid represented in red on the left figure of 1. Let $H$ be the intersection of lines $(AO)$ and $(BC)$ and $M$ such that $\overrightarrow{HM} = \overrightarrow{HA}/6$. We assume that $HA = 1$, so that $HM = 1/6$. The upper side of $O_2$ is such that $M$ is its middle, while the three other sides almost touch

the sides of triangle $OBC$. The obstacle $O_1$ is a small disk of radius $\varepsilon$ centered on the line $(AO)$ and strictly contained in the trapezoid $O_2$. It is not difficult to see that $D(\Omega_2, \Gamma) = MN$ where $N$ is the orthogonal projection of $M$ on the line $(AC)$. A straightforward computation leads to $MN = 5/12 = 1/3 + 1/12$, while clearly $D(\Omega_1, \Gamma) \leq 1/3 + \pi\varepsilon$, so that $D(\Omega_1, \Gamma) < D(\Omega_2, \Gamma)$ for sufficiently small $\varepsilon$. Note that in this example, $G$ and all obstacles $O_1$ and $O_2$ are convex.

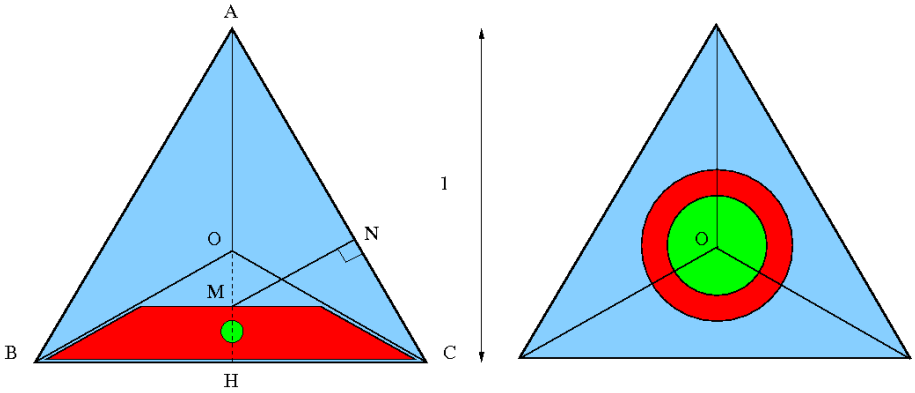



FIGURE 1. Illustration of the non-monotonicity of the mapping $O \mapsto D(\Omega, \Gamma)$

## 7. **Numerical experiments.**

7.1. **Validation of the quasi-reversibility method.** We begin with some preliminary numerical experiments in 2D with the mixed formulation of quasi-reversibility (16) to solve problem (7) when the obstacle $O$ is supposed to be known. The computation domain $\Omega \times (0,T)$, with $\Omega = G \setminus \overline{O}$, is then fixed. The domain $G$ is here the disc of center $(0,0)$ and radius $R = 5$ in $\mathbb{R}^2$, while $\Gamma$ is the whole boundary of $G$. We discretize the space/time domain $Q = \Omega \times (0,T)$ with the help of prismatic finite elements, that is tensor products of $P1$ triangular finite elements in dimension 2 for the spatial domain $\Omega$ and $P1$ finite elements in dimension 1 for the time interval $(0,T)$. We refer to [9] for a description of such finite element and an analysis of the induced discretization error in a slightly different case (heat equation instead of wave equation). Computations are obtained with the Finite Element Library XLife++ [16]. In all our computations related to the inverse problem, the time interval is $4/15$ (whatever $T$ is) while the mesh size with respect to space is such that the number of vertices on $\partial G$ is either 164 (in figures 4 and 7) or 92 (in figures 2, 3, 5 and 6). In order to build some artificial data $(g_0, g_1)$ on $\Gamma \times (0,T)$, we compute the solution to the following forward problem

$$\left\{ \begin{array}{rcll} \partial_t^2 u - \Delta u & = & 0 & \text{in } \Omega \times (0,T) \\ \partial_\nu u & = & g & \text{on } \Gamma \times (0,T) \\ u & = & 0 & \text{on } \partial O \times (0,T) \\ u,\ \partial_t u & = & 0 & \text{on } \Omega \times \{0\}, \end{array} \right. \tag{22}$$

for some Neumann data $g$. This forward computation is based on a classical finite difference (leap-frog) scheme with respect to time and a finite element method with respect to space, in such a way that the meshes and the discretizations of

the forward and inverse problems are different. Our artificial data are then given by $(g_0 = u|_{\Gamma\times(0,T)}, g_1 = g)$. We can then use these lateral Cauchy data $(g_0, g_1)$ as inputs in the mixed formulation of quasi-reversibility (16), in which we set $\varepsilon = 0.001$. We present two cases.

In the first case, the obstacle $O$ is the disc of center $(0,0)$ an radius $r = 2.5$, $T = 5.5$ while $g(x,t) = 10\sin^2(t)$, which means it is a full radial case. For that geometry, the uniqueness domain defined by (4) reduces to

$$Q_0 = \{(x,t) \in Q,\ |x| > R - T + t\}. \tag{23}$$

In the figure 2, we have plotted the discrepancy $u_\varepsilon - u$ between the quasi-reversibility solution and the exact solution for several fixed times $t \in (0,T)$ as a function of the radial distance $|x|$. From Theorem 4.5, the function $u - u_\varepsilon$ is supposed to be small in $Q_0$ for small $\varepsilon$. It can be seen that figure 2 is consistent with (23) since $R - T = -0.5$.

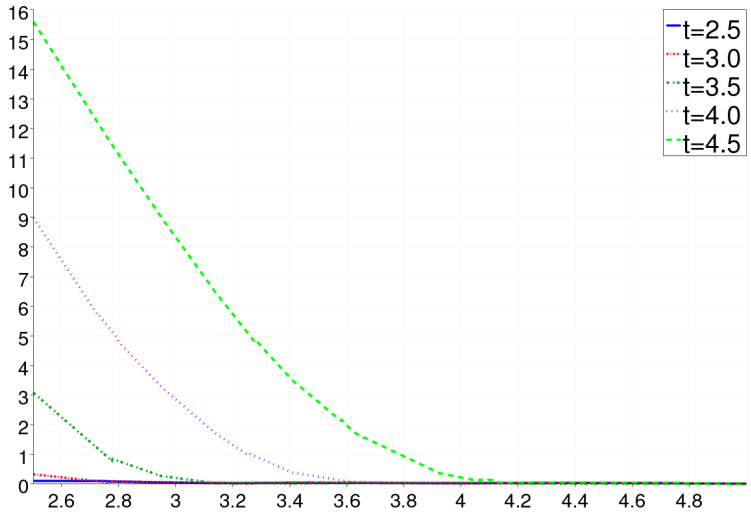



FIGURE 2. Radial case. Discrepancy $|u_\varepsilon - u|$ as a fonction of $|x|$, for $t = 2.5$, $t = 3$, $t = 3.5$, $t = 4$ and $t = 4.5$.

In the second case, the obstacle is the union of two discs, one centered at $(2.5, 1)$ of radius 1 and one centered at $(-2.5, 0)$ of radius 1.5, $T = 15$ and $g(x,t) = (20 + 10\cos(2\theta))\sin^2(t)$. Here $\theta$ is the polar angle, that is some point $x \in \partial G$ has Euclidian coordinates $(5\cos\theta, 5\sin\theta)$. On figure 3 we have plotted the quasi-reversibility solution $u_\varepsilon$ as well as the discrepancy $u_\varepsilon - u$ on the meshed space/time domain $Q = \Omega\times(0,T)$. The visible slice corresponds to $t = T/2$. It can be seen that for $t = T/2$, the error is small near the outer boundary of $\Omega$ and is concentrated near the inner boundary. This is due to the fact that the intersection of the uniqueness domain $Q_0$ and the plane $t = T/2$ is formed by the set $\{x \in \Omega,\ d_\Omega(x,\Gamma) < T/2\}$ and to the fact that the inverse problem in $Q_0$ is exponentially ill-posed.

7.2. **Validation of the exterior approach.** We are now in a position to test the exterior approach, that is the algorithm of section 6 to solve the inverse obstacle problem (2) in the case when the unknown obstacle is the union of the two discs as before. Note that the cylindrical domain $G \times (0,T)$ is meshed once and for all

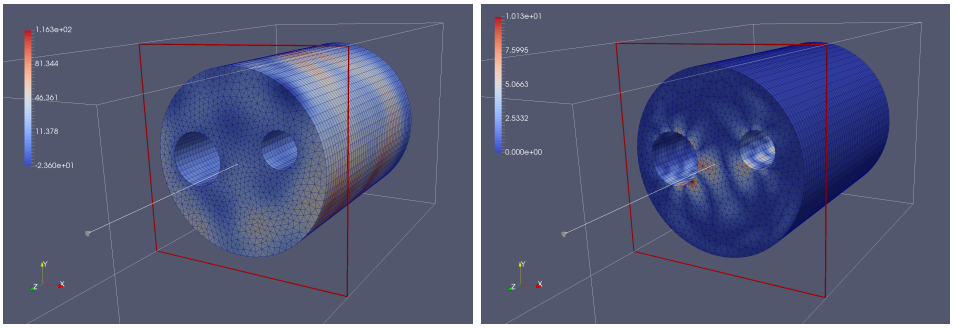


FIGURE 3. Two discs. Left: function $u_\varepsilon$. Right: function $|u_\varepsilon - u|$.

with the help of the prismatic finite elements described above. At iteration $n$, the obstacle $O_n$ and its complementary domain $\Omega_n$ are polygonal domains based on the mesh of $G$. The initial guess $O_0$ is chosen as the disk centered at $(0,0)$ and of radius 4.1. In the step 2 of the algorithm at iteration $n$, the 3D quasi-reversibility problem is solved in domain $\Omega_n \times (0,T)$ with the help of the tensorized $P1 \otimes P1$ finite elements, while in step 3, the 2D level set problem is solved in domain $O_n$ with simple $P1$ finite elements. Let us discuss some critical parameters of the algorithm of section 6.

1. **About the computation of** $T(O_n)$: in view of remark 4, we shall use $2D(\Omega_n,\Gamma)$ instead of $T(O_n)$ in (21), since $2D(\Omega_n,\Gamma)$ may be much smaller than $T(O_n)$. The distance $D(\Omega_n,\Gamma)$ can be computed numerically by solving a forward wave propagation problem in $\Omega_n \times (0,+\infty)$:

$$\left\{\begin{array}{rcll} \partial_t^2 u - \Delta u & = & 0 & \text{in } \Omega_n \times (0,+\infty) \\ u & = & 10\, e^{-25(t-2)^2} & \text{on } \Gamma \times (0,+\infty) \\ \partial_\nu u & = & 0 & \text{on } \tilde{\Gamma} \times (0,+\infty) \\ \partial_\nu u & = & 0 & \text{on } \partial O_n \times (0,+\infty) \\ u,\ \partial_t u & = & 0 & \text{on } \Omega \times \{0\}. \end{array}\right. \tag{24}$$

   Time integration stops once all the points of $\Omega_n$ have been reached by the pulse, let say when $|u| > 0.2 \times 10 = 2$ (20% of the initial pulse amplitude) in order to take the numerical dispersion effects into account. Due to the unit speed of propagation, the minimal time of full impact on $\Omega_n$ (with the pulse shift of 2 taken into account) coincides with $D(\Omega_n,\Gamma)$. In practice, we observe that the only values of $u$ which are involved in (21) are taken on $\partial O_n = \partial\Omega_n$, which leads us to compute

$$D_n = \sup_{x\in\partial O_n} \inf_{y\in\Gamma} d_{\Omega_n}(x,y).$$

   instead of $D(\Omega_n,\Gamma)$. In our numerical experiments, we replace $T(O_n)$ by $2D_n$ in the algorithm.
2. **About the choice of** $f$: in practice the distribution $f$ in (21) is chosen as a constant spatial function which must be sufficiently large in view of (18). Following our algorithm, strictly speaking such constant $f$ does not change during the iterations $n$. However, it may be more efficient to iteratively decrease the value of $f$ in step 3 in order to prevent the algorithm from a preliminary stop due to the fact that the evolution of the level sets may

occur on a scale that cannot be resolved numerically because the mesh size is not small enough. This is why we introduce a refinement of step 3 which in some sense finds the optimal value of $f$. Note that such refinement also includes a stopping criterion. In the following refined step 3, we have set $M_{n+1} = \sup_{x\in\partial O_n} V_n(x)$ for $n \geq 0$ ($M_0 = +\infty$) and $f$ is a large constant.

**Refined step 3 of the algorithm** :

(a) Take $k = 0$, $f^k = f$
(b) Find the solution $\tilde{\varphi}$ to problem (21) with constant $f^k$ and define

$$\tilde{O} = \{x \in O_n, \quad \tilde{\varphi}(x) < 0\}$$

(c) Do the following tests:
  (i) If $\tilde{O} = \emptyset$, restart the complete algorithm with a larger constant $f$
  (ii) If $M_{n+1} > 1.05\, M_n$ or $k = 5$, stop the algorithm and the final retrieved obstacle is $O = O_n$
  (iii) If $\tilde{O} = O_n$, then $f^{k+1} := f^k - f^k/50$, $k + 1 \to k$, go to step (b)
  (iv) Otherwise, $O_{n+1} = \tilde{O}$, $n + 1 \to n$, $0 \to k$ and return to step 2 of the algorithm.

3. **About the choice of** $\varepsilon$: the quasi-reversibility regularization parameter is $\varepsilon = 0.001$ in the following numerical experiments. In practice, this value should be set in accordance with the amplitude of the noise which contaminates the data $(g_0, g_1)$. For instance, the Morozov's discrepancy principe could be applied to the mixed formulation of quasi-reversibility as done in [10] in the simpler case of Helmholtz equation. However, as shown in [10], such procedure requires to take into account the Dirichlet data $g_0$ in a weak way instead of in a strong way. Since such work seems quite significant, we intend to address it in a future contribution.

Before testing the complete algorithm, we wish to test the level set method only. More precisely, in order to illustrate Theorem 5.5, we only apply step 3 of the algorithm of section 6 by using the exact solution $u$ in $\Omega_n \times (0, T - D(\Omega_n, \Gamma))$ instead of the quasi-reversibility solution $u_n$. From Theorem 5.5, the sequence of domains $O_n$ is supposed to converge to the true domain $O$ in the sense of Hausdorff distance. This convergence can be checked, up to the spatial mesh size, on the figure 4. In this computation, the final time is $T = 25$, which is much larger than $2D(\Omega, \Gamma) = 10$, that is the minimal time for convergence in Theorem 5.5. The boundaries of $O_0$ and some selected intermediate obstacles $O_n$ until stationarity of the algorithm are represented on the picture.
Now let us try experiments with the complete algorithm of section 6. In figure 5, we analyze the identification results for exact data and increasing final time. We have tested three values of $T$, namely $T = 2D(\Omega, \Gamma) = 10$, $T = 15$ and $T = 25$, keeping the same discretization time step. We observe that when $T$ increases, the identification results improve: not only the final obstacle is better retrieved but the number of iterations to achieve the final obstacle is smaller. In figure 6, we analyze the identification results for fixed final time $T = 25$ and noisy data. We artificially perturb both Dirichlet data $g_0$ and Neumann data $g_1$ by some pointwise random noise in such a way that we exactly control the $L^2$ relative error $\delta$. In other words, the noisy data $(g_0^\delta, g_1^\delta)$ satisfy

$$\|g_0^\delta - g_0\|_{L^2(\Gamma\times(0,T))} = \delta\|g_0\|_{L^2(\Gamma\times(0,T))}, \quad \|g_1^\delta - g_1\|_{L^2(\Gamma\times(0,T))} = \delta\|g_1\|_{L^2(\Gamma\times(0,T))}.$$

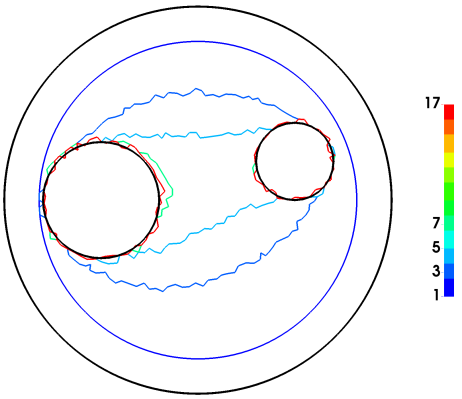



FIGURE 4. Validation of the level set method ($T = 25$)

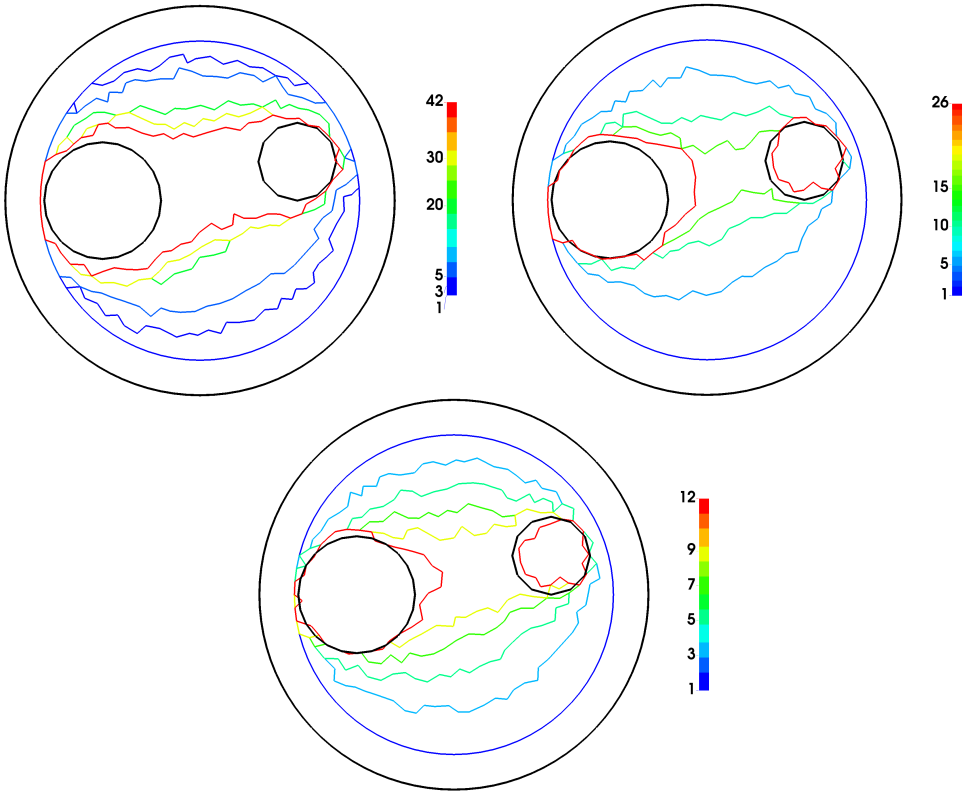



FIGURE 5. Two discs and exact data. Top left: $T = 10$. Top right: $T = 15$. Bottom: $T = 25$

The three pictures of 6 correspond to $\delta = 0$ (exact data), $\delta = 2\%$ and $\delta = 5\%$. In the last picture 7, for exact data and final time $T = 25$, we consider the difficult case of partial data. More precisely, data $(g_0, g_1)$ are only known for $\pi/2 \leq \theta \leq 2\pi$, that is $\Gamma$ is now formed by 3/4-th of the circle $\partial G$. We consider obstacles formed by

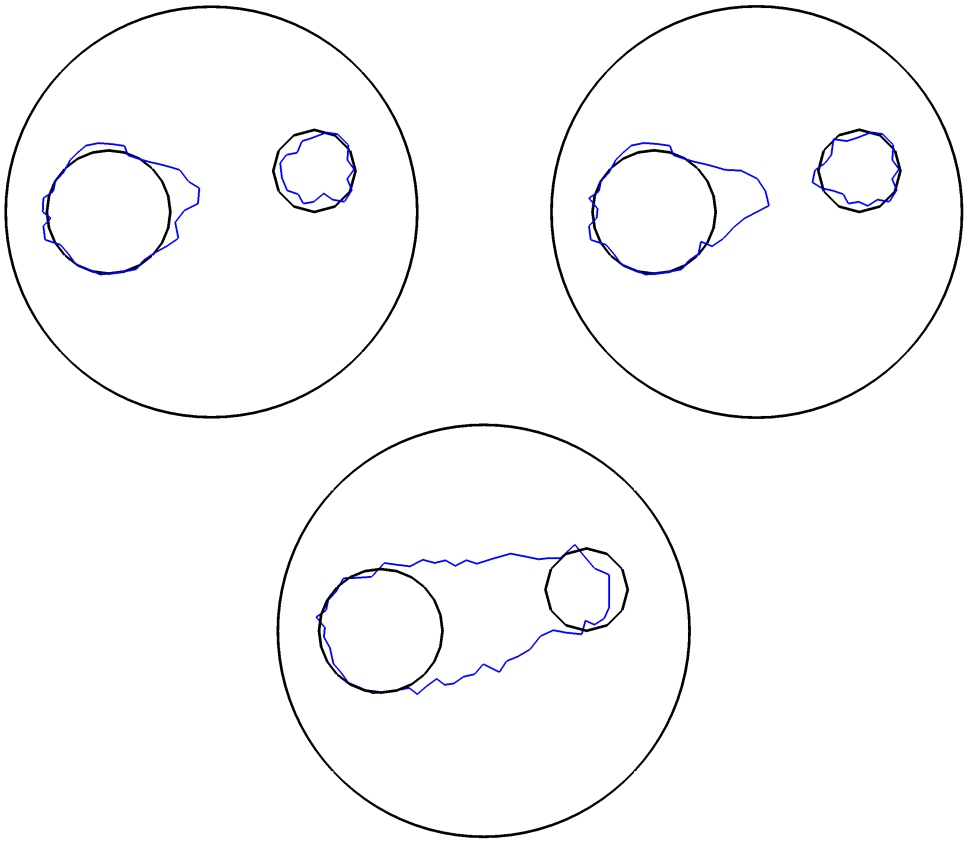


FIGURE 6. Two discs and noisy data. Top left: $\delta = 0$ (exact data). Top right: $\delta = 0.02$. Bottom: $\delta = 0.05$

only one disc, either located far away from the subpart $\tilde{\Gamma}$ of $\partial G$ on which we have no data, or close to $\tilde{\Gamma}$. Clearly, the identification result is better in the first case.

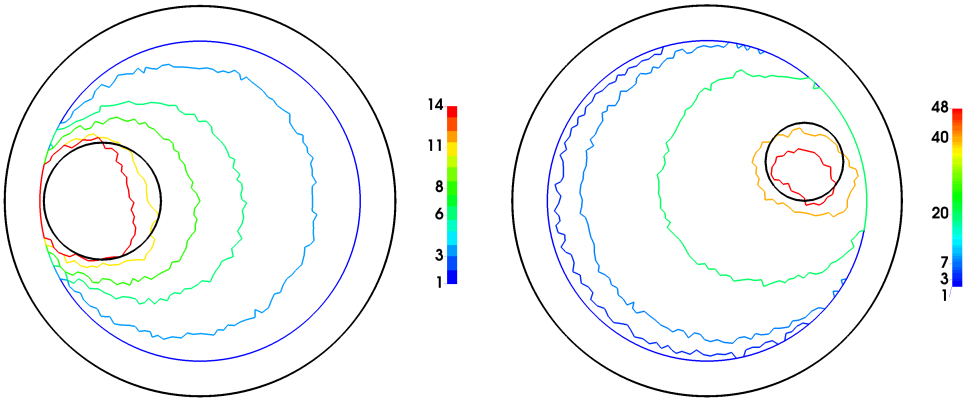



FIGURE 7. Partial (exact) data and one disc. Left: obstacle located far away from $\tilde{\Gamma}$. Right: obstacle located close to $\tilde{\Gamma}$

**Appendix: proofs of unique continuation results.** First, we have the following local result [20], which is itself a consequence of Holmgren's theorem for the wave equation.

**Lemma 7.1.** *Let us consider $\delta, \tau > 0$ and $x_1, x_2 \in \mathbb{R}^d$ such that $\tau > |x_1 - x_2|$. Let us define the open and convex domains*

$$\mathcal{O}_1 = B(x_1, \delta) \times (-\tau, \tau)$$

*and*

$$\mathcal{O}_2 = \bigcup_{\lambda \in [0,1]} B((1-\lambda)x_1 + \lambda x_2, \delta) \times (-\tau + \lambda|x_1 - x_2|, \tau - \lambda|x_1 - x_2|).$$

*If $u \in \mathcal{D}'(\mathcal{O}_2)$ satisfies $\partial_t^2 u - \Delta u = 0$ in the sense of distributions and $u = 0$ in $\mathcal{O}_1$, then $u = 0$ in $\mathcal{O}_2$.*

In order to propagate uniqueness, we will need the concept of $\delta$-sequence of balls introduced in [22].

**Lemma 7.2.** *Consider two points $x_0$ and $x$ in the open and connected domain $\Omega$. For all $\varepsilon, \delta_0 > 0$, there exists some $\delta \in (0, \delta_0)$ and a $\delta$-sequence of balls $B(q_n, \delta)$ for $n = 0, \cdots, N$ that links $x_0$ to $x$, that is*

$$\begin{cases} q_0 = x_0, \\ B(q_{n+1}, \delta) \subset B(q_n, 2\delta), \quad n = 0, \cdots, N-1, \\ B(q_n, 3\delta) \subset \Omega, \quad n = 0, \cdots, N, \\ q_N = x, \end{cases}$$

*with $|q_n - q_{n+1}| \leq \delta$ for $n = 0, \cdots, N-1$ and such that $N\delta \leq d_\Omega(x_0, x) + \varepsilon$.*

Now let us prove the following result.

**Proposition 4.** *Let $\Omega$ be an open and connected domain of $\mathbb{R}^d$, $d \geq 2$. Let $u \in H^1(Q)$ solves the problem*

$$\begin{cases} \partial_t^2 u - \Delta u &= 0 \quad \text{in } \Omega \times (0,T) \\ u,\ \partial_t u &= 0 \quad \text{on } \Omega \times \{0\}. \end{cases} \tag{25}$$

*Assume that $x_0 \in \Omega$ and $\delta_0 > 0$ are such that $B(x_0, \delta_0) \subset \Omega$ and $u = 0$ in $B(x_0, \delta_0) \times (0,T)$. For any $x \in \Omega$ such that $T > d_\Omega(x_0, x)$, for any $\varepsilon \in (0, T - d_\Omega(x_0, x))$, there exists some $\delta \in (0, \delta_0)$ such that $u = 0$ in $B(x, \delta) \times (0, T - d_\Omega(x_0, x) - \varepsilon)$.*

*Proof.* Let us extend $u$ by 0 in $\Omega \times (-T, 0)$ (without change of notation) so that $u \in H^1(\Omega \times (-T,T))$, $\partial_t^2 u - \Delta u = 0$ in $\Omega \times (-T,T)$ and $u = 0$ in $B(x_0, \delta_0) \times (-T,T)$. Let us take $\delta$ and a sequence of balls depending on $\varepsilon$ and $\delta_0$ as in Lemma 7.2. Now we apply $N$ times lemma 7.1 with

$$\mathcal{O}_1 = B(q_n, \delta) \times (-T + S_n, T - S_n)$$

and

$$\mathcal{O}_2 = \bigcup_{\lambda \in [0,1]} B((1-\lambda)q_n + \lambda q_{n+1}, \delta) \times (-T + S_n + \lambda|q_n - q_{n+1}|, T - S_n - \lambda|q_n - q_{n+1}|),$$

for all $0 \leq n \leq N-1$. Here we have denoted (with convention $S_0 = 0$)

$$S_n = \sum_{m=0}^{n-1} |q_m - q_{m+1}|.$$

It is straightforward to check that $\mathcal{O}_2 \subset B(q_n, 2\delta) \times (-T, T) \subset \Omega \times (-T, T)$ for $0 \leq n \leq N-1$. Since $u = 0$ in $B(q_0, \delta) \times (-T, T)$, then $u = 0$ in $B(q_1, \delta) \times (-T + S_1, T - S_1), \cdots$, lastly $u = 0$ in $B(q_N, \delta) \times (-T + S_N, T - S_N)$. We note that

$$T - S_N = T - \sum_{m=0}^{N-1} |q_m - q_{m+1}| \geq T - N\delta \geq T - d_\Omega(x_0, x) - \varepsilon,$$

Hence $u = 0$ in $B(x, \delta) \times (0, T - d_\Omega(x_0, x) - \varepsilon)$, which completes the proof of the proposition. □

We are now able to prove Theorem 3.3.

*Proof of Theorem 3.3.* Let us pick some $x \in \Omega$ and choose $\varepsilon \in (0, \varepsilon_0(x))$ with $\varepsilon_0(x) = (T - d_\Omega(x, \Gamma))/3$. Since $d_\Omega(x, \Gamma) = \inf_{x_0 \in \Gamma} d_\Omega(x, x_0)$, one may find $x_0 \in \Gamma$ such that $d_\Omega(x, \Gamma) \geq d_\Omega(x, x_0) - \varepsilon$. Now let us define $\Omega_+ = \Omega \cup B(x_0, \varepsilon)$, and $u_+$ the extension of $u$ by 0 in $\Omega_+ \times (0, T)$. The function $u_+ \in H^1(\Omega_+ \times (0, T))$ satisfies in $\Omega_+ \times (0, T)$

$$\left\{ \begin{array}{rcll} \partial_t^2 u_+ - \Delta u_+ & = & 0 & \text{in } \Omega_+ \times (0, T) \\ u_+ & = & 0 & \text{on } \Omega_+ \times \{0\} \\ \partial_t u_+ & = & 0 & \text{on } \Omega_+ \times \{0\}, \end{array} \right.$$

as well as $u_+ = 0$ in $(\Omega_+ \setminus \overline{\Omega}) \times (0, T)$. For some $x_+ \in \Omega_+ \setminus \overline{\Omega}$, we have $d_{\Omega_+}(x, x_+) \leq d_{\Omega_+}(x, x_0) + d_{\Omega_+}(x_0, x_+) \leq d_{\Omega_+}(x, x_0) + \varepsilon \leq d_\Omega(x, x_0) + \varepsilon$, since $\Omega \subset \Omega_+$ and by using the very definition of the geodesic distance, so that $d_\Omega(x, \Gamma) \geq d_{\Omega_+}(x, x_+) - 2\varepsilon$. There exists some $\delta_0$ such that $u_+ = 0$ in $B(x_+, \delta_0) \times (0, T)$. By using Proposition 4 in domain $\Omega_+$, we obtain that there exists some $\delta < \delta_0$ such that $u_+ = 0$ in $B(x, \delta) \times (0, T - d_{\Omega_+}(x, x_+) - \varepsilon)$. As a conclusion, for any $\varepsilon \in (0, \varepsilon_0(x))$, there exists $\delta$ such that $u = 0$ in $B(x, \delta) \times (0, T - d_\Omega(x, \Gamma) - 3\varepsilon)$. Since $\varepsilon$ is arbitrarily small, $u$ vanishes in a vicinity of $x$ for $t < T - d_\Omega(x, \Gamma)$. In conclusion $u$ vanishes in $Q_0$. In the particular case when $T > D(\Omega, \Gamma)$, for $x \in \Omega$ we have by definition of $D(\Omega, \Gamma)$,

$$d_\Omega(x, \Gamma) \leq D(\Omega, \Gamma) = T - (T - D(\Omega, \Gamma)),$$

so that $u$ vanishes in $\Omega \times (0, T - D(\Omega, \Gamma))$. □

We complete this appendix by an additional proposition required in the proof of Theorem 3.6.

**Proposition 5.** *Let $\Omega$ be a Lipschitz connected open domain of $\mathbb{R}^d$, with $d \geq 2$, and let $u \in H^1(Q) \cap L^2(0, T; C^0(\overline{\Omega}))$ solve the problem (25). We assume one of the two properties.*

1. *$x_0 \in \Omega$ and $\delta_0 > 0$ are such that $B(x_0, \delta_0) \subset \Omega$ and $u = 0$ in $B(x_0, \delta_0) \times (0, T)$*
2. *$x_0 \in \partial\Omega$ and $\delta_0 > 0$ are such that $u = 0$ and $\partial_\nu u = 0$ in $B(x_0, \delta_0) \cap \partial\Omega \times (0, T)$.*

*We now consider $x \in \partial\Omega$ such that $T > d_\Omega(x_0, x)$. Then $u(x, \cdot) = 0$ in $(0, T - d_\Omega(x_0, x))$.*

*Proof.* Let us prove the part 1 of the Proposition. There exists a continuous path $g$ in $\Omega$ of length $\ell(g) \leq d_\Omega(x_0, x) + \varepsilon$ joining $x_0$ to $x$. For any point $y$ on this path, the sub-path $\tilde{g}$ of $g$ joining $x$ to $y$ satisfies

$$d_\Omega(x, y) \leq \ell(\tilde{g}) \leq \ell(g) \leq d_\Omega(x_0, x) + \varepsilon,$$

By using Proposition 4, we obtain that there exists some $\delta < \delta_0$ such that $u = 0$ in $B(y, \delta) \times (0, T - d_\Omega(x_0, x) - 2\varepsilon)$. Since $\varepsilon$ is arbitrarily small, we have $u(y, \cdot) = 0$ in

$(0, T - d_\Omega(x_0, x))$. Let us introduce a sequence of points $y_n$ located on path $g$ and tending to $x$. We now use the fact that $u \in L^2(0,T;C^0(\overline{\Omega}))$, that is

$$\int_0^T \sup_{y\in\overline{\Omega}} |u(y,t)|^2 \, dt < +\infty.$$

Starting from

$$\int_0^{T-d_\Omega(x_0,x)} |u(y_n,t)|^2 \, dt = 0$$

and passing to the limit $n \to +\infty$ in the above equation with the help of Lebesgue's theorem, we obtain that

$$\int_0^{T-d_\Omega(x_0,x)} |u(x,t)|^2 \, dt = 0,$$

which completes the proof of part 1. The part 2 of the proposition is obtained from part 1 and the same extension argument as used in the proof of Theorem 3.3. □

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