The Linear Sampling Method for Kirchhoff-Love Infinite Plates

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Abstract

This paper addresses the problem of identifying impenetrable obstacles in a Kirchhoff-Love infinite plate from multistatic near-field data. The Linear Sampling Method is introduced in this context. We firstly prove a uniqueness result for such an inverse problem. We secondly provide the classical theoretical foundation of the Linear Sampling Method. We lastly show the feasibility of the method with the help of numerical experiments.

1 Introduction

In this contribution we consider the inverse problem of finding an impenetrable obstacle in an infinite elastic plate from multistatic scattering data in the frequency domain. Assuming that the thickness of the plate is small with respect to the wavelength, we consider that the behavior of the elastic plate is governed by the classical Kirchhoff-Love model in the purely bending case. The impenetrable obstacle $D \subset \mathbb{R}^2$ is supposed to be a bounded open domain of class C^3 which is either characterized by a Dirichlet or a Neumann boundary condition. More precisely, by using the notations of [1], in particular $\Omega = \mathbb{R}^2 \setminus \overline{D}$, the scattered field v^s satisfies in the unbounded domain Ω the problem

$$\left\{\begin{array}{lr} \Delta^2 v^s - k^4 v^s = 0 & \text{in } \Omega \\ B_1(v^s + u^i) = B_2(v^s + u^i) = 0 & \text{on } \partial\Omega \\ \displaystyle\lim_{r\to+\infty} \int_{\partial B_r} \left|\frac{\partial v^s}{\partial n} - ikv^s\right|^2 ds = 0. & \end{array}\right. \tag{1}$$

Here $k > 0$ is the wave number, u^i is an incident field which satisfies $\Delta^2 u^i - k^4 u^i = 0$ in a domain including $\overline{D}$, B_r is the open ball centered at 0 and of radius r, n is the outward normal to B_r and s is the measure on ∂B_r. The reader will refer to [1] for a short justification of how the system (1) is derived. Roughly speaking, the first line describes the motion of the plate in the frequency domain, the second one characterizes the boundary conditions on the boundary of the obstacle while the third one is the radiation condition, which specifies that only outgoing scattering waves are admissible. It is shown in [1] that the classical Sommerfeld condition for the Helmholtz equation is also valid for the Bilaplacian case. In order to specify the surface differential operators B_1 and B_2, we need to introduce some notations. A generic point $x \in \mathbb{R}^2$ has Cartesian coordinates (x_1, x_2). The outward unit normal to Ω is denoted n (note that n is oriented inside D). The unit tangent vector is denoted t and is such that the angle formed by the vectors (n, t) is $\pi/2$. The curvilinear abscissa associated with vector t is denoted s and coincides with the measure on ∂D. With the classical definitions

$$\frac{\partial}{\partial n} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial s} = -n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2},$$

either $(B_1, B_2) = (I, \partial_n)$ (I is the identity), which corresponds to the Dirichlet boundary condition, or $(B_1, B_2) = (M, N)$, which corresponds to the Neumann boundary condition, where the operators M and N are defined as follows:

$$\left\{ \begin{array}{lcl} Mu & = & \nu \Delta u + (1-\nu) M_0 u, \\ Nu & = & -\dfrac{\partial}{\partial n} \Delta u - (1-\nu) \dfrac{\partial}{\partial s} N_0 u. \end{array} \right.$$

Here, $\nu \in [0, 1/2)$ is the Poisson's ratio and M_0 and N_0 are given by

$$\left\{ \begin{array}{lcl} M_0 u & = & \dfrac{\partial^2 u}{\partial {x_1}^2} n_1^2 + 2 \dfrac{\partial^2 u}{\partial x_1 \partial x_2} n_1 n_2 + \dfrac{\partial^2 u}{\partial {x_2}^2} n_2^2, \\ N_0 u & = & \dfrac{\partial^2 u}{\partial x_1 \partial x_2} (n_1^2 - n_2^2) - \left(\dfrac{\partial^2 u}{\partial {x_1}^2} - \dfrac{\partial^2 u}{\partial {x_2}^2} \right) n_1 n_2. \end{array} \right.$$

We mention that the Dirichlet boundary condition amounts to specify the out of plane displacement and the angle of rotation of the plate, while the Neumann boundary condition amounts to specify the bending moment and the shear force. In other words, the Dirichlet boundary condition corresponds to the clamped plate while the Neumann boundary condition corresponds to the free plate. Let us now consider the bounded domain $\Omega_R = \Omega \cap B_R$, where $R > 0$ is such that B_R contains the obstacle D. It is proved in [1] that the problem (1) is equivalent to the following problem set in the bounded domain Ω_R: find $u^s \in H^2(\Omega_R)$ such that

$$\left\{ \begin{array}{cc} \Delta^2 u^s - k^4 u^s = 0 & \text{in } \Omega_R \\ B_1(u^s + u^i) = B_2(u^s + u^i) = 0 & \text{on } \partial\Omega \\ \begin{pmatrix} Nu^s \\ Mu^s \end{pmatrix} = T \begin{pmatrix} u^s|_{\partial B_R} \\ \partial_n u^s|_{\partial B_R} \end{pmatrix} & \text{on } \partial B_R, \end{array} \right. \tag{2}$$

where $T : H^{3/2}(\partial B_R) \times H^{1/2}(\partial B_R) \to H^{-3/2}(\partial B_R) \times H^{-1/2}(\partial B_R)$ is a Dirichlet-to-Neumann operator defined as follows. Assume that $(f, g) \in H^{3/2}(\partial B_R) \times H^{1/2}(\partial B_R)$ has the decomposition

$$(f, g) = \sum_{m \in \mathbb{Z}} (f_m, g_m) \xi_m, \quad \text{with} \quad \xi_m(\theta) = e^{im\theta}/\sqrt{2\pi}.$$

We have

$$T\begin{pmatrix} f \\ g \end{pmatrix} = \sum_{m\in\mathbb{Z}} T_m \begin{pmatrix} f_m \\ g_m \end{pmatrix} \xi_m, \quad T_m = \begin{pmatrix} T_m^{11} & T_m^{12} \\ T_m^{21} & T_m^{22} \end{pmatrix}, \tag{3}$$

with

$$\begin{cases} T_m^{11} & = & -(1-\nu)\dfrac{m^2}{R^3} - 2ik^3 \dfrac{r_m s_m}{r_m - is_m} \\ T_m^{12} = T_m^{21} & = & (1-\nu)\dfrac{m^2}{R^2} + k^2 \dfrac{r_m + is_m}{r_m - is_m} \\ T_m^{22} & = & -\dfrac{1-\nu}{R} - \dfrac{2k}{r_m - is_m} \end{cases} \tag{4}$$

and

$$r_m = \frac{(H_m^1)'(kR)}{H_m^1(kR)}, \quad s_m = \frac{(H_m^1)'(ikR)}{H_m^1(ikR)}.$$

Here, H_m^1 denotes the Hankel function of the first kind and of order m. Well-posedness of the forward problem (1) is the main purpose of [1]. The proof consists of a Fredholm analysis of the equivalent problem (2).

Theorem 1.1. *The problem (1) has a unique solution in $H^2_{\text{loc}}(\Omega)$*

- *for any k in the clamped case,*
- *for $k \notin \mathcal{K}_0$ in the free case, where the set $\mathcal{K}_0$ is formed by some numbers $k_n > 0$, $n \in \mathbb{N}$, such that $k_n \to +\infty$.*

Note that we ignore if the restriction $k \notin \mathcal{K}_0$ is purely technical or not. However, it is proved in [1] that for some particular obstacles D, for example a disc, we have $\mathcal{K}_0 = \emptyset$.

The inverse problem we consider is the following. We assume that D is unknown but a priori contained in B_R. Let us denote by $G(\cdot, y)$ the fundamental solution associated with the operator $\Delta^2 - k^4$, that is the unique solution in $H^2_{\text{loc}}(\mathbb{R}^2)$ of the system

$$\begin{cases} \Delta^2 G(\cdot,y) - k^4 G(\cdot,y) = \delta_y & \text{in } \mathbb{R}^2 \\ \displaystyle\lim_{r\to+\infty} \int_{\partial B_r} \left| \frac{\partial G(\cdot,y)}{\partial n} - ikG(\cdot,y) \right|^2 ds = 0. \end{cases} \tag{5}$$

It is well-known that G is given by

$$G(x,y) = \frac{i}{8k^2}\left(H_0^1(k|x-y|) - H_0^1(ik|x-y|)\right). \tag{6}$$

For sake of self-containment, the well-posedness of problem (5) and the expression (6) are proved in Lemma 2.2 hereafter. The function $G(\cdot, y)$ can be seen as a point source located at y. For some point $y \in \partial B_R$, we denote $u^s(\cdot, y)$ the scattered field which is associated with the incident field $u^i = G(\cdot, y)$ via (1). We also denote $\tilde{u}^s(\cdot, y)$ the scattered field which is associated with the incident field $u^i = \partial_{n_y} G(\cdot, y)$ via (1), where n_y is the outward normal to B_R at point $y \in \partial B_R$. For all points $y \in \partial B_R$, we measure the scattered fields $u^s(\cdot, y)$ and $\tilde{u}^s(\cdot, y)$ as well as their normal derivatives at all points $x \in \partial B_R$. All these measurements constitute the so-called multistatic data. The goal of the inverse problem is to retrieve the obstacle D from those multistatic data. It arises in the framework of Non Destructive Testing, which is for example quite common in the aircraft industry. The Bilaplacian model is interesting when the structure to inspect is thin and the frequency is low, which enables us to replace the $3D$ elastic model by such $2D$ approximate model. For example, Structural Health Monitoring would be a nice application. If we think of the SHM of the fuselage or the wings of an aircraft, even if some small hole in the skin would be visible to the naked eye when the aircraft is on the ground, it cannot be seen when the aircraft flies. Besides, in a view to consider defects such as corrosion or delamination, it would be useful to extend the present study to penetrable obstacles.

In order to address the inverse problem, we adapt the classical Linear Sampling Method introduced by Colton and Kirsch in [2] to the case of plates. Since this pioneering paper, the Linear Sampling Method has been applied in a large number of situations (see for example [3]), in particular in elasticity (see for example [4, 5, 6, 7, 8]). But the case of Kirchhoff-Love plates is new as far as we can judge. The Linear Sampling Method consists in testing, for all point of a sampling grid, if some test function depending on that point belongs to the range of an integral operator, the kernel of which only depends on the multistatic data. The LSM is both simple and efficient. In addition, the main feature of such method is that it works even if the nature of the obstacle is as priori unknown. To the best of our knowledge, the number of articles concerning inverse scattering problems in some infinite Kirchhoff-Love plate is very small. We mention the very recent contribution [9] on that subject in the particular case of a Bilaplacian operator with zero and first order perturbations. In order to build some synthetic data we need to solve the forward problem (1). To do so, we use a Finite Element Method based on the weak formulation associated with the problem (2) and a discretization of the Dirichlet-to-Neumann operator (3). The practical use of such a D-t-N operator to numerically compute a scattering solution in an infinite plate is new, as far as we know. Note that an alternative approach is the use of Perfectly Matched Layers, as in [10].

The paper is organized as follows. The second section is devoted to the treatment of the inverse problem: we first derive an integral representation formula considered here as a preliminary tool, then prove the identifiability of the obstacle from the prescribed data, and lastly provide the justification of the Linear Sampling Method for the Dirichlet case and give some indications for the Neumann case. The third section presents some numerical results: we firstly describe the Finite Element Method we use to produce the artificial data with an example, then show the identification results obtained with the Linear Sampling Method, and lastly complete this numerical section by a short conclusion.

2 The inverse problem

2.1 Integral representation

The integral representation formula is a basic tool in the proof of the identifiability of the obstacle and also in the justification of the Linear Sampling Method to retrieve it. Let us first recall some classical plate-oriented Green formula (see [12]).

Lemma 2.1. *In a bounded domain $\mathcal{O}$ of class C^2, we denote $H^2(\mathcal{O},\Delta^2) = \{u \in H^2(\mathcal{O}),\, \Delta^2 u \in L^2(\mathcal{O})\}$. The linear mapping*

$$H^2(\mathcal{O},\Delta^2) \to H^{-3/2}(\partial\mathcal{O}) \times H^{-1/2}(\partial\mathcal{O})$$

$$u \mapsto \begin{pmatrix} Nu \\ Mu \end{pmatrix}$$

is continuous. Moreover, for all $v \in H^2(\mathcal{O})$, we have the integration by parts formula

$$\int_{\mathcal{O}} \Delta^2 u\, v\, dx = a(u,v) - \int_{\partial\mathcal{O}} \left(\frac{\partial v}{\partial n} Mu + vNu \right) ds,$$

where we have introduced the bilinear form

$$a(u,v) = \int_{\mathcal{O}} \{\nu \Delta u \Delta v + (1-\nu) \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}\}\, dx, \tag{7}$$

and n is the outward normal to $\mathcal{O}$. Here the first part of the integral on $\partial\mathcal{O}$ has the meaning of duality pairing between $H^{1/2}(\partial\mathcal{O})$ and $H^{-1/2}(\partial\mathcal{O})$ while the second part has the meaning of duality pairing between $H^{3/2}(\partial\mathcal{O})$ and $H^{-3/2}(\partial\mathcal{O})$.

Given some functions $(\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$, let us consider the following interior and exterior problems: find $u^- \in H^2(D)$ and $u^+ \in H^2_{\mathrm{loc}}(\Omega)$ such that

$$\begin{cases} \Delta^2 u^- - k^4 u^- = 0 & \text{in } D \\ (u^-, \partial_n u^-) = (\phi, \psi) & \text{on } \partial D \end{cases} \tag{8}$$

and

$$\begin{cases} \Delta^2 u^+ - k^4 u^+ = 0 & \text{in } \Omega \\ (u^+, \partial_n u^+) = (\phi, \psi) & \text{on } \partial\Omega \\ \displaystyle\lim_{r\to+\infty} \int_{\partial B_r} \left| \frac{\partial u^+}{\partial n} - iku^+ \right|^2 ds = 0, & \end{cases} \tag{9}$$

where the unit normal vector n is oriented outside D in both problems (8) and (9). From Theorem 1.1, the exterior problem is well-posed. Concerning the interior problem, it is straightforward that well-posedness holds if and only if $k \notin \mathcal{K}_D$, where $\mathcal{K}_D$ is the set formed by the fourth roots of the Dirichlet eigenvalues of operator Δ^2 in domain D. Before proving our integral representation formula, let us derive the fundamental solution for operator $\Delta^2 - k^4$.

Lemma 2.2. *Problem (5) has a unique solution which is given by (6).*

Proof. Let $u = G(\cdot, y)$ be a solution to problem (5). We use the factorization

$$\Delta^2 u - k^4 u = (\Delta - k^2)(\Delta + k^2)u = (\Delta + k^2)(\Delta - k^2)u.$$

By setting $U = \Delta u + k^2 u$ and $V = \Delta u - k^2 u$, we hence obtain that

$$\Delta U - k^2 U = \delta_y, \quad \Delta V + k^2 V = \delta_y.$$

In addition, by using a decomposition in the form of a series as in [1], it can be seen that if u satisfies the Sommerfeld radiation condition, the functions U and V both satisfy the radiation condition as well. This implies from the case of Helmholtz equation that

$$U(x) = -\frac{i}{4} H^1_0(ik|x-y|), \quad V(x) = -\frac{i}{4} H^1_0(k|x-y|).$$

Since $u = (U - V)/2k^2$, we obtain the expression (6). It remains to prove that $G(\cdot, y)$ belongs to $H^2_{\rm loc}(\mathbb{R}^2)$. Let us compare the regularity of $G(\cdot, y)$ with that of $G_0(\cdot, y)$, that is the classical fundamental solution of the operator Δ^2 (see [12]), the expression of which is

$$G_0(x,y) = \frac{1}{8\pi}|x-y|^2 \log|x-y|. \tag{10}$$

It is clear that the function $G(\cdot, y) - G_0(\cdot, y)$ is infinitely smooth and it is easy to check by using polar coordinates that $G_0(\cdot, y)$ is in $H^2_{\rm loc}(\mathbb{R}^2)$. Then $G(\cdot, y) \in H^2_{\rm loc}(\mathbb{R}^2)$, which completes the proof. □

Remark 1. We also check from the comparison with $G_0(\cdot, y)$ that $G(\cdot, y) \notin H^3_{\rm loc}(\mathbb{R}^2)$.

Proposition 1. *Let us consider a function $u \in D \cap \Omega$ such that $u^- = u|_D$ solves problem (8) and $u^+ = u|_\Omega$ solves problem (9). Then for all $x \in D \cap \Omega$, we have*

$$u(x) = \int_{\partial D} G(x,y)\tau(y)\, ds(y) + \int_{\partial D} \frac{\partial G(x,y)}{\partial n_y}\sigma(y)\, ds(y), \tag{11}$$

where $(\tau, \sigma) = ([Nu], [Mu])$. Here, $[\cdot] = (\cdot)_+ - (\cdot)_-$ denotes the jump across the boundary of D and n_y is oriented inside D. The first integral means duality between $H^{3/2}(\partial D)$ and $H^{-3/2}(\partial D)$ while the second one means duality between $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$.

Proof. Let us consider $x \in \Omega$ and $r > 0$ such that $B(x, 2r) \in \Omega$. Next we define g_x and $\tilde{g}_x$ such that

$$g_x(y) = \begin{cases} G(x,y) & y \in B(x,r) \\ 0 & y \notin B(x,r) \end{cases}$$

and

$$\tilde{g}_x(y) = g_x(y) - G(x,y).$$

Now we take $\varphi \in C_0^\infty(B(x, 2r))$. The function $\tilde{g}_x$ satisfies

$$\langle(\Delta^2 - k^4)\tilde{g}_x, \varphi\rangle = \int_{B(x,2r)} \tilde{g}_x(\Delta^2 - k^4)\varphi\, dy,$$

where we have used the distribution brackets $\langle\cdot,\cdot\rangle$. It follows that

$$\langle(\Delta^2 - k^4)\tilde{g}_x, \varphi\rangle = -\int_{B(x,2r)\setminus B(x,r)} G(x,y)(\Delta^2 - k^4)\varphi(y)\, dy$$

$$= \int_{B(x,2r)\setminus B(x,r)} \left(\varphi(y)(\Delta_y^2 - k^4)G(x,y) - G(x,y)(\Delta^2 - k^4)\varphi(y)\right)\, dy$$

$$= \int_{B(x,2r)\setminus B(x,r)} \left(\varphi(y)\Delta_y^2 G(x,y) - G(x,y)\Delta^2\varphi(y)\right)\, dy.$$

By using Lemma 2.1, we obtain

$$\langle(\Delta^2 - k^4)\tilde{g}_x, \varphi\rangle = -\int_{\partial B(x,r)} \left(M_y G(x,y)\frac{\partial \varphi}{\partial n_y}(y) + N_y G(x,y)\varphi(y)\right)\, ds(y)$$

$$+\int_{\partial B(x,r)} \left(\frac{\partial G(x,y)}{\partial n_y} M\varphi(y) + G(x,y) N\varphi(y)\right) ds(y)$$

where n_y is the unit normal oriented inside $B(x,r)$. As a result,

$$\langle (\Delta^2 - k^4) g_x, \varphi\rangle = \varphi(x) - \int_{\partial B(x,r)} \left(M_y G(x,y) \frac{\partial \varphi}{\partial n_y}(y) + N_y G(x,y)\varphi(y)\right) ds(y)$$

$$+\int_{\partial B(x,r)} \left(\frac{\partial G(x,y)}{\partial n_y} M\varphi(y) + G(x,y) N\varphi(y)\right) ds(y).$$

Now let us choose $\varphi = \theta u$ in the above relationship, for $\theta \in C_0^\infty(B(x,2r))$, with $\theta = 1$ in $\overline{B(x,r)}$. That $\mathrm{supp}(g_x) \subset \overline{B(x,r)}$ implies that

$$\langle (\Delta^2 - k^4) g_x, \theta u\rangle = \int_{B(x,2r)} g_x (\Delta^2 - k^4)(\theta u)\, dx = \int_{B(x,r)} g_x (\Delta^2 - k^4) u \, dx = 0,$$

which implies that

$$u(x) = \int_{\partial B(x,r)} \mathcal{U}(x,y)\, ds(y), \tag{12}$$

where we have used the notation

$$\mathcal{U}(x,y) = M_y G(x,y) \frac{\partial u}{\partial n_y}(y) + N_y G(x,y) u(y) - \frac{\partial G(x,y)}{\partial n_y} Mu(y) - G(x,y) Nu(y).$$

If we now use Lemma 2.1 in the subdomain $\Omega_{r,R} = (\Omega \cap B_R) \setminus \overline{B(x,r)}$, we obtain

$$0 = \int_{\partial B(x,r)} \mathcal{U}(x,y)\, ds(y) + \int_{\partial D} \mathcal{U}(x,y)\, ds(y) + \int_{\partial B_R} \mathcal{U}(x,y)\, ds(y) \tag{13}$$

where the normal n_y involved in $\mathcal{U}(x,y)$ is oriented inside $B(x,r)$ in the first integral, inside D in the second integral and outside B_R in the third one. By using the Dirichlet-to-Neumann operator $T : H^{3/2}(\partial B_R) \times H^{1/2}(\partial B_R) \to H^{-3/2}(\partial B_R) \times H^{-1/2}(\partial B_R)$, we have

$$\int_{\partial B_R} \mathcal{U}(x,y)\, ds(y) = \left\langle (u, \partial_n u)\,, T_y \begin{pmatrix} G(x,y) \\ \partial_{n_y} G(x,y) \end{pmatrix} \right\rangle$$

$$- \left\langle (G(x,y), \partial_{n_y} G(x,y)), T \begin{pmatrix} u \\ \partial_n u \end{pmatrix} \right\rangle,$$

From (4) we observe that the operator T is symmetric, so that

$$\int_{\partial B_R} \mathcal{U}(x,y)\, ds(y) = 0.$$

By using (12) and (13) we obtain that

$$u(x) = -\int_{\partial D} \left(M_y G(x,y) \frac{\partial u^+}{\partial n_y}(y) + N_y G(x,y) u^+(y)\right) ds(y)$$

$$+\int_{\partial D} \left(\frac{\partial G(x,y)}{\partial n_y} Mu^+(y) + G(x,y) Nu^+(y)\right) ds(y),$$

where n_y is oriented inside D. Lastly we use again Lemma 2.1 in domain D, so that

$$0 = \int_{\partial D} \left(M_y G(x,y) \frac{\partial u^-}{\partial n_y}(y) + N_y G(x,y) u^-(y) \right) ds(y)$$

$$- \int_{\partial D} \left(\frac{\partial G(x,y)}{\partial n_y} M u^-(y) + G(x,y) N u^-(y) \right) ds(y),$$

where n_y is oriented inside D. From the two equations above, we obtain that for $x \in \Omega$,

$$u(x) = -\int_{\partial D} \left(M_y G(x,y) \left[\frac{\partial u}{\partial n_y}(y) \right] + N_y G(x,y) \left[u(y) \right] \right) ds(y)$$

$$+ \int_{\partial D} \left(\frac{\partial G(x,y)}{\partial n_y} \left[M u(y) \right] + G(x,y) \left[N u(y) \right] \right) ds(y),$$

where n_y is oriented inside D. It could be similarly proved that the same formula is valid for $x \in D$. Given the boundary conditions satisfied by u^+ and u^- on ∂D, we have that $[u] = 0$ and $[\partial_n u] = 0$ on ∂D, so that the first integral vanishes, which completes the proof. □

2.2 Uniqueness

Before we detail the effective reconstruction, we prove uniqueness of the obstacle from the data. In the sequel, we simply denote $\Gamma = \partial B_R$, which is the support of sources and measurements. More precisely, we have the following result.

Theorem 2.3. *Assume that D_1 and D_2 are two obstacles, either of Dirichlet type (that is $(B_1, B_2) = (I, \partial_n)$) or of Neumann type (that is $(B_1, B_2) = (M, N)$), such that for all $y \in \Gamma$, the corresponding fields $u_1^s(\cdot, y)$ and $u_2^s(\cdot, y)$ coincide on Γ as well as their normal derivative, and such that for all $y \in \Gamma$, the corresponding fields $\tilde{u}_1^s(\cdot, y)$ and $\tilde{u}_2^s(\cdot, y)$ coincide on Γ, as well as the normal derivative. Here, the scattered fields $u^s(\cdot, y)$ and $\tilde{u}^s(\cdot, y)$ are associated via (1) with the incident fields $u^i = G(\cdot, y)$ and $u^i = \partial_{n_y} G(\cdot, y)$, respectively. Then $D_1 = D_2$.*

To prove such theorem, we need the following reciprocity relationships.

Lemma 2.4. *For all $x, y \in \Omega$ and $z \in \Gamma$,*

$$u^s(x,y) = u^s(y,x), \quad \tilde{u}^s(x,z) = \frac{\partial u^s(z,x)}{\partial n_z}.$$

Proof. We detail the proof for the Dirichlet case, the Neumann case follows the same lines. From the proof of Proposition 1, for any $x, z \in \Omega$, we have the integral representation

$$u^s(x,z) = -\int_{\partial D} \left(M_y G(x,y) \frac{\partial u^s(y,z)}{\partial n_y} + N_y G(x,y) u^s(y,z) \right) ds(y)$$

$$+ \int_{\partial D} \left(\frac{\partial G(x,y)}{\partial n_y} M_y u^s(y,z) + G(x,y) N_y u^s(y,z) \right) ds(y),$$

where n_y is oriented inside D. From the Green formula in D we obtain

$$0 = -\int_{\partial D}\left(M_y G(x,y)\frac{\partial G(y,z)}{\partial n_y} + N_y G(x,y)G(y,z)\right) ds(y)$$
$$+\int_{\partial D}\left(\frac{\partial G(x,y)}{\partial n_y}M_y G(y,z) + G(x,y)N_y G(y,z)\right) ds(y). \quad (14)$$

By introducing the total field $u(\cdot,z) = G(\cdot,z)+u^s(\cdot,z)$, which satisfies the Dirichlet condition on ∂D, we obtain by adding the two above relationships that

$$u^s(x,z) = \int_{\partial D}\left(\frac{\partial G(x,y)}{\partial n_y}M_y u(y,z) + G(x,y)N_y u(y,z)\right) ds(y). \quad (15)$$

We now rewrite the first formula by inverting x and z, which yields

$$u^s(z,x) = -\int_{\partial D}\left(M_y G(z,y)\frac{\partial u^s(y,x)}{\partial n_y} + N_y G(z,y)u^s(y,x)\right) ds(y)$$
$$+\int_{\partial D}\left(\frac{\partial G(z,y)}{\partial n_y}M_y u^s(y,x) + G(z,y)N_y u^s(y,x)\right) ds(y). \quad (16)$$

From the Green formula in Ω and the radiation condition we obtain

$$0 = -\int_{\partial D}\left(M_y u^s(y,x)\frac{\partial u^s(y,z)}{\partial n_y} + N_y u^s(y,x)u^s(y,z)\right) ds(y)$$
$$+\int_{\partial D}\left(\frac{\partial u^s(y,x)}{\partial n_y}M_y u^s(y,z) + u^s(y,x)N_y u^s(y,z)\right) ds(y).$$

By subtracting the two above relationships, we obtain by using the fact that $G(y,z) = G(z,y)$,

$$u^s(z,x) = -\int_{\partial D}\left(\frac{\partial u^s(y,x)}{\partial n_y}M_y u(y,z) + u^s(y,x)N_y u(y,z)\right) ds(y). \quad (17)$$

Subtracting (15) and (17) and using the fact that $u(\cdot,x)$ satisfies the Dirichlet condition on ∂D we obtain that $u^s(x,z) - u^s(z,x) = 0$ for all $x,z \in \Omega$.

Let us prove the second relationship. For any $x,z \in \Omega$, we have the integral representation

$$\tilde{u}^s(x,z) = -\int_{\partial D}\left(M_y G(x,y)\frac{\partial \tilde{u}^s(y,z)}{\partial n_y} + N_y G(x,y)\tilde{u}^s(y,z)\right) ds(y)$$
$$+\int_{\partial D}\left(\frac{\partial G(x,y)}{\partial n_y}M_y \tilde{u}^s(y,z) + G(x,y)N_y \tilde{u}^s(y,z)\right) ds(y),$$

where n_y is oriented inside D. By computing the normal derivative of (14) with respect to z at point $z \in \Gamma$, we obtain that for all $x \in \Omega$ and all $z \in \Gamma$,

$$0 = -\int_{\partial D}\left(M_y G(x,y)\frac{\partial^2 G(y,z)}{\partial n_y \partial n_z} + N_y G(x,y)\frac{\partial G(y,z)}{\partial n_z}\right) ds(y)$$
$$+\int_{\partial D}\left(\frac{\partial G(x,y)}{\partial n_y}M_y \frac{\partial G(y,z)}{\partial n_z} + G(x,y)N_y \frac{\partial G(y,z)}{\partial n_z}\right) ds(y).$$

By adding the above relationships, we get

$$\tilde{u}^s(x,z) = \int_{\partial D} \left(\frac{\partial G(x,y)}{\partial n_y} M_y \tilde{u}(y,z) + G(x,y) N_y \tilde{u}(y,z) \right) ds(y), \tag{18}$$

where we have used the fact that the total field $\tilde{u}(\cdot,z) = \partial G(\cdot,z)/\partial n_z + \tilde{u}^s(\cdot,z)$ satisfies the Dirichlet boundary condition on ∂D. Computing the normal derivative of (16) with respect to z at point $z \in \Gamma$, we obtain that for all $x \in \Omega$ and all $z \in \Gamma$,

$$\begin{aligned} \frac{\partial u^s(z,x)}{\partial n_z} = &- \int_{\partial D} \left(M_y \frac{\partial G(z,y)}{\partial n_z} \frac{\partial u^s(y,x)}{\partial n_y} + N_y \frac{\partial G(z,y)}{\partial n_z} u^s(y,x) \right) ds(y) \\ &+ \int_{\partial D} \left(\frac{\partial^2 G(z,y)}{\partial n_y \partial n_z} M_y u^s(y,x) + \frac{\partial G(z,y)}{\partial n_z} N_y u^s(y,x) \right) ds(y). \end{aligned}$$

From the Green formula in Ω and the radiation condition we obtain

$$\begin{aligned} 0 = &- \int_{\partial D} \left(M_y \tilde{u}^s(y,z) \frac{\partial u^s(y,x)}{\partial n_y} + N_y \tilde{u}^s(y,z) u^s(y,x) \right) ds(y) \\ &+ \int_{\partial D} \left(\frac{\partial \tilde{u}^s(y,z)}{\partial n_y} M_y u^s(y,x) + \tilde{u}^s(y,z) N_y u^s(y,x) \right) ds(y). \end{aligned}$$

Adding the two above relationships implies that for all $x \in \Omega$ and all $z \in \Gamma$,

$$\frac{\partial u^s(z,x)}{\partial n_z} = - \int_{\partial D} \left(M_y \tilde{u}(y,z) \frac{\partial u^s(y,x)}{\partial n_y} + N_y \tilde{u}(y,z) u^s(y,x) \right) ds(y). \tag{19}$$

Subtracting (18) and (19) and using the fact that $u(\cdot,x)$ satisfies the Dirichlet condition on ∂D we obtain that $\tilde{u}^s(x,z) - \partial u^s(z,x)/\partial n_z = 0$ for all $x \in \Omega$ and $z \in \Gamma$. □

Proof of Theorem 2.3. Again, we detail the proof for the Dirichlet case, the Neumann case follows the same lines. Let us denote $\tilde{\Omega}$ the unbounded connected component of $\mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$. We have that for all $x, y \in \Gamma$,

$$u_1^s(x,y) = u_2^s(x,y), \quad \frac{\partial u_1^s}{\partial n_x}(x,y) = \frac{\partial u_2^s}{\partial n_x}(x,y)$$

as well as

$$\tilde{u}_1^s(x,y) = \tilde{u}_2^s(x,y), \quad \frac{\partial \tilde{u}_1^s}{\partial n_x}(x,y) = \frac{\partial \tilde{u}_2^s}{\partial n_x}(x,y).$$

From well-posedness of the forward diffraction problem when the obstacle is the ball B_R with Dirichlet boundary condition, we have that for all $x \in \mathbb{R}^2 \setminus \overline{B_R}$, for all $y \in \Gamma$,

$$u_1^s(x,y) = u_2^s(x,y), \quad \tilde{u}_1^s(x,y) = \tilde{u}_2^s(x,y).$$

Unique continuation for the operator $\Delta^2 - k^4$ then implies that for all $x \in \tilde{\Omega}$, for all $y \in \Gamma$,

$$u_1^s(x,y) = u_2^s(x,y), \quad \tilde{u}_1^s(x,y) = \tilde{u}_2^s(x,y).$$

We now use the reciprocity relationships of Lemma 2.4, which imply that for all $x \in \tilde{\Omega}$, for all $y \in \Gamma$,

$$u_1^s(y,x) = u_2^s(y,x), \quad \frac{\partial u_1^s}{\partial n_y}(y,x) = \frac{\partial u_2^s}{\partial n_y}(y,x).$$

By repeating the same arguments as above we obtain that for all $x, y \in \tilde{\Omega}$,

$$u_1^s(x, y) = u_2^s(x, y).$$

Assume that $D_1 \not\subset D_2$. Since $\mathbb{R}^2 \setminus \overline{D_2}$ is connected, there exists some non empty open set $\Gamma_* \subset (\partial D_1 \cap \partial\tilde{\Omega}) \setminus \overline{D_2}$. We now consider some point $x_* \in \Gamma_*$ and the sequence

$$x_m = x_* + \frac{n_1(x_*)}{m}, \quad m \in \mathbb{N} \setminus \{0\},$$

where $n_1(x_*)$ denotes the unit normal to Γ_* at point x_*. For sufficiently large m, $x_m \in \tilde{\Omega}$, so that for all $x \in \tilde{\Omega}$ and all m,

$$u_1^s(x, x_m) = u_2^s(x, x_m),$$

which implies in particular

$$u_1^s(\cdot, x_m)|_{\Gamma_*} = u_2^s(\cdot, x_m)|_{\Gamma_*}, \quad \left.\frac{\partial u_1^s}{\partial n_1}(\cdot, x_m)\right|_{\Gamma_*} = \left.\frac{\partial u_2^s}{\partial n_1}(\cdot, x_m)\right|_{\Gamma_*}.$$

Using the boundary condition for u_1^s, we obtain

$$G(\cdot, x_m)|_{\Gamma_*} = -u_2^s(\cdot, x_m)|_{\Gamma_*}, \quad \left.\frac{\partial G}{\partial n_1}(\cdot, x_m)\right|_{\Gamma_*} = -\left.\frac{\partial u_2^s}{\partial n_1}(\cdot, x_m)\right|_{\Gamma_*}.$$

Passing to the limit $m \to +\infty$, we get

$$G(\cdot, x_*)|_{\Gamma_*} = -u_2^s(\cdot, x_*)|_{\Gamma_*}, \quad \left.\frac{\partial G}{\partial n_1}(\cdot, x_*)\right|_{\Gamma_*} = -\left.\frac{\partial u_2^s}{\partial n_1}(\cdot, x_*)\right|_{\Gamma_*}.$$

The function u_2^s is infinitely smooth in a vicinity of x_*, while Γ_* is a subset of the boundary of the C^3 domain Ω_1. This implies that $(u_2^s(\cdot, x_*)|_{\Gamma_*}, \partial_{n_1} u_2^s(\cdot, x_*)|_{\Gamma_*}) \in H^{5/2}(\Gamma_*) \times H^{3/2}(\Gamma_*)$, which is also the regularity of $(G(\cdot, x_*)|_{\Gamma_*}, \partial_{n_1} G(\cdot, x_*)|_{\Gamma_*})$. Hence the function $G(\cdot, x_*)$, which solves the equation $\Delta^2 G(\cdot, x_*) - k^4 G(\cdot, x_*) = 0$ in $\tilde{\Omega}$ is H^3 in a vicinity of x_* in $\tilde{\Omega}$, from standard regularity results for elliptic problems [14]. But this is a contradiction in view of Remark 1. We conclude that $D_1 \subset D_2$ and we prove the same way that $D_2 \subset D_1$. Eventually, $D_1 = D_2$ $\square$

2.3 Justification of the Linear Sampling Method

We detail the classical theory of the Linear Sampling Method in the Dirichlet case, that is $(B_1, B_2) = (I, \partial_n)$. The Neumann case follows the same lines and will be presented without justification in the next section. Let us start by introducing the mapping $S_D : H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \to H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ such that, for all $x \in \partial D$,

$$S_D \begin{pmatrix} \tau \\ \sigma \end{pmatrix}(x) = \begin{pmatrix} \displaystyle\int_{\partial D} \left(G(x, y)\tau(y) + \frac{\partial G(x, y)}{\partial n_y}\sigma(y) \right) ds(y) \\ \displaystyle\frac{\partial.}{\partial n} \int_{\partial D} \left(G(x, y)\tau(y) + \frac{\partial G(x, y)}{\partial n_y}\sigma(y) \right) ds(y) \end{pmatrix}. \tag{20}$$

We have the following property.

Proposition 2. *The mapping S_D is an isomorphism if $k \notin \mathcal{K}_D$.*

Proof. The proof is based on a comparison between the mapping S_D and the analogue mapping $S_{D,0}$ when the fundamental solution G of operator $\Delta^2 - k^4$ is replaced by the fundamental solution G_0 of operator Δ^2 (see [12]) given by (10). The kernel $G - G_0$ is infinitely smooth, so that since the mapping $S_{D,0} : H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \to H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ is continuous (see for example [12]), this is also the case for S_D. From [11, Lemma 3.1], the operator $S_{D,0}$ is Fredlholm of index 0. Since $S_D - S_{D,0}$ is compact, this implies that the operator S_D is Fredholm of index 0 as well. Let us prove that S_D is surjective. We will then conclude that S_D is an isomorphism. Assume that $(\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$. We have to prove that there exits $(\tau, \sigma) \in H^{-3/2}(\partial D) \times H^{-1/2}(\partial D)$ such that $S_D(\tau, \sigma)^T = (\phi, \psi)^T$. In this view, assuming that $k \notin \mathcal{K}_D$, let us consider the well-defined solutions u^- in D and u^+ in Ω to the interior problem (8) and to the exterior problem (9) associated with the same Dirichlet data (ϕ, ψ). From proposition 1, the solution $u = (u^-, u^+)$ has expression (11). The trace and normal derivative of such solution, namely $(u, \partial_n u)$, are continuous across ∂D and coincide with (ϕ, ψ), so that $(\phi, \psi)^T = S_D(\tau, \sigma)^T$, where we have set $(\tau, \sigma) = ([Nu], [Mu]) \in H^{-3/2}(\partial D) \times H^{-1/2}(\partial D)$. The proof is complete. $\square$

We introduce the following operators: the obstacle-to-data operator $F_D : H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \to L^2(\Gamma) \times L^2(\Gamma)$ such that, for all $x \in \Gamma$,

$$F_D \begin{pmatrix} \tau \\ \sigma \end{pmatrix}(x) = \begin{pmatrix} \displaystyle\int_{\partial D} \left(G(x,y)\tau(y) + \frac{\partial G(x,y)}{\partial n_y}\sigma(y) \right) ds(y) \\ \displaystyle\frac{\partial .}{\partial n} \int_{\partial D} \left(G(x,y)\tau(y) + \frac{\partial G(x,y)}{\partial n_y}\sigma(y) \right) ds(y) \end{pmatrix}, \tag{21}$$

the data-to-obstacle operator $H_D : L^2(\Gamma) \times L^2(\Gamma) \to H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ such that, for all $x \in \partial D$,

$$H_D \begin{pmatrix} h \\ t \end{pmatrix}(x) = \begin{pmatrix} \displaystyle\int_{\Gamma} \left(G(x,y)h(y) + \frac{\partial G(x,y)}{\partial n_y} t(y) \right) ds(y) \\ \displaystyle\frac{\partial .}{\partial n} \int_{\Gamma} \left(G(x,y)h(y) + \frac{\partial G(x,y)}{\partial n_y} t(y) \right) ds(y) \end{pmatrix}, \tag{22}$$

and the near-field operator $N_D : L^2(\Gamma) \times L^2(\Gamma) \to L^2(\Gamma) \times L^2(\Gamma)$ such that, for all $x \in \Gamma$,

$$N_D \begin{pmatrix} h \\ t \end{pmatrix}(x) = \begin{pmatrix} \displaystyle\int_{\Gamma} \left(u^s(x,y)h(y) + \tilde{u}^s(x,y)t(y) \right) ds(y) \\ \displaystyle\frac{\partial .}{\partial n} \int_{\Gamma} \left(u^s(x,y)h(y) + \tilde{u}^s(x,y)t(y) \right) ds(y) \end{pmatrix}, \tag{23}$$

where $u^s(\cdot, y)$ and $\tilde{u}^s(\cdot, y)$ are the solutions to problem (1) with $u^i = G(\cdot, y)$ and $\partial_{n_y} G(\cdot, y)$, respectively. Lastly, let us define the solution operator $B_D : H^{3/2}(\partial D) \times H^{1/2}(\partial D) \to L^2(\Gamma) \times L^2(\Gamma)$ such that for $(\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$, $B(\phi, \psi)^T$ is formed by the trace and normal derivative on Γ of the unique solution $v \in H^2_{\mathrm{loc}}(\Omega)$

to the problem

$$
\left\{
\begin{array}{lr}
\Delta^2 v - k^4 v = 0 & \text{in } \Omega \\
v = \phi, \quad \dfrac{\partial v}{\partial n} = \psi & \text{on } \partial\Omega \\
\displaystyle\lim_{r\to+\infty} \int_{\partial B_r} \left| \frac{\partial v}{\partial n} - ikv \right|^2 ds = 0. &
\end{array}
\right. \tag{24}
$$

Proposition 3. *Let us assume that $k \notin \mathcal{K}_D$. The operators B_D, F_D, H_D and N_D satisfy $F_D = \overline{H}^*_D$, $F_D = B_D S_D$ and $N_D = -B_D H_D$, where A^* denotes the adjoint of operator A. In addition, these four operators are compact, injective with dense range.*

Proof. Let us prove that $F_D = \overline{H}^*_D$. We have, for $(\tau, \sigma) \in H^{-3/2}(\partial D) \times H^{-1/2}(\partial D)$ and $(h, t) \in L^2(\Gamma) \times L^2(\Gamma)$,

$$
\begin{aligned}
\left\langle F_D \begin{pmatrix} \tau \\ \sigma \end{pmatrix}, \begin{pmatrix} h \\ t \end{pmatrix} \right\rangle &= \int_\Gamma \left(\int_{\partial D} G(x,y)\tau(y) + \partial_{n_y} G(x,y)\sigma(y)\, ds(y) \right) \overline{h(x)}\, ds(x) \\
&+ \int_\Gamma \partial_n \left(\int_{\partial D} G(x,y)\tau(y) + \partial_{n_y} G(x,y)\sigma(y)\, ds(y) \right) \overline{t(x)}\, ds(s) \\
&= \int_{\partial D} \tau(y) \left(\int_\Gamma G(x,y)\overline{h(x)} + \partial_{n_x} G(x,y)\overline{t(x)}\, ds(x) \right) ds(y) \\
&+ \int_{\partial D} \sigma(y)\partial_n \left(\int_\Gamma G(x,y)\overline{h(x)} + \partial_{n_x} G(x,y)\overline{t(x)}\, ds(x) \right) ds(y) \\
&= \int_{\partial D} \tau(y) \overline{\left(\int_\Gamma \overline{G(y,x)} h(x) + \overline{\partial_{n_x} G(y,x)} t(x)\, ds(x) \right)} ds(y) \\
&+ \int_{\partial D} \sigma(y)\partial_n \overline{\left(\int_\Gamma \overline{G(y,x)} h(x) + \overline{\partial_{n_x} G(y,x)} t(x)\, ds(x) \right)} ds(y) \\
&= \left\langle \begin{pmatrix} \tau \\ \sigma \end{pmatrix}, \overline{H}_D \begin{pmatrix} h \\ t \end{pmatrix} \right\rangle,
\end{aligned}
$$

that is $F_D = \overline{H}^*_D$. In the computation above, we have used that $G(x,y) = G(y,x)$ in view of (6). The identities $F_D = B_D S_D$ and $N_D = -B_D H_D$ are simple consequences of the very definition of the four operators. That B_D is compact is a consequence of the interior regularity of operator Δ^2. This implies, from $F_D = \overline{H}^*_D$, $F_D = B_D S_D$ and $N_D = -B_D H_D$, that F_D, H_D and N_D are also compact operators. Let us prove that B_D is injective. If $(\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ is such that $B_D(\phi, \psi)^T = 0$, this means that the solution v to problem (24) satisfies $(v|_\Gamma, \partial_n v|_\Gamma) = (0, 0)$. From uniqueness of the scattering problem with Dirichlet boundary condition (see again Theorem 1.1), we have that $v = 0$ in $\mathbb{R}^2 \setminus \overline{B_R}$, and then $v = 0$ in Ω from unique continuation for the operator $\Delta^2 - k^4$. We conclude that $v|_{\partial D}$ and $\partial_n v|_{\partial D} = 0$, that is $(\phi, \psi) = (0, 0)$. From $F_D = B_D S_D$ and the injectivity of S_D, we obtain the injectivity of F_D. Let us prove the injectivity of H_D. Assume that $(h, t) \in L^2(\Gamma) \times L^2(\Gamma)$ is such that $H_D(h, t)^T = 0$. Let us consider the function

$$
v_{h,t}(x) = \int_\Gamma G(x,y)h(y)\, ds + \partial_{n_y} G(x,y)t(y)\, ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma.
$$

Such function is a solution to problem (24) with $(\phi, \psi) = (0, 0)$, so that $v_{h,t} = 0$ in $\Omega \setminus \Gamma$. From the jump relationships $h = [Nv_{h,t}]$ and $t = [Mv_{h,t}]$ across Γ, we obtain that $(h, t) = (0, 0)$. So H_D is injective, as well as N_D since $N_D = -B_D H_D$. That $F_D = \overline{H}_D^*$ implies that F_D has a dense range, as well as $B_D = F_D S_D^{-1}$, $H_D = \overline{F}_D^*$ and $N_D = -B_D H_D$. $\square$

Remark 2. From proposition 3, we derive the classical factorization

$$N_D = -\overline{H}_D^* S_D^{-1} H_D.$$

We also need the fundamental range test property.

Proposition 4. *Assume that $k \notin \mathcal{K}_D$. We have*

$$z \in D \Longleftrightarrow \begin{pmatrix} G(\cdot, z)|_\Gamma \\ \partial_n G(\cdot, z)|_\Gamma \end{pmatrix} \in \mathrm{Range}(F_D).$$

Proof. First we observe that since S_D is an isomorphism, $\mathrm{Range}(F_D) = \mathrm{Range}(B_D)$. If $z \in D$, we observe that the function $G(\cdot, z)$ is the solution to problem (24) with

$$(\phi, \psi) = (G(\cdot, z)|_{\partial D}, \partial_n G(\cdot, z)|_{\partial D}) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D).$$

This implies that

$$\begin{pmatrix} G(\cdot, z)|_\Gamma \\ \partial_n G(\cdot, z)|_\Gamma \end{pmatrix} \in \mathrm{Range}(B_D).$$

If on the contrary $z \notin D$, let us assume that

$$\begin{pmatrix} G(\cdot, z)|_\Gamma \\ \partial_n G(\cdot, z)|_\Gamma \end{pmatrix} = B_D \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

with $(\phi, \psi) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$. The corresponding solution to problem (24) is denoted v. Since the trace $(v|_\Gamma, \partial_n|_\Gamma)$ coincides with $(G(\cdot, z)|_\Gamma, \partial_n G(\cdot, z)|_\Gamma)$, we have that $G(\cdot, z)$ and v coincide outside the ball B_R, and then in $\Omega \setminus \{z\}$ by a unique continuation argument. The contradiction comes from a regularity comparison at point z: the function v in locally infinitely smooth while from Remark 1, $G(\cdot, z)$ is not locally H^3. $\square$

From propositions 2, 3 and 4, we have the following Theorem. Since the proof mimics the one of [13] and is classical, it is left to the reader.

Theorem 2.5. *We assume that $k \notin \mathcal{K}_D$.*

- *If $z \in D$, then for all $\varepsilon > 0$ there exists a solution $(h_\varepsilon(\cdot, z), t_\varepsilon(\cdot, z)) \in L^2(\Gamma) \times L^2(\Gamma)$ of the inequality*

$$\left\| N_D \begin{pmatrix} h_\varepsilon(\cdot, z) \\ t_\varepsilon(\cdot, z) \end{pmatrix} - \begin{pmatrix} G(\cdot, z)|_\Gamma \\ \partial_n G(\cdot, z)|_\Gamma \end{pmatrix} \right\|_{L^2(\Gamma) \times L^2(\Gamma)} \leq \varepsilon \tag{25}$$

  such that the function $H_D(h_\varepsilon(\cdot, z), t_\varepsilon(\cdot, z))^T$ converges in $H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ as $\varepsilon \to 0$.
  Furthermore, for a given fixed $\varepsilon > 0$, the couple of functions $(h_\varepsilon(\cdot, z), t_\varepsilon(\cdot, z))$ satisfies

$$\lim_{z \to \partial D} \|(h_\varepsilon(\cdot, z), t_\varepsilon(\cdot, z))\|_{L^2(\Gamma) \times L^2(\Gamma)} = +\infty$$

  and

$$\lim_{z \to \partial D} \left\| H_D \begin{pmatrix} h_\varepsilon(\cdot, z) \\ t_\varepsilon(\cdot, z) \end{pmatrix} \right\|_{H^{3/2}(\partial D) \times H^{1/2}(\partial D)} = +\infty.$$

- *If $z \notin D$, then every solution $(h_\varepsilon(\cdot,z), t_\varepsilon(\cdot,z))$ of the inequality (25) satisfies*

$$\lim_{\varepsilon\to 0} \|(h_\varepsilon(\cdot,z), t_\varepsilon(\cdot,z))\|_{L^2(\Gamma)\times L^2(\Gamma)} = +\infty$$

and

$$\lim_{\varepsilon\to 0} \left\| H_D \begin{pmatrix} h_\varepsilon(\cdot,z) \\ t_\varepsilon(\cdot,z) \end{pmatrix} \right\|_{H^{3/2}(\partial D)\times H^{1/2}(\partial D)} = +\infty.$$

2.4 The case of Neumann obstacle

In the Neumann case, that is $(B_1, B_2) = (M, N)$, the important point is that the near-field operator N_N is unchanged with respect to N_D, provided in this case the scattered fields $u^s(\cdot,y)$ and $\tilde{u}^s(\cdot,y)$ for $y \in \Gamma$ are obtained for a Neumann obstacle instead of a Dirichlet obstacle. However the operators S_N, F_N, H_N and B_N which are involved in the theoretical justification of the Linear Sampling Method are modified as follows: $S_N : H^{3/2}(\partial D) \times H^{1/2}(\partial D) \to H^{-3/2}(\partial D) \times H^{-1/2}(\partial D)$ is such that, for all $x \in \partial D$,

$$S_N \begin{pmatrix} \phi \\ \psi \end{pmatrix}(x) = \begin{pmatrix} N \displaystyle\int_{\partial D} (N_y G(x,y)\phi(y) + M_y G(x,y)\psi(y))\, ds(y) \\ M \displaystyle\int_{\partial D} (N_y G(x,y)\phi(y) + M_y G(x,y)\psi(y))\, ds(y) \end{pmatrix}, \tag{26}$$

the obstacle-to-data operator is $F_N : H^{3/2}(\partial D) \times H^{1/2}(\partial D) \to L^2(\Gamma) \times L^2(\Gamma)$ such that, for all $x \in \Gamma$,

$$F_N \begin{pmatrix} \phi \\ \psi \end{pmatrix}(x) = \begin{pmatrix} \displaystyle\int_{\partial D} (N_y G(x,y)\phi(y) + M_y G(x,y)\psi(y))\, ds(y) \\ \dfrac{\partial \cdot}{\partial n} \displaystyle\int_{\partial D} (N_y G(x,y)\phi(y) + M_y G(x,y)\psi(y))\, ds(y) \end{pmatrix}, \tag{27}$$

the data-to-obstacle operator is $H_N : L^2(\Gamma) \times L^2(\Gamma) \to H^{-3/2}(\partial D) \times H^{-1/2}(\partial D)$ such that, for all $x \in \partial D$,

$$H_N \begin{pmatrix} h \\ t \end{pmatrix}(x) = \begin{pmatrix} N \displaystyle\int_{\Gamma} \left(G(x,y)h(y) + \frac{\partial G(x,y)}{\partial n_y} t(y) \right) ds(y) \\ M \displaystyle\int_{\Gamma} \left(G(x,y)h(y) + \frac{\partial G(x,y)}{\partial n_y} t(y) \right) ds(y) \end{pmatrix}. \tag{28}$$

Lastly, the solution operator $B_N : H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \to L^2(\Gamma) \times L^2(\Gamma)$ is such that for $(\tau, \sigma) \in H^{-3/2}(\partial D) \times H^{-1/2}(\partial D)$, $B(\tau,\sigma)^T$ is formed by the trace and normal derivative on Γ of the unique solution v in $H^2_{\mathrm{loc}}(\Omega)$ to the problem

$$\begin{cases} \Delta^2 v - k^4 v = 0 & \text{in } \Omega \\ Mv = \sigma, \quad Nv = \tau & \text{on } \partial\Omega \\ \displaystyle\lim_{r\to+\infty} \int_{\partial B_r} \left| \frac{\partial v}{\partial n} - ikv \right|^2 ds = 0. & \end{cases} \tag{29}$$

It can be checked that the Propositions 2, 3, 4 and Theorem 2.5 are also valid for operators S_N, B_N, F_N, H_N and N_N provided $k \notin \mathcal{K}_N$, where $\mathcal{K}_N$ is the union of $\mathcal{K}_0$ (see Theorem 1.1) and of the fourth roots of Neumann eigenvalues of operator Δ^2 in domain D.

3 Numerical experiments

3.1 Producing artificial data

In all the numerics, the Poisson's ratio is $\nu = 0.3$. In order to obtain artificial data for the inverse problem, we need to solve the problem (2) numerically in Ω_R. We use a Finite Element Method based on the Morley's finite element and by limiting the infinite sum (3) which defines the Dirichlet-to-Neumann operator T to a finite number M of terms. The finite element of Morley was introduced in [15] and analyzed for example in [16, 17]. This non-conforming finite element is probably the most simple element which enables one to solve bilaplacian problems. If Ω_R is approximated by a polygonal domain and meshed with triangles, the finite element space is formed by functions u_h such that their restriction on each triangle is given by a second degree polynomial, so that the number of degrees of freedom is 6 by triangle: the values of u_h at the three vertices of the triangle and the values of the normal derivatives of u_h at the middle of the three sides of the triangle. We use a very refined mesh, which is in particular consistent with the largest value of k ($k = 30$) that is used later on. The integer M has also to be sufficiently large with respect to k, we take $M = \lfloor k/R \rfloor + 20$. In the figure 1, we check the validity of the discretized Dirichlet-to-Neumann operator by computing the scattering solution either in Ω_1 or in Ω_2, that is Ω_R for $R = 1$ and $R = 2$. We verify that the two solutions coincide in the numerical sense in the intersection $\Omega_1 \cap \Omega_2 = \Omega_1$.

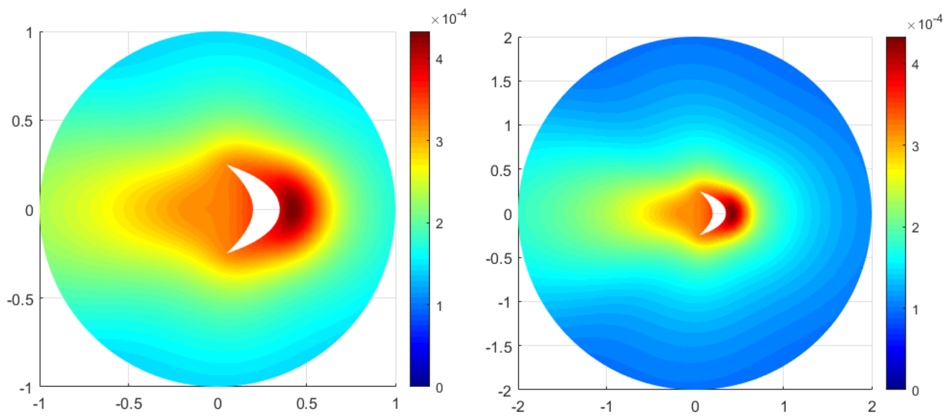

Figure 1: Validation of the artificial boundary condition. Left: scattering solution computed in Ω_1. Right: scattering solution computed in Ω_2.

3.2 Identification results

The procedure described in section 3.1 enables us to compute, for all $y \in \Gamma$, the solutions $u^s(\cdot, y)$ and $\tilde{u}^s(\cdot, y)$ to the problem (1) with $u^i = G(\cdot, y)$ and $\partial_{n_y} G(\cdot, y)$, respectively. For each sampling point $z \in \Omega_R$ we wish to solve in $L^2(\Gamma) \times L^2(\Gamma)$ the near-field equation

$$N_D \begin{pmatrix} h(\cdot, z) \\ t(\cdot, z) \end{pmatrix} = \begin{pmatrix} G(\cdot, z)|_\Gamma \\ \partial_n G(\cdot, z)|_\Gamma \end{pmatrix} \tag{30}$$

for Dirichlet data. That the operator N_D is compact implies that the previous equation is always ill-posed, this is why we solve it in the Tikhonov sense. In the realistic case when the data are perturbed by noise, the regularization parameter in the Tikhonov regularization is chosen as a function of the amplitude of noise by using the Morozov discrepancy principle exactly as in [18]. Of course the operator N_D has to be replaced by N_N for Neumann data. As it is done classically by LSM users, for each $z \in \Omega_R$ we plot

$$\Psi(z) = \log\left(\frac{1}{\sqrt{\|h_\varepsilon(\cdot,z)\|^2_{L^2(\Gamma)} + \|t_\varepsilon(\cdot,z)\|^2_{L^2(\Gamma)}}}\right),$$

where $(h_\varepsilon, t_\varepsilon)$ is the Tikhonov regularized solution. Following Theorem 2.5, the function Ψ happens to be finite inside the unknown defect D and $-\infty$ outside D, which means that imaging the defect D amounts to plotting the level sets of the function Ψ. In practice, the set of data is finite. In other words, we have to handle discretized versions of operators N_D to N_N corresponding to multistatic data $u^s(x_i, y_j)$ and $\tilde{u}^s(x_i, y_j)$ where the points x_i and y_j for $i, j = 1, \cdots, I$ are equally distributed on the circle Γ (the locations of points x_i and points y_j are the same). Here we have $I = 500$ in all the identification experiments. In the figure 2, we have represented the function Ψ for a Dirichlet obstacle D formed by the union of two discs and by using exact data, for various wave numbers k, that is $k = 10$, $k = 20$ and $k = 30$. In the figure 3, we show the identification results for an obstacle with the same geometry but with Neumann boundary condition, for $k = 10$, $k = 20$ and $k = 30$. In the figure 4, the identification results are presented for a Dirichlet obstacle formed by three circles with $k = 30$ and for a kite-shaped Neumann obstacle with $k = 20$. Data are exact in all the previous cases.

Now, let us analyze the impact of noise on the data $u^s(x_i, y_j)$ and $\tilde{u}^s(x_i, y_j)$ for $i, j = 1, \cdots, I$. This noise is such that for each j, the scattered fields $u^s(\cdot, y_j)$ and $\tilde{u}^s(\cdot, y_j)$ as well as their normal derivatives are perturbed at each point x_i by a Gaussian noise which is then rescaled in such a way that the artificial relative errors for the L^2 norm of all these fields have a prescribed value σ. We show in figure 5 the obtained results for the Dirichlet obstacle formed by two circles for $k = 20$ when the amplitude of noise is 5% and 10% (that is $\sigma = 0.05$ and $\sigma = 0.1$). These results have to be compared with the top right part of figure 2 obtained with unperturbed data.

We complete this numerical study by some identification attempts with much less data. Indeed, from a practical point of view, it could be considered as a complicated task to produce the data $\tilde{u}^s(\cdot, y)$, which for each y is the diffractive response due to the solicitation consisting in the normal derivative of a point source, namely $\partial_{n_y} G(\cdot, y)$. In addition, not only our sampling method needs the trace of fields $u^s(\cdot, y)$ and $\tilde{u}^s(\cdot, y)$ on Γ but also their normal derivatives. Hence it is natural to wonder if one can use a sampling method using only $u^s(x, y)$ for $(x, y) \in \Gamma \times \Gamma$, exactly as if the scattered field u^s solved the Helmholtz equation $\Delta u^s + k^2 u^s = 0$ instead of the true equation $\Delta^2 u^s - k^4 u^s = 0$. In [1], we observed that in Ω, any solution u to the equation $\Delta^2 u - k^4 u = 0$ is given by $u = u_{\rm pr} + u_{\rm ev}$, where $u_{\rm pr}$ satisfies $\Delta u_{\rm pr} + k^2 u_{\rm pr} = 0$ and $u_{\rm ev}$ satisfies $\Delta u_{\rm ev} - k^2 u_{\rm ev} = 0$. If we assume in addition that the function u is radiating, is was proved that outside some ball B_R, the function $u_{\rm pr}$ is an infinite linear combination of the functions $H^1_n(kr)e^{in\theta}$, which are oscillating and slowly decaying at infinity, while $u_{\rm ev}$ is an infinite linear combination of the

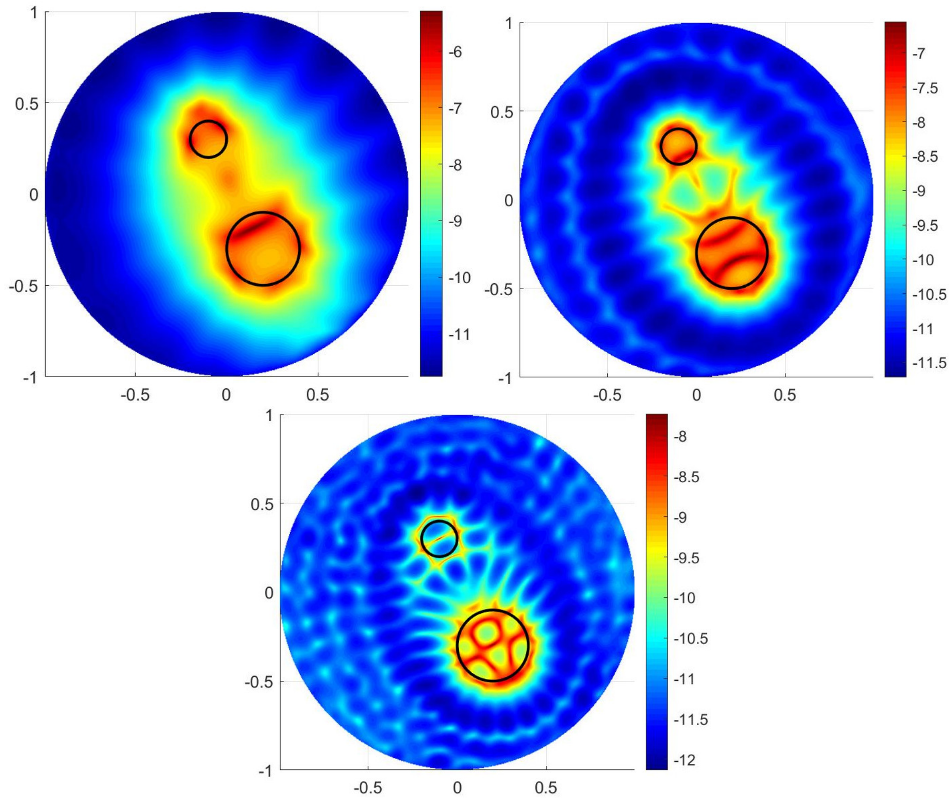

Figure 2: Dirichlet obstacle, exact data and various wave numbers k. Top left: $k = 10$. Top right: $k = 20$. Bottom: $k = 30$

functions $H_n^1(ikr)e^{in\theta}$, which are exponentially decaying at infinity. This is why we call $u_{\rm pr}$ the propagating part of the radiating solution u and $u_{\rm ev}$ the evanescent part. As a conclusion, at a long distance of the obstacle, the scattered field u can be approximated by its propagating part $u_{\rm pr}$, which solves the Helmholtz equation (the evanescent part $u_{\rm ev}$ of u is neglected). Similarly, the fundamental solution $G(\cdot, y)$ given by (6) can be approximated at long distance of y by its propagating part $i\,H_0^1(k|\cdot - y|)/8k^2$, which up to a constant coincides with the fundamental solution $\mathscr{G}(\cdot, y)$ to the Helmholtz equation. In the Dirichlet case, it is then tempting, rather than solving the near-field equation (30), to solve the classical near-field equation corresponding to the Helmholtz equation. It consists, for each sampling point $z \in \Omega_R$, to solve in $L^2(\Gamma)$ the near-field equation

$$\mathscr{N} h = \mathscr{G}(\cdot, z)|_\Gamma, \tag{31}$$

where the operator $\mathscr{N} : L^2(\Gamma) \to L^2(\Gamma)$ is defined, for $h \in L^2(\Gamma)$, by

$$\mathscr{N} h(x) = \int_\Gamma u^s(x, y) h(y)\, dy, \quad x \in \Gamma.$$

We here use the fact that the Dirichlet boundary conditions for problem (1) implies the Dirichlet boundary condition for the Helmholtz equation. Comparing the operator $\mathscr{N}$ and the operator N_D, we see that solving the equation (31) rather than

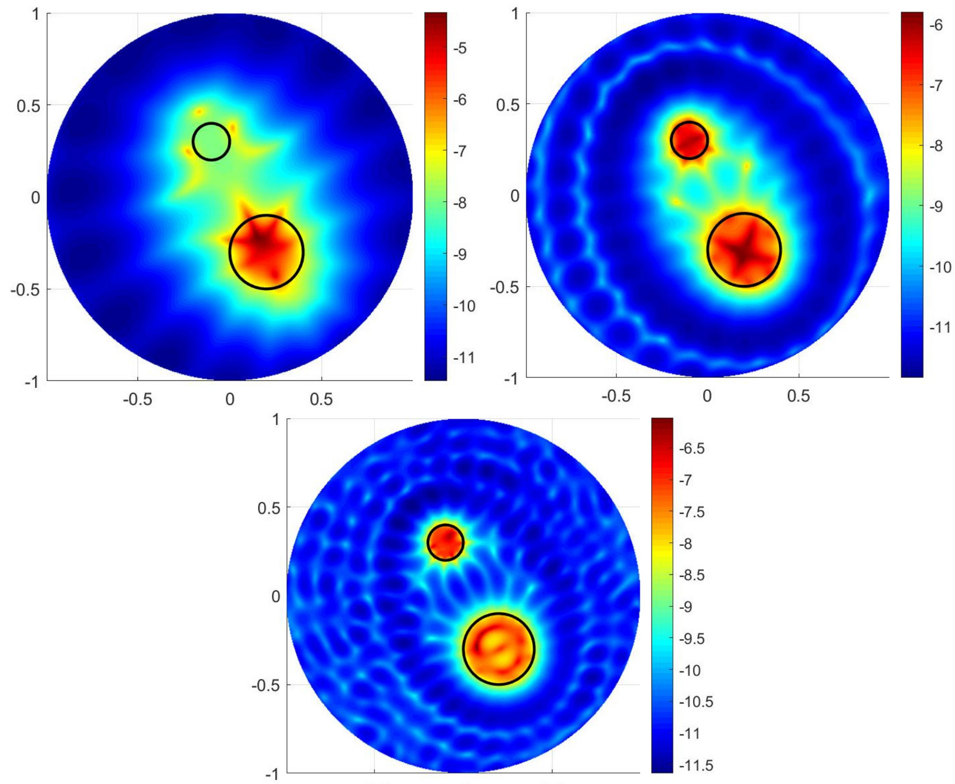

Figure 3: Neumann obstacle, exact data and various wave numbers k. Top left: $k = 10$. Top right: $k = 20$. Bottom: $k = 30$.

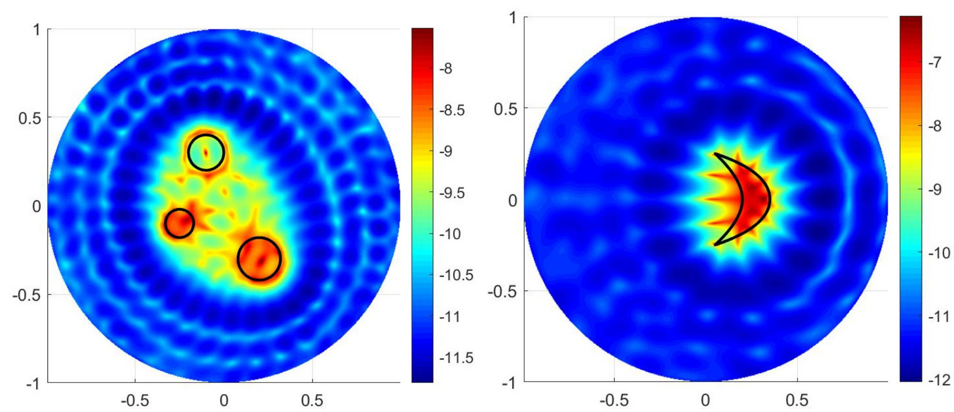

Figure 4: Left: Dirichlet obstacle formed by 3 circles, $k = 30$ and exact data. Right: Neumann kite-shaped obstacle, $k = 20$ and exact data.

the equation (30) leads us to use one block of data out of four. On the picture 6, we present the identification results for the Dirichlet obstacles formed by two or three circles, in the case of unperturbed data and $k = 30$, when we solve the equation

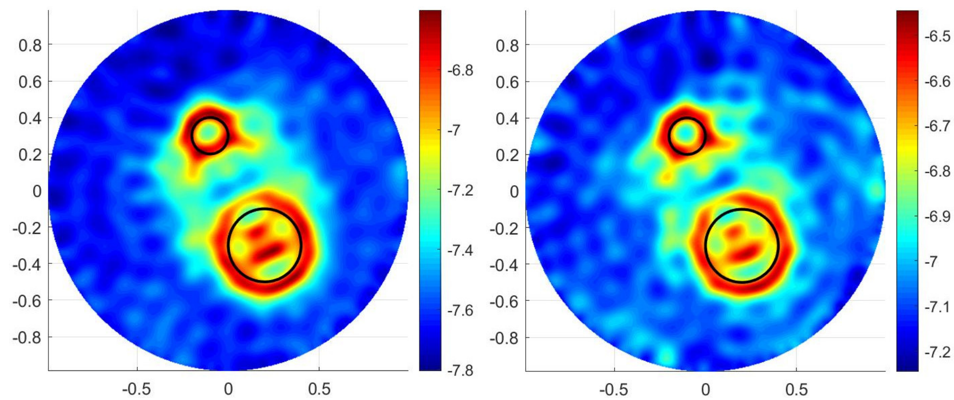

Figure 5: Dirichlet obstacle, $k = 20$. Left: noise of amplitude 5%. Right: noise of amplitude 10%.

(31) instead of (30). These results have to be compared with the bottom part of picture 2 and the left part of picture 4. We now consider the case when we have

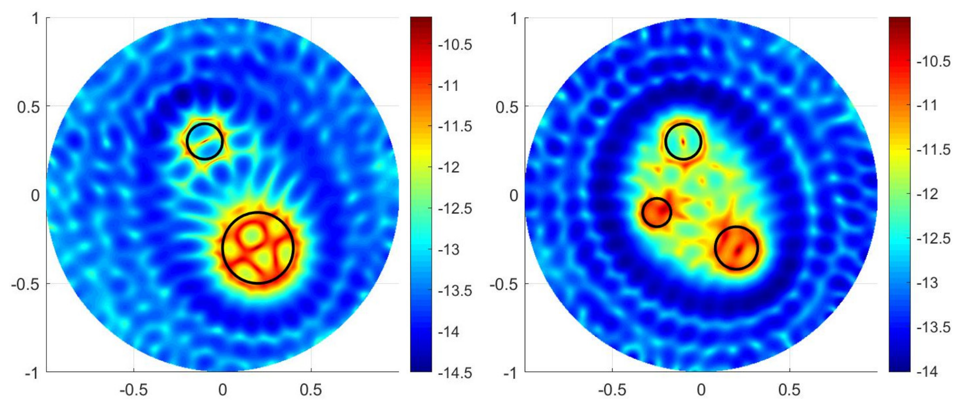

Figure 6: Dirichlet obstacle with less (exact) data, $k = 30$. Left: two circles. Right: three circles.

less data and when those data are noisy. The corresponding results are presented on picture 7 for the Dirichlet obstacle formed by two circles, for $k = 20$, when the amplitude of noise is 5% and 10%. These results have to be directly compared with the ones presented on picture 5.

3.3 Conclusion

The numerical experiments of the previous section seem to show that the Linear Sampling Method is effective for Kirchhoff-Love plates, at least with data given on a circle Γ surrounding the unknown obstacle. It works both for a Dirichlet obstacle and for a Neumann obstacle, even in the presence of noisy data. As usual, the identification results improve when the wave number increases. Maybe we notice a slight degradation of the quality of the identification if we compare the results

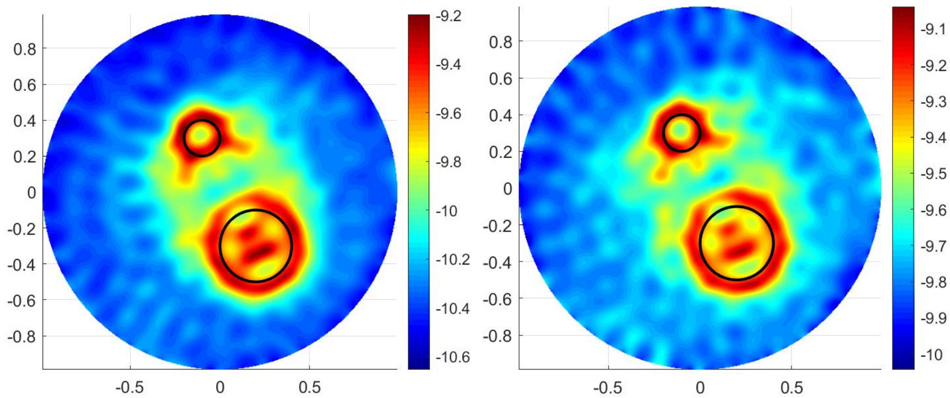

Figure 7: Dirichlet obstacle with less data, $k = 20$. Left: noise of amplitude 5%. Right: noise of amplitude 10%.

with those classically obtained for the Helmholtz equation in two dimensions. If we compare to such case of Helmholtz equation in two dimensions, the LSM requires more data in the sense that we need the trace and the normal derivative on Γ of the scattered fields associated with both the point source $G(\cdot, y)$ and its normal derivative $\partial_{n_y} G(\cdot, y)$ for all $y \in \Gamma$. However, we have shown experimentally in the Dirichlet case that using only the trace on Γ of the scattered field associated with the point source $G(\cdot, y)$ for all $y \in \Gamma$ and hence proceeding exactly as if we solved the Dirichlet case for the Helmholtz equation, produces almost as good results as when using the complete data. This simplification would deserve a more quantitative justification.

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