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Crandall-Lions Viscosity Solutions for Path-Dependent PDEs: the Case of Heat Equation

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We address our interest to the development of a theory of viscosity solutions à la Crandall-Lions for path-dependent partial differential equations (PDEs), namely PDEs in the space of continuous paths $C([0, T]; \mathbb{R}^d)$. Path-dependent PDEs can play a central role in the study of certain classes of optimal control problems, as for instance optimal control problems with delay. Typically, they do not admit a smooth solution satisfying the corresponding HJB equation in a classical sense, it is therefore natural to search for a weaker notion of solution. While other notions of generalized solution have been proposed in the literature, the extension of the Crandall-Lions framework to the path-dependent setting is still an open problem. The question of uniqueness of the solutions, which is the most delicate issue, will be based on early ideas from the theory of viscosity solutions and a suitable variant of Ekeland's variational principle. This latter is based on the construction of a smooth gauge-type function, where smooth is meant in the horizontal/vertical (rather than Fréchet) sense. In order to make the presentation more readable, we address the path-dependent heat equation, which in particular simplifies the smoothing of its natural "candidate" solution. Finally, concerning the existence part, we provide a functional Itô formula under general assumptions, extending earlier results in the literature.

MSC 2010 subject classifications: 35D40, 35R15, 60H30.

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1. Introduction

Path-dependent heat equation refers to the second-order partial differential equation in the space of continuous paths

$$\begin{cases} -\partial_t^H v(t, \mathbf{x}) - \frac{1}{2} \text{tr}[\partial_{\mathbf{x}\mathbf{x}}^V v(t, \mathbf{x})] = 0, & (t, \mathbf{x}) \in [0, T) \times C([0, T]; \mathbb{R}^d), \\ v(T, \mathbf{x}) = \xi(\mathbf{x}), & \mathbf{x} \in C([0, T]; \mathbb{R}^d). \end{cases} \quad (1.1)$$

Here $C([0, T]; \mathbb{R}^d)$ denotes the Banach space of continuous paths $\mathbf{x}: [0, T] \rightarrow \mathbb{R}^d$ equipped with the supremum norm $\|\mathbf{x}\|_\infty := \sup_{t \in [0, T]} |\mathbf{x}(t)|$, with $|\cdot|$ denoting the Euclidean norm

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on \mathbb{R}^d . The terminal condition $\xi: C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}$ is assumed to be continuous and bounded. We refer to equation (1.1) as path-dependent *heat* equation. Similarly as for the usual heat equation, it admits the following Feynman-Kac representation formula in terms of the d -dimensional Brownian motion $\mathbf{W} = (\mathbf{W}_s)_{s \in [0, T]}$:

$$v(t, \mathbf{x}) = \mathbb{E}[\xi(\mathbf{W}^{t, \mathbf{x}})], \quad \forall (t, \mathbf{x}) \in [0, T] \times C([0, T]; \mathbb{R}^d), \quad (1.2)$$

where

$$\mathbf{W}_s^{t, \mathbf{x}} := \begin{cases} \mathbf{x}(s), & s \leq t, \\ \mathbf{x}(t) + \mathbf{W}_s - \mathbf{W}_t, & s > t. \end{cases}$$

In the case of the classical heat equation ξ only depends on the terminal value $\mathbf{W}_T^{t, \mathbf{x}}$.

The peculiarity of equation (1.1) is the presence of the so-called functional or pathwise derivatives $\partial_t^H v$, $\partial_{\mathbf{x}\mathbf{x}}^V v$, where $\partial_t^H v$ is known as horizontal derivative, while $\partial_{\mathbf{x}\mathbf{x}}^V v$ is the matrix of second-order vertical derivatives. Those derivatives appeared in [56, 57] (under the name of coinvariant derivatives) as building block of the so-called *i*-smooth analysis, and independently in [1], where they were denoted Clio derivatives; later, they were rediscovered by [31] (from which we borrow terminology and definitions), which adopted a slightly different definition based on the space of càdlàg paths and in addition developed a related stochastic calculus, known as functional Itô calculus, including in particular the so-called functional Itô formula. Differently from the classical Fréchet derivative on $C([0, T]; \mathbb{R}^d)$, the distinguished features of the pathwise derivatives are their finite-dimensional nature and the property of being non-anticipative, which follow from the interpretation of t in $\mathbf{x}(t)$ as time variable. This means that $v(t, \mathbf{x})$ only depends on the values of the path \mathbf{x} up to time t ; moreover, the horizontal and vertical derivatives at time t are computed keeping the past values frozen, while only the present value of the path (that is $\mathbf{x}(t)$) can vary. The related functional Itô calculus was rigorously investigated in [12, 13, 14]. [15, 18] also gave a contribution in this direction, exploring the relation between pathwise derivatives and Banach space stochastic calculus, built on an appropriate notion of Fréchet type derivative and firstly conceived in [26], see also [27, 28, 29, 25].

Partial differential equations in the space of continuous paths (also known as functional or Clio or path-dependent partial differential equations) are mostly motivated by optimal control problems of deterministic and stochastic systems with delay (or path-dependence) in the state variable. Such control systems arise in many fields, as for instance optimal advertising theory [49, 50], chemical engineering [46], financial management [39, 74], economic growth theory [2], mean field game theory [5], biomedicine [47, 84], systemic risk [10]. The underlying deterministic or stochastic controlled differential equations with delay can be studied in two ways: first using a direct approach (see for instance [52, 86, 56, 53, 58]), second by lifting them into a suitable infinite-dimensional framework, leading to evolution equations in Hilbert (as in [11, 24, 42]) or Banach spaces (as in [71, 72, 26]). The latter methodology turned out to be preferable to address general optimal control problems with delay (see for instance [92, 51, 49, 40, 41, 45, 38]), although such an infinite-dimensional reformulation may require some additional artificial assumptions to be imposed on the original control problem. On the other hand, the

direct approach was adopted for special problems where the Hamilton-Jacobi-Bellman equation reduces to a finite-dimensional differential equation, as in [36, 59]. This approach can now regain relevance thanks to a well-grounded theory of path-dependent partial differential equations. To this regard, the path-dependent heat equation represents the primary test for such a theory, it indeed requires the main building blocks of the methodology, without overloading the proofs with additional technicalities.

Path-dependent partial differential equations represent a quite recent area of research. Typically, they do not admit a smooth solution satisfying the equation in a classical sense, mainly because of the awkward nature of the underlying space $C([0, T]; \mathbb{R}^d)$. This happens also for the path-dependent heat equation, which in particular does not have the smoothing effect characterizing the classical heat equation, except in some specific cases (as shown in [26, 30]) with ξ belonging to the class of so-called cylinder or tame functions (therefore depending specifically on a finite number of integrals with respect to the path) or ξ being smoothly Fréchet differentiable. It is indeed quite easy, relying on the probabilistic representation formula (1.2), to see that the function v is not smooth (in the horizontal/vertical sense mentioned above) for terminal conditions of the form

$$\xi(\mathbf{x}) = \sup_{0 \leq t \leq T} \mathbf{x}(t), \quad \xi(\mathbf{x}) = \mathbf{x}(t_0),$$

for some fixed $t_0 \in (0, T)$. For a detailed analysis of the first case above we refer to Section 3.2 in [17], see also Remark 3.8 in [18]. Concerning the second case, see for instance Example 11.1.3 of [93]. It is however worth mentioning that some positive results on smooth solutions were obtained in [18, 78]. We also refer to Chapter 9 of [26] and [30], where smooth solutions were investigated using a Fréchet type derivative formulation.

It is therefore natural to search for a weaker notion of solution, as the notion of viscosity solution, commonly used in the standard finite-dimensional case. The theory of viscosity solutions, firstly introduced in [21, 22] for first-order equations in finite dimension and later extended to the second-order case in [61, 62, 63], provides a well-suited framework guaranteeing the desired existence, uniqueness, and stability properties (for a comprehensive account see [20]). The extension of such a theory to equations in infinite dimension was initiated by [23, 64, 65, 66, 87, 90]. One of the structural assumption is that the state space has to be a Hilbert space or, slightly more general, certain Banach space with smooth norm, not including for instance the Banach space $C([0, T]; \mathbb{R}^d)$ (notice however that in this paper we do not directly generalize those results to $C([0, T]; \mathbb{R}^d)$, as we adopt horizontal/vertical, rather than Fréchet, derivatives on $C([0, T]; \mathbb{R}^d)$).

First-order path-dependent partial differential equations were deeply investigated in [70] using a viscosity type notion of solution, which differs from the Crandall-Lions definition as the maximum/minimum condition is formulated on the subset of absolutely continuous paths. Such a modification does not affect existence in the first-order case, however it is particularly convenient for uniqueness, which is indeed established under general conditions. Other notions of generalized solution designed for first-order equations were adopted in [1] as well as in [67, 68, 69], where the minimax framework introduced in [88, 89] was implemented. We also mention [4], where such a minimax approach was extended to first-order path-dependent Hamilton-Jacobi-Bellman equations in infinite di-

mension. Concerning the second-order case, a first attempt to extend the Crandall-Lions framework to the path-dependent case was carried out in [76], even though a technical condition on the semi-jets was imposed, namely condition (16) in [76], which narrows down the applicability of such a result. In the literature, this was perceived as an almost insurmountable obstacle, so that the Crandall-Lions definition was not further investigated, while other notions of generalized solution were devised, see [33, 77, 91, 19, 60, 3, 9]. We mention in particular the framework designed in [33] and further investigated in [34, 35, 81, 82, 83, 16], where the notion of sub/supersolution adopted differs from the Crandall-Lions definition as the tangency condition is not pointwise but in the sense of expectation with respect to an appropriate class of probability measures. On the other hand, in [19] we introduced the so-called strong-viscosity solution, which is quite similar to the notion of good solution for partial differential equations in finite dimension, that in turn is known to be equivalent to the definition of L^p -viscosity solution, see for instance [54]. We also mention [3], where the authors deal with semilinear path-dependent equations and propose the notion of decoupled mild solution, formulated in terms of generalized transition semigroups; such a notion also adapts to path-dependent equations with integro-differential terms.

In the present paper we adopt the natural generalization of the well-known definition of viscosity solution à la Crandall-Lions given in terms of test functions and, under this notion, we establish existence and uniqueness for the path-dependent heat equation (1.1). The uniqueness property is derived, as usual, from the comparison theorem. The proof of this latter, which is the most delicate issue, is known to be quite involved even in the classical finite-dimensional case (see for instance [20]), and in its latest form is based on Ishii's lemma. Here we follow instead an earlier approach (see for instance Theorem II.1 in [63] or Theorem IV.1 in [64]), which in principle can be applied to any path-dependent equation admitting a "candidate" solution v , for which a probabilistic representation formula holds. This is the case for equation (1.1), where the candidate solution is given by formula (1.2), but it is also the case for Kolmogorov type equations or, more generally, for Hamilton-Jacobi-Bellman equations. This latter is the class of equations studied in [63] and [64], whose methodology in a nutshell can be described as follows. Let u (resp. w) be a viscosity subsolution (resp. supersolution) of the same path-dependent equation. The desired inequality $u \leq w$ follows if we compare both u and w to the "candidate" solution v , that is if we prove the two inequalities $u \leq v$ and $v \leq w$. Let us consider for instance the first inequality $u \leq v$. In the non-path-dependent and finite-dimensional case (as in [63]), this is proved proceeding as follows: firstly, performing a smoothing of v through its probabilistic representation formula; secondly, taking a local maximum of $u - v_n$ (here it is used the local compactness of the finite-dimensional underlying space), with v_n being a smooth approximation of v ; finally, the inequality $u \leq v_n$ is proved proceeding as in the so-called partial comparison theorem (comparison between a viscosity subsolution/supersolution and a smooth supersolution/subsolution), namely exploiting the viscosity subsolution property of u with v_n playing the role of test function. In [64], where such a methodology was extended to the infinite-dimensional case, the existence of a maximum of $u - v_n$ is achieved relying on Ekeland's variational principle, namely exploiting the completeness of the space instead of the missing local compactness.

In this paper we generalize the methodology sketched above to the path-dependent case. There are however at least two crucial mathematical issues required by such a proof, still not at disposal in the path-dependent framework.

Firstly, given a candidate solution v , it is not a priori obvious how to perform a smooth approximation of v itself starting from its probabilistic representation formula. Here we exploit the results proved in [18] (Theorem 3.5) and [19] (Theorem 3.12), which are reported and adapted to the present framework in Appendix D (Lemma D.1 and Lemma D.2, respectively). Notice that such results apply to the case of the path-dependent heat equation (1.1), where there is only the terminal condition ξ in the probabilistic representation formula (1.2) for v . More general results are at disposal in [18] and [19], which cover the case of semilinear path-dependent partial differential equations, characterized by the presence of four coefficients b , σ , F , ξ (see, in particular, Theorem 3.16 in [19] for more details). However, when those other coefficients appear in the path-dependent partial differential equation, we need more information on the sequence $\{v_n\}_n$ approximating v . For instance, we also have to estimate the derivatives of v_n in order to proceed as in [63] or [64]. Since such results are still not at disposal in the path-dependent setting, in order to make the paper more readable and not excessively lengthy, here we address the case of the path-dependent heat equation.

Secondly, concerning the existence of a maximum of $u - v_n$, we rely on a generalized version of Ekeland's variational principle for which we need a smooth gauge-type function with bounded derivatives, as explained below. Our equation is in fact formulated on the non-locally compact space $[0, T] \times C([0, T]; \mathbb{R}^d)$ endowed with the pseudometric

$$d_\infty((t, \mathbf{x}), (t', \mathbf{x}')) := |t - t'| + \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}'(\cdot \wedge t')\|_\infty.$$

Recall that Ekeland's variational principle, in its original form, applied to $([0, T] \times C([0, T]; \mathbb{R}^d), d_\infty)$ states that a perturbation $u(\cdot, \cdot) - v_n(\cdot, \cdot) - \delta d_\infty((\cdot, \cdot), (\bar{t}, \bar{\mathbf{x}}))$ of $u(\cdot, \cdot) - v_n(\cdot, \cdot)$ has a strict global maximum, with the perturbation being expressed in terms of the distance d_∞ (the point $(\bar{t}, \bar{\mathbf{x}})$ is fixed). As the map $(t, \mathbf{x}) \mapsto d_\infty((t, \mathbf{x}), (\bar{t}, \bar{\mathbf{x}}))$ is not smooth, it cannot be a test function. In order to have a smooth map instead of d_∞ , we need a smooth variational principle on $[0, T] \times C([0, T]; \mathbb{R}^d)$. To this end, the starting point is a generalization of the so-called Borwein-Preiss smooth variant of Ekeland's variational principle (see for instance [8]), which works when d_∞ is replaced by a so-called gauge-type function (see Definition 3.1). For the proof of the comparison theorem, we have to construct a gauge-type function which is also smooth and with bounded derivatives, recalling that *smooth* in the present context means in the horizontal/vertical (rather than in the Fréchet) sense. In Section 3 such a gauge-type function is built through a smoothing of d_∞ itself (more precisely, of the part concerning the supremum norm). This latter smoothing is performed by convolution, firstly in the vertical direction, that is in the direction of the map $1_{[t, T]}$ (Lemma 3.1), then in the horizontal direction (Lemma 3.2), the ordering of smoothings being crucial. Notice in particular that the supremum norm is already smooth in the horizontal direction; however, after the vertical smoothing, we lose in general the horizontal regularity because of the presence of the term $1_{[t, T]}$; for this reason we have also to perform the horizontal smoothing. The resulting smooth

gauge-type function with bounded derivatives corresponds to the function ρ_∞ defined in (3.8).

Regarding existence, we prove that the candidate solution v in (1.2) solves in the viscosity sense equation (1.1). We proceed essentially as in the classical non-path-dependent case, relying as usual on Itô's formula, which in the present context corresponds to the functional Itô formula. Such a formula was firstly stated in [31] and then rigorously proved in [12, 13], see also [14, 43, 18, 60, 73]. In the present paper we provide a functional Itô formula under general assumptions (Theorem 2.2). In particular, we do not require any boundedness assumption on the functional $u: [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}$, thus improving (when the semimartingale process is continuous) the results stated in [12, 13].

The paper is organized as follows. Section 2 is devoted to pathwise derivatives and functional Itô calculus. In particular, there is the functional Itô formula (Theorem 2.2) whose complete proof is reported in Appendix A. In Section 3 we prove the smooth variational principle on $[0, T] \times C([0, T]; \mathbb{R}^d)$, constructing the smooth gauge-type function with bounded derivatives. In Section 4 we provide the (path-dependent) Crandall-Lions definition of viscosity solution for a general path-dependent partial differential equation. We then study in detail the path-dependent heat equation. In particular, we prove existence showing that the so-called candidate solution v solves in the viscosity sense the path-dependent heat equation (Theorem 4.1). We conclude Section 4 proving the comparison theorem (Theorem 4.2) and uniqueness (Corollary 4.1).

2. Pathwise derivatives and functional Itô calculus

In the present section we define the pathwise derivatives and prove the functional Itô formula under general assumptions.

2.1. Maps on càdlàg paths

Given $T > 0$ and $d \in \mathbb{N}^*$, we denote by $D([0, T]; \mathbb{R}^d)$ the set of càdlàg functions $\hat{\mathbf{x}}: [0, T] \rightarrow \mathbb{R}^d$. We denote by $\hat{\mathbf{x}}(t)$ the value of $\hat{\mathbf{x}}$ at $t \in [0, T]$. We also denote by $\mathbf{0}$ the function $\hat{\mathbf{x}}: [0, T] \rightarrow \mathbb{R}^d$ identically equal to zero. We consider on $D([0, T]; \mathbb{R}^d)$ the supremum norm $\|\cdot\|_\infty$, namely $\|\hat{\mathbf{x}}\|_\infty := \sup_{t \in [0, T]} |\hat{\mathbf{x}}(t)|$, where $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^d (we use the same symbol $|\cdot|$ to denote the Euclidean norm on \mathbb{R}^k , for any $k \in \mathbb{N}$). We refer to Chapter V in [79] and to Section 15 of Chapter 3 in [6] for a study of the set of càdlàg functions endowed with the uniform metric and a comparison with the Skorokhod space.

We set $\hat{\mathbf{\Lambda}} := [0, T] \times D([0, T]; \mathbb{R}^d)$ and define $\hat{d}_\infty: \hat{\mathbf{\Lambda}} \times \hat{\mathbf{\Lambda}} \rightarrow [0, \infty)$ as

$$\hat{d}_\infty((t, \hat{\mathbf{x}}), (t', \hat{\mathbf{x}}')) := |t - t'| + \|\hat{\mathbf{x}}(\cdot \wedge t) - \hat{\mathbf{x}}'(\cdot \wedge t')\|_\infty.$$

Notice that \hat{d}_∞ is a *pseudometric* on $\hat{\mathbf{\Lambda}}$, that is \hat{d}_∞ is not a true metric because one may have $\hat{d}_\infty((t, \hat{\mathbf{x}}), (t', \hat{\mathbf{x}}')) = 0$ even if $(t, \hat{\mathbf{x}}) \neq (t', \hat{\mathbf{x}}')$. We recall that one can construct a

true metric space $(\hat{\mathbf{\Lambda}}^*, \hat{d}_\infty^*)$, called the metric space induced by the pseudometric space $(\hat{\mathbf{\Lambda}}, \hat{d}_\infty)$, by means of the equivalence relation which follows from the vanishing of the pseudometric. We also observe that $(\hat{\mathbf{\Lambda}}, \hat{d}_\infty)$ is a complete pseudometric space. Finally, we denote by $\mathcal{B}(\hat{\mathbf{\Lambda}})$ the Borel σ -algebra on $\hat{\mathbf{\Lambda}}$ induced by \hat{d}_∞ .

Definition 2.1. A map (or functional) $\hat{u}: \hat{\mathbf{\Lambda}} \rightarrow \mathbb{R}$ is said to be **non-anticipative** (on $\hat{\mathbf{\Lambda}}$) if it satisfies

$$\hat{u}(t, \hat{\mathbf{x}}) = \hat{u}(t, \hat{\mathbf{x}}(\cdot \wedge t)),$$

for all $(t, \hat{\mathbf{x}}) \in \hat{\mathbf{\Lambda}}$.

Remark 2.1. (i) The property of being non-anticipative is crucial and automatically true if the map $\hat{u}: \hat{\mathbf{\Lambda}} \rightarrow \mathbb{R}$ is continuous with respect to \hat{d}_∞ .

(ii) More generally, it holds that whenever $\hat{u}: \hat{\mathbf{\Lambda}} \rightarrow \mathbb{R}$ is Borel measurable, namely \hat{u} is measurable with respect to $\mathcal{B}(\hat{\mathbf{\Lambda}})$, then \hat{u} is non-anticipative on $\hat{\mathbf{\Lambda}}$. As a matter of fact, notice that every open subset B of $\hat{\mathbf{\Lambda}}$, endowed with \hat{d}_∞ , satisfies the following property: if $(t, \hat{\mathbf{x}}) \in B$ then $(t, \hat{\mathbf{x}}(\cdot \wedge t)) \in B$ (this follows from the fact that $\hat{d}_\infty((t, \hat{\mathbf{x}}), (t, \hat{\mathbf{x}}(\cdot \wedge t))) = 0$). As a consequence, by a monotone class argument, the same property holds true for every Borel subset of $\hat{\mathbf{\Lambda}}$. Now, let $\hat{u}: \hat{\mathbf{\Lambda}} \rightarrow \mathbb{R}$ be Borel measurable. For every $(t, \hat{\mathbf{x}}) \in \hat{\mathbf{\Lambda}}$, denote

$$B_{\hat{u}(t, \hat{\mathbf{x}})} := \{(s, \hat{\mathbf{y}}) \in \hat{\mathbf{\Lambda}} : \hat{u}(s, \hat{\mathbf{y}}) = \hat{u}(t, \hat{\mathbf{x}})\}.$$

Notice that $B_{\hat{u}(t, \hat{\mathbf{x}})} \in \mathcal{B}(\hat{\mathbf{\Lambda}})$ and since $(t, \hat{\mathbf{x}}) \in B_{\hat{u}(t, \hat{\mathbf{x}})}$ we deduce that $(t, \hat{\mathbf{x}}(\cdot \wedge t)) \in B_{\hat{u}(t, \hat{\mathbf{x}})}$. This means that $\hat{u}(t, \hat{\mathbf{x}}(\cdot \wedge t)) = \hat{u}(t, \hat{\mathbf{x}})$, namely the map \hat{u} is non-anticipative.

Definition 2.2. We denote by $C(\hat{\mathbf{\Lambda}})$ the set of maps $\hat{u}: \hat{\mathbf{\Lambda}} \rightarrow \mathbb{R}$ which are continuous on $\hat{\mathbf{\Lambda}}$ with respect to \hat{d}_∞ .

Definition 2.3 (Pathwise derivatives). Let $\hat{u}: \hat{\mathbf{\Lambda}} \rightarrow \mathbb{R}$ be non-anticipative.

- (i) Given $(t, \hat{\mathbf{x}}) \in \hat{\mathbf{\Lambda}}$, with $t < T$, the **horizontal derivative** of \hat{u} at $(t, \hat{\mathbf{x}})$ (if the corresponding limit exists) is defined as

$$\partial_t^H \hat{u}(t, \hat{\mathbf{x}}) := \lim_{\delta \rightarrow 0^+} \frac{\hat{u}(t + \delta, \hat{\mathbf{x}}(\cdot \wedge t)) - \hat{u}(t, \hat{\mathbf{x}})}{\delta}.$$

At $t = T$ the horizontal derivative is defined as

$$\partial_t^H \hat{u}(T, \hat{\mathbf{x}}) := \lim_{t \rightarrow T^-} \partial_t^H \hat{u}(t, \hat{\mathbf{x}}).$$

- (ii) Given $(t, \hat{\mathbf{x}}) \in \hat{\mathbf{\Lambda}}$, the **vertical derivatives** of first and second-order of \hat{u} at $(t, \hat{\mathbf{x}})$ (if the corresponding limits exist) are defined as

$$\partial_{x_i}^V \hat{u}(t, \hat{\mathbf{x}}) := \lim_{h \rightarrow 0} \frac{\hat{u}(t, \hat{\mathbf{x}} + h \mathbf{e}_i 1_{[t, T]}) - \hat{u}(t, \hat{\mathbf{x}})}{h},$$

$$\partial_{x_i x_j}^V \hat{u}(t, \hat{\mathbf{x}}) := \partial_{x_j}^V (\partial_{x_i}^V \hat{u})(t, \hat{\mathbf{x}}),$$

where $\mathbf{e}_1, \dots, \mathbf{e}_d$ is the standard orthonormal basis of \mathbb{R}^d .

Finally, we denote $\partial_{\mathbf{x}}^V \hat{u} = (\partial_{x_1}^V \hat{u}, \dots, \partial_{x_d}^V \hat{u})$ and $\partial_{\mathbf{x}\mathbf{x}}^V \hat{u} = (\partial_{x_i x_j}^V \hat{u})_{i,j=1, \dots, d}$.

Definition 2.4. We denote by $C^{1,2}(\hat{\Lambda})$ the set of $\hat{u} \in C(\hat{\Lambda})$ such that $\partial_t^H \hat{u}$, $\partial_{\mathbf{x}}^V \hat{u}$, $\partial_{\mathbf{x}\mathbf{x}}^V \hat{u}$ exist everywhere on $\hat{\Lambda}$ and are continuous.

For later use, we also introduce the following set of maps on càdlàg paths.

Definition 2.5. We denote by $C^{0,2}(\hat{\Lambda})$ the set of $\hat{u} \in C(\hat{\Lambda})$ such that $\partial_{\mathbf{x}}^V \hat{u}$, $\partial_{\mathbf{x}\mathbf{x}}^V \hat{u}$ exist everywhere on $\hat{\Lambda}$ and are continuous.

We can finally state the functional Itô formula for maps on càdlàg paths, whose proof is reported in Appendix A.

Theorem 2.1. Let $\hat{u} \in C^{1,2}(\hat{\Lambda})$. Then, for every d -dimensional continuous semimartingale $\mathbf{X} = (\mathbf{X}_t)_{t \in [0, T]}$, where $\mathbf{X} = (X^1, \dots, X^d)$, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, with $(\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions, the following **functional Itô formula** holds:

$$\begin{aligned} \hat{u}(t, \mathbf{X}) &= \hat{u}(0, \mathbf{X}) + \int_0^t \partial_t^H \hat{u}(s, \mathbf{X}) ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{x_i x_j}^V \hat{u}(s, \mathbf{X}) d[X^i, X^j]_s \\ &\quad + \sum_{i=1}^d \int_0^t \partial_{x_i}^V \hat{u}(s, \mathbf{X}) dX_s^i, \end{aligned} \quad \text{for all } 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

Proof. See Appendix A. □

2.2. Maps on continuous paths

Let $C([0, T]; \mathbb{R}^d)$ denote the set of continuous functions $\mathbf{x}: [0, T] \rightarrow \mathbb{R}^d$. Notice that $C([0, T]; \mathbb{R}^d)$ is a subset of $D([0, T]; \mathbb{R}^d)$. We set $\Lambda := [0, T] \times C([0, T]; \mathbb{R}^d)$ and denote \mathbf{d}_∞ the restriction of $\hat{\mathbf{d}}_\infty$ to $\Lambda \times \Lambda$. Then, \mathbf{d}_∞ is a pseudometric on Λ and $(\Lambda, \mathbf{d}_\infty)$ is a complete pseudometric space. We denote by $\mathcal{B}(\Lambda)$ the Borel σ -algebra on Λ induced by \mathbf{d}_∞ .

Definition 2.6. Let $\hat{u}: \hat{\Lambda} \rightarrow \mathbb{R}$ be non-anticipative and consider $u: \Lambda \rightarrow \mathbb{R}$. We say that \hat{u} is **consistent** with u if

$$u(t, \mathbf{x}) = \hat{u}(t, \mathbf{x}),$$

for all $(t, \mathbf{x}) \in \Lambda$.

The following consistency property is crucial as it implies that, given u admitting two maps \hat{u}_1 and \hat{u}_2 , both being consistent with u , their pathwise derivatives coincide on continuous paths (see also Remark 2.2).

Lemma 2.1. *If $\hat{u}_1, \hat{u}_2 \in C^{1,2}(\hat{\Lambda})$ satisfy*

$$\hat{u}_1(t, \mathbf{x}) = \hat{u}_2(t, \mathbf{x}), \quad \forall (t, \mathbf{x}) \in \Lambda,$$

then

$$\begin{aligned} \partial_t^H \hat{u}_1(t, \mathbf{x}) &= \partial_t^H \hat{u}_2(t, \mathbf{x}), \\ \partial_{\mathbf{x}}^V \hat{u}_1(t, \mathbf{x}) &= \partial_{\mathbf{x}}^V \hat{u}_2(t, \mathbf{x}), \\ \partial_{\mathbf{x}\mathbf{x}}^V \hat{u}_1(t, \mathbf{x}) &= \partial_{\mathbf{x}\mathbf{x}}^V \hat{u}_2(t, \mathbf{x}), \end{aligned}$$

for all $(t, \mathbf{x}) \in \Lambda$.

Proof. See Appendix B. □

Thanks to Lemma 2.1 we can now give the following definition (see also Remark 2.2).

Definition 2.7. *Let $u: \Lambda \rightarrow \mathbb{R}$. We say that $u \in C^{1,2}(\Lambda)$ if there exists $\hat{u}: \hat{\Lambda} \rightarrow \mathbb{R}$ consistent with u and satisfying $\hat{u} \in C^{1,2}(\hat{\Lambda})$. Moreover, we define*

$$\begin{aligned} \partial_t^H u(t, \mathbf{x}) &:= \partial_t^H \hat{u}(t, \mathbf{x}), \\ \partial_{\mathbf{x}}^V u(t, \mathbf{x}) &:= \partial_{\mathbf{x}}^V \hat{u}(t, \mathbf{x}), \\ \partial_{\mathbf{x}\mathbf{x}}^V u(t, \mathbf{x}) &:= \partial_{\mathbf{x}\mathbf{x}}^V \hat{u}(t, \mathbf{x}), \end{aligned}$$

for all $(t, \mathbf{x}) \in \Lambda$.

Remark 2.2. Notice that, by Lemma 2.1, if $u \in C^{1,2}(\Lambda)$ then the definition of the pathwise derivatives of u is independent of the map $\hat{u} \in C^{1,2}(\hat{\Lambda})$ consistent with u .

Theorem 2.2. *Let $u \in C^{1,2}(\Lambda)$. Then, for every d -dimensional continuous semimartingale $\mathbf{X} = (\mathbf{X}_t)_{t \in [0, T]}$, where $\mathbf{X} = (X^1, \dots, X^d)$, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, with $(\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions, the following **functional Itô formula** holds:*

$$\begin{aligned} u(t, \mathbf{X}) &= u(0, \mathbf{X}) + \int_0^t \partial_t^H u(s, \mathbf{X}) ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{x_i x_j}^V u(s, \mathbf{X}) d[X^i, X^j]_s \\ &\quad + \sum_{i=1}^d \int_0^t \partial_{x_i}^V u(s, \mathbf{X}) dX_s^i, \end{aligned} \quad \text{for all } 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \quad (2.1)$$

Proof. Since $u \in C^{1,2}(\Lambda)$, by Definition 2.7 there exists a map $\hat{u}: \hat{\Lambda} \rightarrow \mathbb{R}$ consistent with u and satisfying $\hat{u} \in C^{1,2}(\hat{\Lambda})$. Then, by Theorem 2.1, the following functional Itô formula holds:

$$\begin{aligned} \hat{u}(t, \mathbf{X}) &= \hat{u}(0, \mathbf{X}) + \int_0^t \partial_t^H \hat{u}(s, \mathbf{X}) ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{x_i x_j}^V \hat{u}(s, \mathbf{X}) d[X^i, X^j]_s \\ &\quad + \sum_{i=1}^d \int_0^t \partial_{x_i}^V \hat{u}(s, \mathbf{X}) dX_s^i, \end{aligned} \quad \text{for all } 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

The claim follows identifying the pathwise derivatives of \hat{u} with those of u . \square

3. Smooth variational principle on Λ

The goal of the present section is the proof of a smooth variational principle on Λ , which plays a crucial role in the proof of the comparison theorem (Theorem 4.2). To this end, we begin recalling a generalization of the so-called Borwein-Preiss smooth variant ([7]) of Ekeland's variational principle ([32]), corresponding to Theorem 3.1 below. We state it for the case of real-valued (rather than $\mathbb{R} \cup \{+\infty\}$ -valued as in [8]) maps on Λ . We firstly recall the definition of gauge-type function for the specific set Λ .

Definition 3.1. We say that $\Psi: \Lambda \times \Lambda \rightarrow [0, +\infty)$ is a ***gauge-type function*** provided that the properties below hold:

- a) Ψ is continuous on $\Lambda \times \Lambda$;
- b) $\Psi((t, \mathbf{x}), (t, \mathbf{x})) = 0$, for every $(t, \mathbf{x}) \in \Lambda$;
- c) for every $\varepsilon > 0$ there exists $\eta > 0$ such that, for all $(t', \mathbf{x}'), (t'', \mathbf{x}'') \in \Lambda$, the inequality $\Psi((t', \mathbf{x}'), (t'', \mathbf{x}'')) \leq \eta$ implies $d_\infty((t', \mathbf{x}'), (t'', \mathbf{x}'')) < \varepsilon$.

Theorem 3.1. Let $G: \Lambda \rightarrow \mathbb{R}$ be an upper semicontinuous map, bounded from above. Suppose that $\Psi: \Lambda \times \Lambda \rightarrow [0, +\infty)$ is a gauge-type function (according to Definition 3.1) and $\{\delta_n\}_{n \geq 0}$ is a sequence of strictly positive real numbers. For every $\varepsilon > 0$, let $(t_0, \mathbf{x}_0) \in \Lambda$ such that

$$\sup G - \varepsilon \leq G(t_0, \mathbf{x}_0).$$

Then, there exists a sequence $\{(t_n, \mathbf{x}_n)\}_{n \geq 1} \subset \Lambda$ which converges to some $(\bar{t}, \bar{\mathbf{x}}) \in \Lambda$ satisfying the following properties.

- i) $\Psi((\bar{t}, \bar{\mathbf{x}}), (t_n, \mathbf{x}_n)) \leq \frac{\varepsilon}{2^n \delta_0}$, for every $n \geq 0$.
- ii) $G(t_0, \mathbf{x}_0) \leq G(\bar{t}, \bar{\mathbf{x}}) - \sum_{n=0}^{+\infty} \delta_n \Psi((\bar{t}, \bar{\mathbf{x}}), (t_n, \mathbf{x}_n))$.
- iii) For every $(t, \mathbf{x}) \neq (\bar{t}, \bar{\mathbf{x}})$,

$$G(t, \mathbf{x}) - \sum_{n=0}^{+\infty} \delta_n \Psi((t, \mathbf{x}), (t_n, \mathbf{x}_n)) < G(\bar{t}, \bar{\mathbf{x}}) - \sum_{n=0}^{+\infty} \delta_n \Psi((\bar{t}, \bar{\mathbf{x}}), (t_n, \mathbf{x}_n)).$$

Proof. Theorem 3.1 follows trivially from Theorem 2.5.2 in [8], the only difference being that the latter result is stated on complete *metric* spaces, while here Λ is a complete *pseudometric* space. \square

The main ingredient of Theorem 3.1 is the gauge-type function Ψ . In the proof of the comparison theorem we need such a gauge-type function to be also smooth as a map of its first pair, namely $(t, \mathbf{x}) \mapsto \Psi((t, \mathbf{x}), (t_0, \mathbf{x}_0))$, and with bounded derivatives. The most important example of gauge-type function is the pseudometric d_∞ itself, which unfortunately is not smooth enough. The major contribution of the present section is the construction of such a smooth gauge-type function with bounded derivatives, which corresponds to the function ρ_∞ in (3.8). In order to do it, we perform a smoothing of the pseudometric d_∞ itself (more precisely of the part concerning the supremum norm), first in the vertical direction, and then in the horizontal direction. In particular, the next result concerns the smoothing in the vertical direction. The precise form of the mollifier ζ in (3.1) is used to get explicit bounds on $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ and its derivatives.

Lemma 3.1. *Let $\zeta: \mathbb{R}^d \rightarrow \mathbb{R}$ be the probability density function of the standard normal multivariate distribution*

$$\zeta(\mathbf{z}) := \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{1}{2}|\mathbf{z}|^2}, \quad \forall \mathbf{z} \in \mathbb{R}^d. \quad (3.1)$$

For every fixed $(t_0, \mathbf{x}_0) \in \Lambda$, define the map $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}: \hat{\Lambda} \rightarrow \mathbb{R}$ as

$$\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) := \int_{\mathbb{R}^d} \|\hat{\mathbf{x}}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t, T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z}, \quad (3.2)$$

for all $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$. Moreover, let $\kappa_\infty^{(t_0, \mathbf{x}_0)}: \Lambda \rightarrow \mathbb{R}$ be given by

$$\kappa_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}) := \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}),$$

for every $(t, \mathbf{x}) \in \Lambda$. Then, the following properties hold.

- 1) For every $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$, the vertical derivatives of first and second-order of $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ at $(t, \hat{\mathbf{x}})$ (namely $\partial_{x_i}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$ and $\partial_{x_i x_j}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$, for every $i, j = 1, \dots, d$) exist.
- 2) For every $i, j = 1, \dots, d$, $\partial_{x_i}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ is bounded by the constant 1 and $\partial_{x_i x_j}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ is bounded by the constant $\sqrt{\frac{2}{\pi}}$.
- 3) $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)} \geq -C_\zeta$ and $\kappa_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}) \geq \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty - C_\zeta$, for every $(t, \mathbf{x}) \in \Lambda$, with

$$C_\zeta := \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} = \sqrt{2} \frac{\Gamma(\frac{d}{2} + \frac{1}{2})}{\Gamma(\frac{d}{2})} > 0, \quad (3.3)$$

where $\Gamma(\cdot)$ is the Gamma function.

4) For every fixed d , there exists some constant $\alpha_d > 0$ such that

$$\begin{aligned} \alpha_d \left(\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty^{d+1} \wedge \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty \right) \\ \leq \kappa_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}) \leq \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty, \end{aligned} \quad (3.4)$$

for all $(t, \mathbf{x}) \in \mathbf{\Lambda}$. In particular, it holds that $\kappa_\infty^{(t_0, \mathbf{x}_0)} \geq 0$.

Proof. See Appendix C, Section C.1. \square

We now address the problem of smoothing the map $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ in the horizontal direction. This is required by the fact that the presence of $1_{[t, T]}$ in the definition of $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ is an obstruction to horizontal regularity, therefore a further convolution in the time variable t is needed. The latter convolution also provides the continuity on $\hat{\mathbf{\Lambda}}$ (notice that the map $(t, \hat{\mathbf{x}}) \mapsto \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$ is not continuous on $\hat{\mathbf{\Lambda}}$, see Remark 3.1).

We perform such a horizontal smoothing to $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)} / (1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)})$. We apply it to such a map (rather than to $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ directly) in order to have bounded derivatives (see item 3 of Lemma 3.2). Moreover, we consider $1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ (instead of $1 + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$) in order to have a denominator greater than or equal to 1 (this follows from inequality $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)} \geq -C_\zeta$, see item 3 of Lemma 3.1). The precise form of the mollifier η in (3.5) is used to get explicit bounds on $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ and its derivatives.

Remark 3.1 ([48]). Notice that the map $(t, \hat{\mathbf{x}}) \mapsto \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$ is not continuous on $\hat{\mathbf{\Lambda}}$. As a matter of fact, consider the following example. Take $d = 1$, $T = 2$, $t_0 = 0$, $\mathbf{x}_0 \equiv 0$, $t = 1$, $\hat{\mathbf{x}} = 1_{[1, 2]}$. Then, it holds that

$$\begin{aligned} \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) &= \int_{\mathbb{R}} \|\hat{\mathbf{x}}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t, T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} \\ &= \int_{\mathbb{R}} |1 - \mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z}. \end{aligned}$$

Now, take $\delta \in (0, 1)$, then

$$\begin{aligned} \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t + \delta, \hat{\mathbf{x}}) &= \int_{\mathbb{R}} \|\hat{\mathbf{x}}(\cdot \wedge (t + \delta)) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t + \delta, T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} \\ &= \int_{\mathbb{R}} \max\{1, |1 - \mathbf{z}|\} \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z}. \end{aligned}$$

In conclusion, we have

$$\begin{aligned} |\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t + \delta, \hat{\mathbf{x}}) - \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})| &= \int_{\mathbb{R}} \left\{ \max\{1, |1 - \mathbf{z}|\} - |1 - \mathbf{z}| \right\} \zeta(\mathbf{z}) d\mathbf{z} \\ &= \int_0^2 (1 - |1 - \mathbf{z}|) \zeta(\mathbf{z}) d\mathbf{z} =: \varepsilon_* > 0, \end{aligned}$$

where ε_* is a constant independent of δ . This proves that $|\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t+\delta, \hat{\mathbf{x}}) - \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})| \not\rightarrow 0$ as $\delta \rightarrow 0^+$ and shows that $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ is not continuous on $\hat{\Lambda}$.

Lemma 3.2. *Let $\eta: \mathbb{R} \rightarrow \mathbb{R}$ be given by*

$$\eta(s) := s e^{-s}, \quad \forall s \in \mathbb{R}. \quad (3.5)$$

For every fixed $(t_0, \mathbf{x}_0) \in \Lambda$, let $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ be the map defined in Lemma 3.1 and define the map $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}: \hat{\Lambda} \rightarrow \mathbb{R}$ as

$$\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) := \int_0^{+\infty} \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))} \eta(s) ds,$$

for all $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$, with C_ζ as in (3.3), where we recall that $1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)} \geq 1$ (see item 3 of Lemma 3.1). Moreover, let $\chi_\infty^{(t_0, \mathbf{x}_0)}: \Lambda \rightarrow \mathbb{R}$ be given by

$$\chi_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}) := \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}), \quad \forall (t, \mathbf{x}) \in \Lambda. \quad (3.6)$$

Then, the following properties hold.

- 1) *For every $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$, the horizontal and vertical derivatives of first and second-order of $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ at $(t, \hat{\mathbf{x}})$ (namely $\partial_t^H \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$, $\partial_{x_i}^V \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$ and $\partial_{x_i x_j}^V \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$, for every $i, j = 1, \dots, d$) exist.*
- 2) *$\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)} \in C^{1,2}(\hat{\Lambda})$ and the map $((t_0, \mathbf{x}_0), (t, \hat{\mathbf{x}})) \mapsto \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$ is continuous on $\Lambda \times \hat{\Lambda}$.*
- 3) *The horizontal derivative of $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ is bounded by the constant $\frac{2}{e}$; the first-order vertical derivatives of $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ are bounded by the constant $1 + C_\zeta$; the second-order vertical derivatives of $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ are bounded by the constant $(1 + C_\zeta)(\sqrt{\frac{2}{\pi}} + 2)$.*
- 4) *For every $(t, \mathbf{x}) \in \Lambda$,*

$$\begin{aligned} \alpha_d \frac{\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty^{d+1} \wedge \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty}{1 + C_\zeta + \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty} \\ \leq \chi_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}) \leq \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty \wedge 1, \end{aligned} \quad (3.7)$$

with the same constant α_d as in (3.4). In particular, it holds that $\chi_\infty^{(t_0, \mathbf{x}_0)} \geq 0$.

Proof. See Appendix C, Section C.2. □

In conclusion, by Lemma 3.2 it follows that the map $\rho_\infty: \Lambda \times \Lambda \rightarrow [0, +\infty)$ given by

$$\rho_\infty((t, \mathbf{x}), (t_0, \mathbf{x}_0)) = |t - t_0|^2 + \chi_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}), \quad \forall (t, \mathbf{x}), (t_0, \mathbf{x}_0) \in \Lambda, \quad (3.8)$$

with χ_∞ as in (3.6), is a gauge-type function, which is also smooth as a map of the first pair, namely $(t, \mathbf{x}) \mapsto \rho_\infty((t, \mathbf{x}), (t_0, \mathbf{x}_0))$, and with bounded derivatives.

We now apply Theorem 3.1 to the smooth gauge-type function ρ_∞ with bounded derivatives defined by (3.8), taking $\delta_0 := \delta > 0$ and $\delta_n := \delta/2^n$, for every $n \geq 1$.

Theorem 3.2 (Smooth variational principle on Λ). *Let $\delta > 0$ and $G: \Lambda \rightarrow \mathbb{R}$ be an upper semicontinuous map, bounded from above. For every $\varepsilon > 0$, let $(t_0, \mathbf{x}_0) \in \Lambda$ satisfy*

$$\sup G - \varepsilon \leq G(t_0, \mathbf{x}_0).$$

Then, there exists a sequence $\{(t_n, \mathbf{x}_n)\}_{n \geq 1} \subset \Lambda$ which converges to some $(\bar{t}, \bar{\mathbf{x}}) \in \Lambda$ fulfilling the properties below.

- i) $\rho_\infty((\bar{t}, \bar{\mathbf{x}}), (t_n, \mathbf{x}_n)) \leq \frac{\varepsilon}{2^n \delta}$, for every $n \geq 0$.
- ii) $G(t_0, \mathbf{x}_0) \leq G(\bar{t}, \bar{\mathbf{x}}) - \delta \varphi_\varepsilon(t, \mathbf{x})$, where the map $\varphi_\varepsilon: \Lambda \rightarrow [0, +\infty)$ is defined as

$$\varphi_\varepsilon(t, \mathbf{x}) := \sum_{n=0}^{+\infty} \frac{1}{2^n} \rho_\infty((t, \mathbf{x}), (t_n, \mathbf{x}_n)), \quad \forall (t, \mathbf{x}) \in \Lambda.$$

- iii) For every $(t, \mathbf{x}) \neq (\bar{t}, \bar{\mathbf{x}})$, $G(t, \mathbf{x}) - \delta \varphi_\varepsilon(t, \mathbf{x}) < G(\bar{t}, \bar{\mathbf{x}}) - \delta \varphi_\varepsilon(\bar{t}, \bar{\mathbf{x}})$.

Finally, the map φ_ε satisfies the following properties.

- 1) $\varphi_\varepsilon \in C^{1,2}(\Lambda)$ and is bounded.
- 2) $\partial_t^H \varphi_\varepsilon$ is bounded by the constant $2(2T + \frac{2}{\varepsilon})$.
- 3) For every $i, j = 1, \dots, d$, $\partial_{x_i}^V \varphi_\varepsilon$ is bounded by the constant $2(1 + C_\zeta)$ and $\partial_{x_i x_j}^V \varphi_\varepsilon$ is bounded by the constant $2(1 + C_\zeta) \left(\sqrt{\frac{2}{\pi}} + 2 \right)$.

Proof. Items i)-ii)-iii) follow directly from Theorem 3.1, while items 1)-2)-3) follow easily from items 2)-3)-4) of Lemma 3.2. \square

4. Crandall-Lions (path-dependent) viscosity solutions

4.1. Viscosity solutions

In the present section we consider the second-order path-dependent partial differential equation

$$\begin{cases} \partial_t^H u(t, \mathbf{x}) = F(t, \mathbf{x}, u(t, \mathbf{x}), \partial_{\mathbf{x}}^V u(t, \mathbf{x}), \partial_{\mathbf{x}\mathbf{x}}^V u(t, \mathbf{x})), & (t, \mathbf{x}) \in [0, T] \times C([0, T]; \mathbb{R}^d), \\ u(T, \mathbf{x}) = \xi(\mathbf{x}), & \mathbf{x} \in C([0, T]; \mathbb{R}^d), \end{cases} \quad (4.1)$$

with $F: [0, T] \times C([0, T]; \mathbb{R}^d) \times \mathbb{R}^d \times \mathcal{S}(d) \rightarrow \mathbb{R}$ and $\xi: C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}$, where $\mathcal{S}(d)$ is the set of symmetric $d \times d$ matrices.

Definition 4.1. We denote by $C_{\text{pol}}^{1,2}(\Lambda)$ the set of $\varphi \in C^{1,2}(\Lambda)$ such that φ , $\partial_t^H \varphi$, $\partial_{\mathbf{x}}^V \varphi$, $\partial_{\mathbf{x}\mathbf{x}}^V \varphi$ satisfy a polynomial growth condition.

Definition 4.2. We say that an upper semicontinuous map $u: \mathbf{\Lambda} \rightarrow \mathbb{R}$ is a **(path-dependent) viscosity subsolution** of equation (4.1) if the following holds.

- $u(T, \mathbf{x}) \leq \xi(\mathbf{x})$, for all $\mathbf{x} \in C([0, T]; \mathbb{R}^d)$;
- for any $(t, \mathbf{x}) \in [0, T) \times C([0, T]; \mathbb{R}^d)$ and $\varphi \in C_{\text{pol}}^{1,2}(\mathbf{\Lambda})$, satisfying

$$(u - \varphi)(t, \mathbf{x}) = \sup_{(t', \mathbf{x}') \in \mathbf{\Lambda}} (u - \varphi)(t', \mathbf{x}'),$$

we have

$$-\partial_t^H \varphi(t, \mathbf{x}) + F(t, \mathbf{x}, u(t, \mathbf{x}), \partial_{\mathbf{x}}^V \varphi(t, \mathbf{x}), \partial_{\mathbf{x}\mathbf{x}}^V \varphi(t, \mathbf{x})) \leq 0.$$

We say that a lower semicontinuous map $u: \mathbf{\Lambda} \rightarrow \mathbb{R}$ is a **(path-dependent) viscosity supersolution** of equation (4.1) if:

- $u(T, \mathbf{x}) \geq \xi(\mathbf{x})$, for all $\mathbf{x} \in C([0, T]; \mathbb{R}^d)$;
- for any $(t, \mathbf{x}) \in [0, T) \times C([0, T]; \mathbb{R}^d)$ and $\varphi \in C_{\text{pol}}^{1,2}(\mathbf{\Lambda})$, satisfying:

$$(u - \varphi)(t, \mathbf{x}) = \inf_{(t', \mathbf{x}') \in \mathbf{\Lambda}} (u - \varphi)(t', \mathbf{x}'),$$

we have

$$-\partial_t^H \varphi(t, \mathbf{x}) + F(t, \mathbf{x}, u(t, \mathbf{x}), \partial_{\mathbf{x}}^V \varphi(t, \mathbf{x}), \partial_{\mathbf{x}\mathbf{x}}^V \varphi(t, \mathbf{x})) \geq 0.$$

We say that a continuous map $u: \mathbf{\Lambda} \rightarrow \mathbb{R}$ is a **(path-dependent) viscosity solution** of equation (4.1) if u is both a (path-dependent) viscosity subsolution and a (path-dependent) viscosity supersolution of (4.1).

4.2. Path-dependent heat equation

In the present section we focus on the path-dependent heat equation, namely when $F(t, \mathbf{x}, r, p, M) = -\frac{1}{2}\text{tr}[M]$

$$\begin{cases} \partial_t^H u(t, \mathbf{x}) + \frac{1}{2}\text{tr}[\partial_{\mathbf{x}\mathbf{x}}^V u(t, \mathbf{x})] = 0, & (t, \mathbf{x}) \in [0, T) \times C([0, T]; \mathbb{R}^d), \\ u(T, \mathbf{x}) = \xi(\mathbf{x}), & \mathbf{x} \in C([0, T]; \mathbb{R}^d). \end{cases} \quad (4.2)$$

In the sequel we denote

$$\mathcal{L}u(t, \mathbf{x}) := \partial_t^H u(t, \mathbf{x}) + \frac{1}{2}\text{tr}[\partial_{\mathbf{x}\mathbf{x}}^V u(t, \mathbf{x})]. \quad (4.3)$$

On the terminal condition ξ , we impose the assumption

(A) The function $\xi: C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}$ is continuous and bounded.

4.2.1. Existence

The “candidate solution” to equation (4.2) is

$$v(t, \mathbf{x}) := \mathbb{E}[\xi(\mathbf{W}^{t, \mathbf{x}})], \quad \text{for all } (t, \mathbf{x}) \in \mathbf{\Lambda}, \quad (4.4)$$

where $\mathbf{W} = (\mathbf{W}_s)_{s \in [0, T]}$ is a d -dimensional Brownian motion on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the stochastic process $\mathbf{W}^{t, \mathbf{x}} = (\mathbf{W}_s^{t, \mathbf{x}})_{s \in [0, T]}$ is given by

$$\mathbf{W}_s^{t, \mathbf{x}} := \begin{cases} \mathbf{x}(s), & s \leq t, \\ \mathbf{x}(t) + \mathbf{W}_s - \mathbf{W}_t, & s > t. \end{cases} \quad (4.5)$$

Remark 4.1. The boundedness of ξ in Assumption (A) will be used in the proof of (the comparison) Theorem 4.2. On the other hand, the proof that the function v in (4.4) is continuous and is a viscosity solution of equation (4.2) (see the proof of Theorem 4.1) holds under weaker growth condition on ξ (for instance, ξ having polynomial growth).

Theorem 4.1. *Under Assumption (A), the function v in (4.4) is continuous and bounded. Moreover, v is a (path-dependent) viscosity solution of equation (4.2).*

Proof. STEP I. *Continuity of v .* Given $(t, \mathbf{x}), (t', \mathbf{x}') \in \mathbf{\Lambda}$, with $t \leq t'$, from (4.5) we have

$$\mathbf{W}_s^{t, \mathbf{x}} - \mathbf{W}_s^{t', \mathbf{x}'} = \begin{cases} \mathbf{x}(s) - \mathbf{x}'(s), & s \leq t, \\ \mathbf{x}(t) - \mathbf{x}'(s) + \mathbf{W}_s - \mathbf{W}_t, & t < s \leq t', \\ \mathbf{x}(t) - \mathbf{x}'(t') + \mathbf{W}_{t'} - \mathbf{W}_t, & s > t'. \end{cases}$$

Hence

$$\begin{aligned} \sup_{s \in [0, T]} |\mathbf{W}_s^{t, \mathbf{x}} - \mathbf{W}_s^{t', \mathbf{x}'}| &\leq \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}'(\cdot \wedge t')\|_\infty + \sup_{s \in [t, t']} |\mathbf{W}_s - \mathbf{W}_t| \\ &\leq \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}'(\cdot \wedge t')\|_\infty + \sum_{i=1}^d \sup_{s \in [t, t']} |W_s^i - W_t^i|, \end{aligned}$$

where $\mathbf{W} = (W^1, \dots, W^d)$ and the second inequality follows from the fact the Euclidean norm on \mathbb{R}^d is estimated by the 1-norm. By the reflection principle, $\sup_{s \in [t, t']} |W_s^i - W_t^i|$ has the same law as $|W_{t'}^i - W_t^i|$, therefore

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, T]} |\mathbf{W}_s^{t, \mathbf{x}} - \mathbf{W}_s^{t', \mathbf{x}'}| \right] &\leq \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}'(\cdot \wedge t')\|_\infty + \sum_{i=1}^d \mathbb{E}[|W_{t'}^i - W_t^i|] \\ &= \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}'(\cdot \wedge t')\|_\infty + d \sqrt{\frac{2}{\pi}} \sqrt{|t - t'|}. \end{aligned}$$

Then, since ξ is bounded and continuous, the continuity of v follows from the above estimate together with the Lebesgue dominated convergence theorem.

STEP II. v is a viscosity solution of equation (4.2). For every $t \in [0, T]$, let $\mathbb{F}^t = (\mathcal{F}_s)_{s \in [t, T]}$ be the completion of the filtration generated by $(\mathbf{W}_s - \mathbf{W}_t)_{s \in [t, T]}$. Now, fix $(t, \mathbf{x}) \in \mathbf{\Lambda}$ and $t' \in [t, T]$. We first prove that

$$v(t, \mathbf{x}) = \mathbb{E}[v(t', \mathbf{W}^{t, \mathbf{x}})]. \quad (4.6)$$

To this end, we begin noticing that by (4.5) we have

$$\mathbf{W}_{\cdot \wedge t}^{t, \mathbf{x}} = \mathbf{x}(\cdot \wedge t) + \mathbf{W}_{\cdot \vee t} - \mathbf{W}_t. \quad (4.7)$$

Therefore

$$v(t, \mathbf{x}) = \mathbb{E}[\xi(\mathbf{x}(\cdot \wedge t) + \mathbf{W}_{\cdot \vee t} - \mathbf{W}_t)]. \quad (4.8)$$

Now, notice that, by (4.7),

$$\mathbf{W}_{\cdot \wedge t'}^{t', \mathbf{W}^{t, \mathbf{x}}} = \mathbf{W}_{\cdot \wedge t'}^{t, \mathbf{x}} + \mathbf{W}_{\cdot \vee t'} - \mathbf{W}_{t'} = \mathbf{W}_{\cdot \wedge t'}^{t, \mathbf{x}}.$$

This proves the flow property $\mathbf{W}_{\cdot \wedge t'}^{t, \mathbf{x}} = \mathbf{W}_{\cdot \wedge t'}^{t', \mathbf{W}^{t, \mathbf{x}}}$. Then, by the freezing lemma for conditional expectation and formula (4.8), we obtain

$$\begin{aligned} v(t, \mathbf{x}) &= \mathbb{E}[\xi(\mathbf{W}^{t, \mathbf{x}})] = \mathbb{E}[\xi(\mathbf{W}^{t', \mathbf{W}^{t, \mathbf{x}}})] = \mathbb{E}[\xi(\mathbf{W}_{\cdot \wedge t'}^{t, \mathbf{x}} + \mathbf{W}_{\cdot \vee t'} - \mathbf{W}_{t'})] \\ &= \mathbb{E}[\mathbb{E}[\xi(\mathbf{W}_{\cdot \wedge t'}^{t, \mathbf{x}} + \mathbf{W}_{\cdot \vee t'} - \mathbf{W}_{t'}) | \mathcal{F}_{t'}^t]] = \mathbb{E}[v(t', \mathbf{W}_{\cdot \wedge t'}^{t, \mathbf{x}})]. \end{aligned}$$

Finally, recalling that v is non-anticipative we deduce that $v(t', \mathbf{W}_{\cdot \wedge t'}^{t, \mathbf{x}}) = v(t', \mathbf{W}^{t, \mathbf{x}})$, which concludes the proof of formula (4.6).

Let us now prove that v is a viscosity solution of equation (4.2). We only prove the viscosity subsolution property, as the supersolution property can be proved in a similar way. We proceed along the same lines as in the proof of the subsolution property in Theorem 3.66 of [38]. Let $(t, \mathbf{x}) \in [0, T] \times C([0, T]; \mathbb{R}^d)$ and $\varphi \in C_{\text{pol}}^{1,2}(\mathbf{\Lambda})$, satisfying:

$$(v - \varphi)(t, \mathbf{x}) = \sup_{(t', \mathbf{x}') \in \mathbf{\Lambda}} (v - \varphi)(t', \mathbf{x}').$$

We suppose that $(v - \varphi)(t, \mathbf{x}) = 0$ (if this is not the case, we replace φ by $\psi(\cdot, \cdot) := \varphi(\cdot, \cdot) + v(t, \mathbf{x}) - \varphi(t, \mathbf{x})$). Take

$$\varphi(t, \mathbf{x}) = v(t, \mathbf{x}) = \mathbb{E}[v(t + \varepsilon, \mathbf{W}^{t, \mathbf{x}})] \leq \mathbb{E}[\varphi(t + \varepsilon, \mathbf{W}^{t, \mathbf{x}})], \quad (4.9)$$

where the latter inequality follows from the fact that $\sup(v - \varphi) = 0$, so that $v \leq \varphi$ on $\mathbf{\Lambda}$. Notice that the last expectation in (4.9) is finite, as φ has polynomial growth. Now, by the functional Itô formula (2.1), we have

$$\varphi(t + \varepsilon, \mathbf{W}^{t, \mathbf{x}}) = \varphi(t, \mathbf{x}) + \int_t^{t+\varepsilon} \mathcal{L}\varphi(s, \mathbf{W}^{t, \mathbf{x}}) ds + \sum_{i=1}^d \int_t^{t+\varepsilon} \partial_{x_i}^V \varphi(s, \mathbf{W}^{t, \mathbf{x}}) dW_s^i,$$

where \mathcal{L} was defined in (4.3). Since $\partial_{x_i}^V \varphi$ has polynomial growth, the corresponding stochastic integral is a martingale. Then, plugging the above formula into (4.9) and dividing by ε , we find

$$-\mathbb{E} \left[\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathcal{L}\varphi(s, \mathbf{W}^{t,\mathbf{x}}) ds \right] \leq 0.$$

Letting $\varepsilon \rightarrow 0^+$, we conclude that

$$-\mathcal{L}\varphi(t, \mathbf{x}) \leq 0,$$

which proves the viscosity subsolution property. \square

4.2.2. Comparison theorem and uniqueness

Theorem 4.2. *Suppose that Assumption (A) holds. Let $u, w: \mathbf{\Lambda} \rightarrow \mathbb{R}$ be respectively upper and lower semicontinuous, satisfying*

$$\sup u < +\infty, \quad \inf w > -\infty.$$

Suppose that u (resp. w) is a (path-dependent) viscosity subsolution (resp. supersolution) of equation (4.2). Then $u \leq w$ on $\mathbf{\Lambda}$.

Proof. The proof consists in showing that $u \leq v$ and $v \leq w$ on $\mathbf{\Lambda}$ (with v given by (4.4)), from which we immediately deduce the claim. In what follows, we only report the proof of the inequality $u \leq v$, as the other inequality (that is $v \leq w$) can be deduced from the first one replacing u, v, ξ with $-w, -v, -\xi$, respectively.

We proceed by contradiction and assume that $\sup(u - v) > 0$. Then, there exists $(t_0, \mathbf{x}_0) \in \mathbf{\Lambda}$ such that

$$(u - v)(t_0, \mathbf{x}_0) > 0.$$

Notice that $t_0 < T$, since $u(T, \cdot) \leq \xi(\cdot) = v(T, \cdot)$. We split the rest of the proof into five steps.

STEP I. Let $\{\xi_N\}_N$ be the sequence given by Lemma D.2. Since ξ is bounded, we have that ξ_N is bounded uniformly with respect to N . Now, denote

$$v_N(t, \mathbf{x}) := \mathbb{E}[\xi_N(\mathbf{W}^{t,\mathbf{x}})], \quad \text{for all } (t, \mathbf{x}) \in \mathbf{\Lambda}.$$

Then, v_N is bounded uniformly with respect to N . Moreover, by Lemma D.1 it follows that, for every N , $v_N \in C^{1,2}(\mathbf{\Lambda})$ and is a classical (smooth) solution of equation (4.2) with terminal condition ξ_N . Finally, recalling from Lemma D.2 that $\{\xi_N\}_N$ converges pointwise to ξ as $N \rightarrow +\infty$, it follows from the Lebesgue dominated convergence theorem that $\{v_N\}_N$ converges pointwise to v as $N \rightarrow +\infty$. Then, we notice that there exists $N_0 \in \mathbb{N}$ such that

$$(u - v_{N_0})(t_0, \mathbf{x}_0) > 0. \tag{4.10}$$

We also suppose that (possibly enlarging N_0)

$$|\xi(\mathbf{x}_0) - \xi_{N_0}(\mathbf{x}_0)| \leq \frac{1}{2}(u - v_{N_0})(t_0, \mathbf{x}_0). \tag{4.11}$$

STEP II. For every $\lambda > 0$, we set

$$u^\lambda(t, \mathbf{x}) := e^{\lambda t} u(t, \mathbf{x}), \quad \xi^\lambda(\mathbf{x}) := e^{\lambda T} \xi(\mathbf{x}), \quad v_{N_0}^\lambda(t, \mathbf{x}) := e^{\lambda t} v_{N_0}(t, \mathbf{x}), \quad \xi_{N_0}^\lambda(\mathbf{x}) := e^{\lambda T} \xi_{N_0}(\mathbf{x}).$$

for all $(t, \mathbf{x}) \in \mathbf{\Lambda}$. Notice that u^λ is a (path-dependent) viscosity subsolution of the path-dependent partial differential equation

$$\begin{cases} \partial_t^H u^\lambda(t, \mathbf{x}) + \frac{1}{2} \text{tr}[\partial_{\mathbf{x}}^V u^\lambda(t, \mathbf{x})] = \lambda u^\lambda(t, \mathbf{x}), & (t, \mathbf{x}) \in [0, T) \times C([0, T]; \mathbb{R}^d), \\ u^\lambda(T, \mathbf{x}) = \xi^\lambda(\mathbf{x}), & \mathbf{x} \in C([0, T]; \mathbb{R}^d). \end{cases} \quad (4.12)$$

Similarly, $v_{N_0}^\lambda$ is a classical (smooth) solution of equation (4.12) with ξ^λ replaced by $\xi_{N_0}^\lambda$. We finally notice that by (4.10) we have

$$(u^\lambda - v_{N_0}^\lambda)(t_0, \mathbf{x}_0) > 0.$$

So, in particular,

$$\sup(u^\lambda - v_{N_0}^\lambda) - \varepsilon = (u^\lambda - v_{N_0}^\lambda)(t_0, \mathbf{x}_0) \leq \sup(u^\lambda - v_{N_0}^\lambda), \quad (4.13)$$

where $\varepsilon := \sup(u^\lambda - v_{N_0}^\lambda) - (u^\lambda - v_{N_0}^\lambda)(t_0, \mathbf{x}_0)$.

STEP III. Notice that $u^\lambda - v_{N_0}^\lambda$ is upper semicontinuous and bounded from above. Then, by (4.13) and the smooth variational principle (Theorem 3.2) with $G = u^\lambda - v_{N_0}^\lambda$, we deduce that for every $\delta > 0$ there exists a sequence $\{(t_n, \mathbf{x}_n)\}_{n \geq 1} \subset \mathbf{\Lambda}$ converging to some $(\bar{t}, \bar{\mathbf{x}}) \in \mathbf{\Lambda}$ (possibly depending on $\varepsilon, \delta, \lambda, N_0$) such that the following holds.

- i) $\rho_\infty((t_n, \mathbf{x}_n), (\bar{t}, \bar{\mathbf{x}})) \leq \frac{\varepsilon}{2^n \delta}$, for every $n \geq 0$, where ρ_∞ is the smooth gauge-type function with bounded derivatives defined by (3.8).
- ii) $(u^\lambda - v_{N_0}^\lambda)(t_0, \mathbf{x}_0) \leq (u^\lambda - (v_{N_0}^\lambda + \delta \varphi_\varepsilon))(\bar{t}, \bar{\mathbf{x}})$, where

$$\varphi_\varepsilon(t, \mathbf{x}) := \sum_{n=0}^{+\infty} \frac{1}{2^n} \rho_\infty((t, \mathbf{x}), (t_n, \mathbf{x}_n)) \quad \forall (t, \mathbf{x}) \in \mathbf{\Lambda}.$$

iii) It holds that

$$(u^\lambda - (v_{N_0}^\lambda + \delta \varphi_\varepsilon))(\bar{t}, \bar{\mathbf{x}}) = \sup_{(t, \mathbf{x}) \in \mathbf{\Lambda}} (u^\lambda - (v_{N_0}^\lambda + \delta \varphi_\varepsilon))(t, \mathbf{x}). \quad (4.14)$$

We also recall from Theorem 3.2 that φ_ε satisfies the following properties.

- 1) $\varphi_\varepsilon \in C^{1,2}(\mathbf{\Lambda})$ and is bounded.
- 2) $|\partial_t^H \varphi_\varepsilon(t, \mathbf{x})| \leq 2(2T + \frac{2}{\varepsilon})$, for every $(t, \mathbf{x}) \in [0, T) \times C([0, T]; \mathbb{R}^d)$.
- 3) For every $i, j = 1, \dots, d$, $\partial_{x_i}^V \varphi_\varepsilon$ is bounded by the constant $2(1 + C_\zeta)$ and $\partial_{x_i x_j}^V \varphi_\varepsilon$ is bounded by the constant $2(1 + C_\zeta) \left(\sqrt{\frac{2}{\pi}} + 2 \right)$.

In particular, $\varphi_\varepsilon \in \mathbf{C}_{\text{pol}}^{1,2}(\mathbf{\Lambda})$.

STEP IV. We prove below that $\bar{t} < T$. As a matter of fact, by item ii) of STEP III we have

$$(u^\lambda - (v_{N_0}^\lambda + \delta\varphi_\varepsilon))(\bar{t}, \bar{\mathbf{x}}) \geq (u^\lambda - v_{N_0}^\lambda)(t_0, \mathbf{x}_0). \quad (4.15)$$

On the other hand, if $\bar{t} = T$ we obtain

$$(u^\lambda - (v_{N_0}^\lambda + \delta\varphi_\varepsilon))(\bar{t}, \bar{\mathbf{x}}) = e^{\lambda T}(\xi(\bar{\mathbf{x}}) - \xi_{N_0}(\bar{\mathbf{x}})) - \delta\varphi_\varepsilon(T, \bar{\mathbf{x}}) \leq e^{\lambda T}(\xi(\bar{\mathbf{x}}) - \xi_{N_0}(\bar{\mathbf{x}})), \quad (4.16)$$

where the latter inequality comes from the fact that $\varphi_\varepsilon \geq 0$. Hence, by (4.15) and (4.16) we get

$$e^{\lambda t_0}(u - v_{N_0})(t_0, \mathbf{x}_0) \leq e^{\lambda T}(\xi(\bar{\mathbf{x}}) - \xi_{N_0}(\bar{\mathbf{x}})).$$

Letting $\varepsilon \rightarrow 0$, it follows from item i) above with $n = 0$ and (3.7) that $d_\infty((\bar{t}, \bar{\mathbf{x}}), (t_0, \mathbf{x}_0)) \rightarrow 0$. Therefore, letting $\varepsilon \rightarrow 0$ in the previous inequality, we obtain

$$e^{\lambda t_0}(u - v_{N_0})(t_0, \mathbf{x}_0) \leq e^{\lambda T}(\xi(\mathbf{x}_0) - \xi_{N_0}(\mathbf{x}_0)).$$

By (4.11), we end up with $e^{\lambda t_0} \leq \frac{1}{2} e^{\lambda T}$. Letting $\lambda \rightarrow 0$, we find a contradiction.

STEP V. Here again $\lambda > 0$ is fixed. By (4.14) and the definition of viscosity subsolution of (4.12) applied to u^λ at the point $(\bar{t}, \bar{\mathbf{x}})$ with test function $v_{N_0}^\lambda + \delta\varphi_\varepsilon$, we obtain

$$-\mathcal{L}(v_{N_0}^\lambda + \delta\varphi_\varepsilon)(\bar{t}, \bar{\mathbf{x}}) + \lambda u^\lambda(\bar{t}, \bar{\mathbf{x}}) \leq 0.$$

Recalling that $v_{N_0}^\lambda$ is a classical (smooth) solution of equation (4.12) with ξ^λ replaced by $\xi_{N_0}^\lambda$, we find

$$\lambda(u^\lambda - v_{N_0}^\lambda)(\bar{t}, \bar{\mathbf{x}}) \leq \delta \mathcal{L}\varphi_\varepsilon(\bar{t}, \bar{\mathbf{x}}).$$

By item ii) in STEP III (namely (4.15)), subtracting from both sides the quantity $\lambda\delta\varphi_\varepsilon(\bar{t}, \bar{\mathbf{x}})$, we obtain

$$\lambda(u^\lambda - v_{N_0}^\lambda)(t_0, \mathbf{x}_0) \leq \lambda(u^\lambda - (v_{N_0}^\lambda + \delta\varphi_\varepsilon))(\bar{t}, \bar{\mathbf{x}}) \leq \delta \mathcal{L}\varphi_\varepsilon(\bar{t}, \bar{\mathbf{x}}) - \lambda\delta\varphi_\varepsilon(\bar{t}, \bar{\mathbf{x}}).$$

Recalling that $\varphi_\varepsilon \geq 0$, we see that

$$\lambda(u^\lambda - v_{N_0}^\lambda)(t_0, \mathbf{x}_0) \leq \lambda(u^\lambda - (v_{N_0}^\lambda + \delta\varphi_\varepsilon))(\bar{t}, \bar{\mathbf{x}}) \leq \delta \mathcal{L}\varphi_\varepsilon(\bar{t}, \bar{\mathbf{x}}).$$

From items 2) and 3) above, it follows that $\mathcal{L}\varphi_\varepsilon(\bar{t}, \bar{\mathbf{x}})$ is bounded by a constant (not depending on $\varepsilon, \delta, \lambda$). Therefore, letting $\delta \rightarrow 0^+$, taking into account the notations of STEP II, we have

$$\lambda e^{\lambda t_0}(u - v_{N_0})(t_0, \mathbf{x}_0) = \lambda(u^\lambda - v_{N_0}^\lambda)(t_0, \mathbf{x}_0) \leq 0,$$

which gives a contradiction to (4.10). \square

As a direct consequence of the comparison theorem (Theorem 4.2), we obtain the following uniqueness result.

Corollary 4.1. *Under Assumption (A), the function v in (4.4) is the unique (path-dependent) viscosity solution of equation (4.2), where uniqueness holds in the class of all continuous and bounded functions from Λ to \mathbb{R} .*

Proof. By Theorem 4.1 we know that v is continuous and bounded, moreover it is a (path-dependent) viscosity solution of equation (4.2).

Now, let $u: \Lambda \rightarrow \mathbb{R}$ be a continuous and bounded function such that u is a (path-dependent) viscosity solution of equation (4.2). Then, in particular, u (resp. v) is a (path-dependent) viscosity subsolution (resp. supersolution) of equation (4.2). As a consequence, by the comparison theorem (Theorem 4.2) we deduce that $u \leq v$ on Λ . Changing the roles of u and v we get the opposite inequality, from which we conclude that $u \equiv v$. \square

Appendix A: Functional Itô's formula

We start with a definition and a technical result.

Definition A.1. *We denote by $C^{1,0}(\hat{\Lambda})$ the set of $\hat{u} \in C(\hat{\Lambda})$ such that $\partial_t^H \hat{u}$ exists everywhere on $\hat{\Lambda}$ and is continuous.*

Lemma A.1. *Let $\hat{u} \in C^{1,0}(\hat{\Lambda})$. Then, for every $(t, \hat{x}) \in [0, T) \times D([0, T]; \mathbb{R}^d)$, the map $\phi: [0, T-t] \rightarrow \mathbb{R}$, defined as*

$$\phi(a) := \hat{u}(t+a, \hat{x}(\cdot \wedge t)), \quad \forall a \in [0, T-t],$$

is in $C^1([0, T-t])$ and

$$\phi'(a) = \partial_t^H \hat{u}(t+a, \hat{x}(\cdot \wedge t)), \quad \forall a \in [0, T-t].$$

Proof. Let $a \in [0, T-t)$. We have, for any $\delta \in (0, T-t-a]$,

$$\frac{\phi(a+\delta) - \phi(a)}{\delta} = \frac{\hat{u}(t+a+\delta, \hat{x}(\cdot \wedge t)) - \hat{u}(t+a, \hat{x}(\cdot \wedge t))}{\delta} \xrightarrow{\delta \rightarrow 0^+} \partial_t^H \hat{u}(t+a, \hat{x}(\cdot \wedge t)).$$

This shows that ϕ is right-differentiable on $[0, T-t)$ and that such a right-derivative is continuous on $[0, T-t)$. Then, it follows for instance from Corollary 1.2, Chapter 2, in [75] that $\phi \in C^1([0, T-t])$. Finally, at $a = T-t$, we have

$$\lim_{a \rightarrow (T-t)^-} \phi'(a) = \lim_{a \rightarrow (T-t)^-} \partial_t^H \hat{u}(t+a, \hat{x}(\cdot \wedge t)) = \partial_t^H \hat{u}(T, \hat{x}(\cdot \wedge t)).$$

This implies that $\phi \in C^1([0, T-t])$ and concludes the proof. \square

We now report the proof of Theorem 2.1.

Proof of Theorem 2.1. Fix $t \in (0, T]$. For every $\hat{\mathbf{x}} \in D([0, T]; \mathbb{R}^d)$, denote by $\hat{\mathbf{x}}(\cdot \wedge t-)$ the path

$$\hat{\mathbf{x}}(s \wedge t-) := \begin{cases} \hat{\mathbf{x}}(s), & 0 \leq s < t, \\ \hat{\mathbf{x}}(t-), & t \leq s \leq T. \end{cases} \quad (\text{A.1})$$

When $t = 0$, we set $\hat{\mathbf{x}}(\cdot \wedge 0-) := \hat{\mathbf{x}}(\cdot \wedge 0)$. We split the rest of the proof into several steps.

Step 1. Piecewise constant approximation of \mathbf{X} . For every $n \in \mathbb{N}$, consider the dyadic partition $\{t_0^n, \dots, t_\ell^n, \dots, t_{2^n}^n\}$ of $[0, t]$, with $t_\ell^n = \frac{\ell}{2^n}t$, for every $\ell = 0, \dots, 2^n$. Then, we consider the piecewise constant approximation of \mathbf{X}

$$\mathbf{X}_s^n = \sum_{\ell=0}^{2^n-1} \mathbf{X}_{t_\ell^n} 1_{[t_\ell^n, t_{\ell+1}^n)}(s) + \mathbf{X}_t 1_{[t, T]}(s), \quad \forall s \in [0, T]. \quad (\text{A.2})$$

Now, recalling the definition of $\hat{\mathbf{x}}(\cdot \wedge t-)$ in (A.1), we notice that

$$\hat{u}(t, \mathbf{X}_{\cdot \wedge t-}^n) - \hat{u}(0, \mathbf{X}) = \sum_{\ell=0}^{2^n-1} \left(\hat{u}(t_{\ell+1}^n, \mathbf{X}_{\cdot \wedge t_{\ell+1}^n-}^n) - \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_\ell^n-}^n) \right), \quad (\text{A.3})$$

where we have used the telescoping property and the equality $\hat{u}(0, \mathbf{X}) = \hat{u}(0, \mathbf{X}^n)$. For every $\ell = 0, \dots, 2^n - 1$, we have

$$\hat{u}(t_{\ell+1}^n, \mathbf{X}_{\cdot \wedge t_{\ell+1}^n-}^n) - \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_\ell^n-}^n) = I_\ell^n + J_\ell^n,$$

where

$$\begin{aligned} I_\ell^n &:= \hat{u}(t_{\ell+1}^n, \mathbf{X}_{\cdot \wedge t_{\ell+1}^n-}^n) - \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell+1}^n-}^n) = \hat{u}(t_{\ell+1}^n, \mathbf{X}_{\cdot \wedge t_\ell^n}^n) - \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_\ell^n}^n), \\ J_\ell^n &:= \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell+1}^n-}^n) - \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_\ell^n-}^n). \end{aligned}$$

Notice that when $\ell = 0$ we have $J_\ell^n = 0$. Moreover, when $\ell = 1, \dots, 2^n - 1$, J_ℓ^n can be written as

$$J_\ell^n = \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_\ell^n}^n) - \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n).$$

Step 2. Definitions of I^n , $J^{n,1}$, $J^{n,2}$, $J^{n,3}$, $J^{n,4}$. By Lemma A.1 we have

$$I_\ell^n = \int_{t_\ell^n}^{t_{\ell+1}^n} \partial_t^H \hat{u}(s, \mathbf{X}_{\cdot \wedge t_\ell^n}^n) ds = \int_{t_\ell^n}^{t_{\ell+1}^n} \partial_t^H \hat{u}(s, \mathbf{X}_{\cdot \wedge s}^n) ds.$$

Then, we define

$$I^n := \sum_{\ell=0}^{2^n-1} I_\ell^n = \sum_{\ell=0}^{2^n-1} \int_{t_\ell^n}^{t_{\ell+1}^n} \partial_t^H \hat{u}(s, \mathbf{X}_{\cdot \wedge s}^n) ds = \int_0^t \partial_t^H \hat{u}(s, \mathbf{X}_{\cdot \wedge s}^n) ds. \quad (\text{A.4})$$

On the other hand, when $\ell = 1, \dots, 2^n - 1$ the term J_ℓ^n can be written as

$$J_\ell^n = J_\ell^{n,1} + J_\ell^{n,2} + J_\ell^{n,3} + J_\ell^{n,4},$$

where (notice that $\mathbf{X}^n = (X^{n,1}, \dots, X^{n,d})$)

$$\begin{aligned}
J_\ell^{n,1} &= \sum_{i=1}^d \partial_{x_i}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n) (X_{t_\ell^n}^{n,i} - X_{t_{\ell-1}^n}^{n,i}) = \sum_{i=1}^d \partial_{x_i}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n) (X_{t_\ell^n}^i - X_{t_{\ell-1}^n}^i), \\
J_\ell^{n,2} &= \frac{1}{2} \sum_{i,j=1}^d \partial_{x_i x_j}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n) (X_{t_\ell^n}^i - X_{t_{\ell-1}^n}^i) (X_{t_\ell^n}^j - X_{t_{\ell-1}^n}^j), \\
J_\ell^{n,3} &= \frac{1}{2} \sum_{i,j=1}^d \left(\partial_{x_i x_j}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n) - \partial_{x_i x_j}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}) \right) (X_{t_\ell^n}^i - X_{t_{\ell-1}^n}^i) (X_{t_\ell^n}^j - X_{t_{\ell-1}^n}^j), \\
J_\ell^{n,4} &= \frac{1}{2} \sum_{i,j=1}^d \int_0^1 \left(\partial_{x_i x_j}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n + a(\mathbf{X}_{t_\ell^n} - \mathbf{X}_{t_{\ell-1}^n}) 1_{[t_\ell^n, T]}) \right. \\
&\quad \left. - \partial_{x_i x_j}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n) \right) (X_{t_\ell^n}^i - X_{t_{\ell-1}^n}^i) (X_{t_\ell^n}^j - X_{t_{\ell-1}^n}^j) da.
\end{aligned}$$

Then, we define

$$J^{n,k} := \sum_{\ell=1}^{2^n-1} J_\ell^{n,k}, \quad (\text{A.5})$$

for every $k = 1, 2, 3, 4$. As a consequence, by (A.3), (A.4), (A.5), we obtain

$$\hat{u}(t, \mathbf{X}_{\cdot \wedge t-}^n) - \hat{u}(0, \mathbf{X}) = I^n + J^{n,1} + J^{n,2} + J^{n,3} + J^{n,4}. \quad (\text{A.6})$$

Notice that

$$J^{n,1} = \sum_{\ell=1}^{2^n-1} \sum_{i=1}^d \partial_{x_i}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n) (X_{t_\ell^n}^i - X_{t_{\ell-1}^n}^i) = \sum_{i=1}^d \int_0^t Z_s^{n,i} dX_s^i, \quad (\text{A.7})$$

where

$$Z_s^{n,i} := \sum_{\ell=1}^{2^n-1} \partial_{x_i}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge s}^n) 1_{[t_{\ell-1}^n, t_\ell^n)}(s), \quad \forall s \in [0, t].$$

Step 3. Convergence. Before studying the convergence of the various terms in (A.6), we make some observations. Recall that for a.e. $\omega \in \Omega$ the trajectory $\mathbf{X}(\omega)$ is continuous on $[0, T]$, so that it admits a modulus of continuity $\rho_{\mathbf{X}(\omega)}: [0, +\infty) \rightarrow [0, +\infty)$. Then, it holds that

$$\hat{d}_\infty((t, \mathbf{X}_{\cdot \wedge t-}^n(\omega)), (t, \mathbf{X}_{\cdot \wedge t}(\omega))) = \|\mathbf{X}_{\cdot \wedge t-}^n(\omega) - \mathbf{X}_{\cdot \wedge t}(\omega)\|_\infty \leq \rho_{\mathbf{X}(\omega)}(2^{-n}), \quad (\text{A.8})$$

for a.e. $\omega \in \Omega$. Similarly, we have

$$\hat{d}_\infty((s, \mathbf{X}_{\cdot \wedge s}^n(\omega)), (s, \mathbf{X}_{\cdot \wedge s}(\omega))) = \|\mathbf{X}_{\cdot \wedge s}^n(\omega) - \mathbf{X}_{\cdot \wedge s}(\omega)\|_\infty \leq \rho_{\mathbf{X}(\omega)}(2^{-n}), \quad (\text{A.9})$$

for a.e. $\omega \in \Omega$ and for every $s \in [0, t]$. Moreover, for every $\ell = 1, \dots, 2^n - 1$ and $t_{\ell-1}^n \leq s \leq t_\ell^n$, it holds that

$$\hat{d}_\infty((t_\ell^n, \mathbf{X}_{\cdot \wedge s}^n(\omega)), (s, \mathbf{X}_{\cdot \wedge s}(\omega))) = |t_\ell^n - s| + \|\mathbf{X}_{\cdot \wedge s}^n(\omega) - \mathbf{X}_{\cdot \wedge s}(\omega)\|_\infty$$

$$\leq 2^{-n} + \rho_{\mathbf{X}(\omega)}(2^{-n}), \quad (\text{A.10})$$

for a.e. $\omega \in \Omega$. Now, for every $\mathbf{x} \in C([0, T]; \mathbb{R}^d)$ and $n \in \mathbb{N}$, set (as in (A.2))

$$\mathbf{x}^n(s) = \sum_{\ell=0}^{2^n-1} \mathbf{x}(t_\ell^n) 1_{[t_\ell^n, t_{\ell+1}^n)}(s) + \mathbf{x}(t) 1_{[t, T]}(s), \quad \forall s \in [0, T].$$

For such an \mathbf{x} , define

$$K_1(\mathbf{x}) := \{(s, \hat{\mathbf{x}}) \in \hat{\Lambda} : \hat{\mathbf{x}} = \mathbf{x}^n(\cdot \wedge s), \text{ for some } n \in \mathbb{N}\}, \quad (\text{A.11})$$

$$K_2(\mathbf{x}) := \{(r, \hat{\mathbf{x}}) \in \hat{\Lambda} : (r, \hat{\mathbf{x}}) = (t_\ell^n, \mathbf{x}^n(\cdot \wedge s)), \text{ for some } n \in \mathbb{N}, \\ \ell = 1, \dots, 2^n - 1, s \in [t_{\ell-1}^n, t_\ell^n)\}, \quad (\text{A.12})$$

$$K_3(\mathbf{x}) := \{(r, \hat{\mathbf{x}}) \in \hat{\Lambda} : (r, \hat{\mathbf{x}}) = (t_\ell^n, \mathbf{x}^n(\cdot \wedge t_{\ell-1}^n) + a(\mathbf{x}(t_\ell^n) - \mathbf{x}(t_{\ell-1}^n)) 1_{[t_\ell^n, T]}(\cdot)), \\ \text{for some } n \in \mathbb{N}, \ell = 1, \dots, 2^n - 1, a \in [0, 1]\}. \quad (\text{A.13})$$

Notice that $K_2(\mathbf{x}) \subset K_3(\mathbf{x})$ (observe that $\mathbf{x}^n(\cdot \wedge s) = \mathbf{x}^n(\cdot \wedge t_{\ell-1}^n)$, whenever $s \in [t_{\ell-1}^n, t_\ell^n)$, and consider the pairs in $K_3(\mathbf{x})$ with $a = 0$). It is easy to see that $K_1(\mathbf{x})$ and $K_3(\mathbf{x})$ (and a fortiori $K_2(\mathbf{x})$) are relatively compact subsets of $(\hat{\Lambda}, \hat{d}_\infty)$. Now, we study separately the convergence of the various terms appearing in (A.6).

Substep 3.1. *Convergence of $\hat{u}(t, \mathbf{X}_{\cdot \wedge t-}^n) - \hat{u}(0, \mathbf{X})$.* Since \hat{u} is continuous on $(\hat{\Lambda}, \hat{d}_\infty)$ and (A.8) holds, we deduce that

$$\hat{u}(t, \mathbf{X}_{\cdot \wedge t-}^n) - \hat{u}(0, \mathbf{X}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} \hat{u}(t, \mathbf{X}_{\cdot \wedge t}) - \hat{u}(0, \mathbf{X}).$$

Substep 3.2. *Convergence of I^n .* Recalling (A.11), we see that for a.e. $\omega \in \Omega$ the set $K_1(\mathbf{X}(\omega))$ is a relatively compact subset of $\hat{\Lambda}$. Since $\partial_t^H \hat{u}$ is continuous on $(\hat{\Lambda}, \hat{d}_\infty)$, for such an ω we see that $\partial_t^H \hat{u}(s, \mathbf{X}_{\cdot \wedge s}^n(\omega))$ is uniformly bounded with respect to s and n . In addition, using again the continuity of $\partial_t^H \hat{u}$ and (A.9), we deduce that for a.e. $\omega \in \Omega$ the sequence of maps $s \mapsto \partial_t^H \hat{u}(s, \mathbf{X}_{\cdot \wedge s}^n(\omega))$ converges pointwise to $s \mapsto \partial_t^H \hat{u}(s, \mathbf{X}_{\cdot \wedge s}(\omega))$. In conclusion, we can apply Lebesgue's dominated convergence theorem, which yields

$$I^n = \int_0^t \partial_t^H \hat{u}(s, \mathbf{X}_{\cdot \wedge s}^n) ds \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} \int_0^t \partial_t^H \hat{u}(s, \mathbf{X}_{\cdot \wedge s}) ds.$$

Substep 3.3. *Convergence of $J^{n,1}$.* In order to study the convergence of the sum of stochastic integrals in (A.7), we write each X^i as $V^i + M^i$, where V^i is a bounded variation process and M^i is a continuous local martingale, so that the stochastic integral with respect to X^i can be written as the sum of two integrals with respect to V^i and M^i , respectively.

Convergence of $\int_0^t Z_s^{n,i} dV_s^i$. Recalling (A.12), we see that for a.e. $\omega \in \Omega$ the set $K_2(\mathbf{X}(\omega))$ is a relatively compact subset of $\hat{\Lambda}$. Then, reasoning as in **Substep 3.2**, using the

continuity of $\partial_{x_i}^V \hat{u}$ and also (A.10), we see that we can apply Lebesgue's dominated convergence theorem, from which we get

$$\int_0^t Z_s^{n,i} dV_s^i \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} \int_0^t \partial_{x_i}^V \hat{u}(s, \mathbf{X}_{\cdot \wedge s}) dV_s^i.$$

Convergence of $\int_0^t Z_s^{n,i} dM_s^i$. Let $\langle M^i \rangle$ denote the quadratic variation of M^i . Reasoning as in the proof of the convergence of $\int_0^t Z_s^{n,i} dV_s^i$, we obtain by Lebesgue's dominated convergence theorem

$$\int_0^t |Z_s^{n,i} - \hat{u}(s, \mathbf{X}_{\cdot \wedge s})|^2 d\langle M^i \rangle_s \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} 0.$$

By Proposition 2.26, Chapter 3, in [55], we deduce the convergence in probability (even u.c.p.)

$$\int_0^t Z_s^{n,i} dM_s^i \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t \partial_{x_i}^V \hat{u}(s, \mathbf{X}_{\cdot \wedge s}) dM_s^i.$$

Substep 3.4. *Convergence of $J^{n,2}$.* For every $n \in \mathbb{N}$ and $i, j = 1, \dots, d$, consider the process $\langle X^i, X^j \rangle^n$ defined as

$$\langle X^i, X^j \rangle_s^n := \sum_{\ell=1}^{2^n} (X_{t_{\ell}^n}^i - X_{t_{\ell-1}^n}^i)(X_{t_{\ell}^n}^j - X_{t_{\ell-1}^n}^j) 1_{[t_{\ell-1}^n, T]}(s), \quad \forall s \in [0, T].$$

Notice that

$$J_{\ell}^{n,2} = \frac{1}{2} \sum_{i,j=1}^d \int_{[t_{\ell-1}^n, t_{\ell}^n)} \partial_{x_i x_j}^V \hat{u}(s, \mathbf{X}_{\cdot \wedge s}) d\langle X^i, X^j \rangle_s^n$$

and, therefore,

$$J^{n,2} = \frac{1}{2} \sum_{i,j=1}^d \int_{[0, (1-2^{-n})t)} \partial_{x_i x_j}^V \hat{u}(s, \mathbf{X}_{\cdot \wedge s}) d\langle X^i, X^j \rangle_s^n.$$

From Theorem 23 in Section II.6 of [80], it is easy to deduce that

$$\sup_{s \in [0, t]} |\langle X^i, X^j \rangle_s^n - \langle X^i, X^j \rangle_s| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

where $\langle X^i, X^j \rangle$ is the covariation of X^i and X^j . Then, up to a subsequence, there exists a \mathbb{P} -null set N such that, whenever $\omega \notin N$, $\langle X^i, X^j \rangle_s^n(\omega) \rightarrow \langle X^i, X^j \rangle_s(\omega)$, for all $s \in [0, t]$. For such an ω , this implies that the measure $d\langle X^i, X^j \rangle^n(\omega)$ converges weakly to the measure $d\langle X^i, X^j \rangle(\omega)$. Since $\partial_{x_i x_j}^V \hat{u}$ is continuous, we deduce (up to a subsequence)

$$J^{n,2} \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{x_i x_j}^V \hat{u}(s, \mathbf{X}_{\cdot \wedge s}) d\langle X^i, X^j \rangle_s.$$

Substep 3.5. *Convergence of $J^{n,3}$.* We have (we set $\langle X^i \rangle^n := \langle X^i, X^i \rangle^n$, for every $i = 1, \dots, d$)

$$|J^{n,3}| \leq \frac{1}{4} \sum_{i,j=1}^d \sup_{\ell=1, \dots, 2^n-1} \left| \partial_{x_i x_j}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n) - \partial_{x_i x_j}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}) \right| (\langle X^i \rangle_T^n + \langle X^j \rangle_T^n).$$

Recalling (A.12), we see that for a.e. $\omega \in \Omega$ the set $K_2(\mathbf{X}(\omega))$ is a relatively compact subset of $\hat{\mathbf{A}}$. Since $\partial_{x_i x_j}^V \hat{u}$ is continuous on $(\hat{\mathbf{A}}, \hat{d}_\infty)$, for a.e. $\omega \in \Omega$ there exists a (non-decreasing) modulus of continuity $\rho_{i,j}^\omega: [0, +\infty) \rightarrow [0, +\infty)$ such that

$$|J^{n,3}(\omega)| \leq \frac{1}{4} \sum_{i,j=1}^d \sup_{\ell=1, \dots, 2^n-1} \rho_{i,j}^\omega(\|\mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n(\omega) - \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}(\omega)\|_\infty) (\langle X^i \rangle_T^n(\omega) + \langle X^j \rangle_T^n(\omega)).$$

From **Substep 3.4** we know that, up to a subsequence, $\langle X^i \rangle_T^n(\omega)$ converges to $\langle X^i \rangle_T(\omega)$, for a.e. $\omega \in \Omega$ and for every $i = 1, \dots, d$. On the other hand, by (A.9) we have that $\sup_{\ell=1, \dots, 2^n-1} \rho_{i,j}^\omega(\|\mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n(\omega) - \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}(\omega)\|_\infty) \leq \rho_{i,j}^\omega(\rho_{\mathbf{X}(\omega)}(2^{-n}))$, for a.e. $\omega \in \Omega$. Hence, up to a subsequence,

$$J^{n,3} \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} 0.$$

Substep 3.6. *Convergence of $J^{n,4}$.* We have (as in **Substep 3.5** we set $\langle X^i \rangle^n := \langle X^i, X^i \rangle^n$, for every $i = 1, \dots, d$)

$$|J^{n,4}| \leq \frac{1}{4} \sum_{i,j=1}^d (\langle X^i \rangle_T^n + \langle X^j \rangle_T^n) \int_0^1 \sup_{\ell=1, \dots, 2^n-1} \left| \partial_{x_i x_j}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n) - \partial_{x_i x_j}^V \hat{u}(t_\ell^n, \mathbf{X}_{\cdot \wedge t_{\ell-1}^n}^n + a(\mathbf{X}_{t_\ell^n} - \mathbf{X}_{t_{\ell-1}^n}) 1_{[t_\ell^n, T]}) \right| da.$$

Recalling (A.13), we see that for a.e. $\omega \in \Omega$ the set $K_3(\mathbf{X}(\omega))$ is a relatively compact subset of $\hat{\mathbf{A}}$. Since $\partial_{x_i x_j}^V \hat{u}$ is continuous on $(\hat{\mathbf{A}}, \hat{d}_\infty)$, for a.e. $\omega \in \Omega$ there exists a (non-decreasing) modulus of continuity $\bar{\rho}_{i,j}^\omega: [0, +\infty) \rightarrow [0, +\infty)$ such that

$$|J^{n,4}(\omega)| \leq \frac{1}{4} \sum_{i,j=1}^d (\langle X^i \rangle_T^n(\omega) + \langle X^j \rangle_T^n(\omega)) \int_0^1 \sup_{\ell=1, \dots, 2^n-1} \bar{\rho}_{i,j}^\omega(a \|\mathbf{X}_{t_\ell^n}(\omega) - \mathbf{X}_{t_{\ell-1}^n}(\omega)\|) da.$$

We know from **Substep 3.4** that, up to a subsequence, $\langle X^i \rangle_T^n(\omega)$ converges to $\langle X^i \rangle_T(\omega)$, for a.e. $\omega \in \Omega$ and for every $i = 1, \dots, d$. On the other hand, $\sup_{\ell=1, \dots, 2^n-1} \bar{\rho}_{i,j}^\omega(a \|\mathbf{X}_{t_\ell^n}(\omega) - \mathbf{X}_{t_{\ell-1}^n}(\omega)\|) \leq \bar{\rho}_{i,j}^\omega(\rho_{\mathbf{X}(\omega)}(2^{-n}))$, for a.e. $\omega \in \Omega$, where we recall that $\rho_{\mathbf{X}(\omega)}$ is the modulus of continuity of the trajectory $\mathbf{X}(\omega)$. Hence, up to a subsequence,

$$J^{n,4} \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} 0.$$

□

Appendix B: Consistency

Proof of Lemma 2.1. The claim concerning the horizontal derivatives follows directly from their definition (Definition 2.3-(i)).

It remains to prove the claim concerning the vertical derivatives. To this end, let $\mathbf{X} = (\mathbf{X}_t)_{t \in [0, T]}$, with $\mathbf{X} = (X^1, \dots, X^d)$, be a d -dimensional continuous semimartingale on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \in [0, T]}$ satisfies the usual conditions. Since $\hat{u}_1, \hat{u}_2 \in C^{1,2}(\hat{\mathbf{A}})$, by Theorem 2.1 the functional Itô formulae

$$\begin{aligned} \hat{u}_1(t, \mathbf{X}) &= \hat{u}_1(0, \mathbf{X}) + \int_0^t \partial_t^H \hat{u}_1(s, \mathbf{X}) ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{x_i x_j}^V \hat{u}_1(s, \mathbf{X}) d[X^i, X^j]_s \\ &\quad + \sum_{i=1}^d \int_0^t \partial_{x_i}^V \hat{u}_1(s, \mathbf{X}) dX_s^i, \end{aligned} \quad \text{for all } 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

and

$$\begin{aligned} \hat{u}_2(t, \mathbf{X}) &= \hat{u}_2(0, \mathbf{X}) + \int_0^t \partial_t^H \hat{u}_2(s, \mathbf{X}) ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{x_i x_j}^V \hat{u}_2(s, \mathbf{X}) d[X^i, X^j]_s \\ &\quad + \sum_{i=1}^d \int_0^t \partial_{x_i}^V \hat{u}_2(s, \mathbf{X}) dX_s^i, \end{aligned} \quad \text{for all } 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

hold. Recalling that \hat{u}_1 and \hat{u}_2 , together with their horizontal derivatives, coincide on continuous paths, identifying bounded variation and local martingale parts in the above formulae, (up to a \mathbb{P} -null set), for every $i, j = 1, \dots, d$ and any $t \in [0, T]$, we have

$$\partial_{x_i}^V \hat{u}_1(t, \mathbf{X}) = \partial_{x_i}^V \hat{u}_2(t, \mathbf{X}), \quad \partial_{x_i x_j}^V \hat{u}_1(t, \mathbf{X}) = \partial_{x_i x_j}^V \hat{u}_2(t, \mathbf{X}). \quad (\text{B.1})$$

Now, recall that $C([0, T]; \mathbb{R}^d)$ is endowed with the uniform topology and let $\mathcal{B}(C([0, T]; \mathbb{R}^d))$ denote the Borel σ -algebra on $C([0, T]; \mathbb{R}^d)$. We also recall that given a probability measure $\mu: \mathcal{B}(C([0, T]; \mathbb{R}^d)) \rightarrow [0, 1]$, the *support* of μ is the smallest closed set $\mathcal{S}_\mu \subset C([0, T]; \mathbb{R}^d)$ such that $\mu(\mathcal{S}_\mu) = 1$. Consider a continuous semimartingale $\mathbf{X} = (\mathbf{X}_s)_{s \in [0, T]}$ whose law has support equal to $C([0, T]; \mathbb{R}^d)$. An example of such an \mathbf{X} is given by $\mathbf{X}_s := \boldsymbol{\eta} + \mathbf{W}_s$, $s \in [0, T]$, where $\mathbf{W} = (\mathbf{W}_s)_{s \in [0, T]}$ is a d -dimensional Brownian motion, while $\boldsymbol{\eta}: \Omega \rightarrow \mathbb{R}^d$ is independent of \mathbf{W} and has a standard normal multivariate distribution. Then, for every fixed $t \in [0, T]$, using equalities (B.1) with such a semimartingale \mathbf{X} , and exploiting the continuity of $\partial_{\mathbf{x}}^V \hat{u}_1$, $\partial_{\mathbf{x}}^V \hat{u}_2$, $\partial_{\mathbf{x}\mathbf{x}}^V \hat{u}_1$, $\partial_{\mathbf{x}\mathbf{x}}^V \hat{u}_2$, we obtain

$$\partial_{\mathbf{x}}^V \hat{u}_1(t, \mathbf{x}) = \partial_{\mathbf{x}}^V \hat{u}_2(t, \mathbf{x}), \quad \partial_{\mathbf{x}\mathbf{x}}^V \hat{u}_1(t, \mathbf{x}) = \partial_{\mathbf{x}\mathbf{x}}^V \hat{u}_2(t, \mathbf{x}),$$

for every $\mathbf{x} \in C([0, T]; \mathbb{R}^d)$. Since the above equalities hold for every $t \in [0, T]$, the claim follows. \square

Appendix C: Smooth variational principle on Λ

C.1. Proof of Lemma 3.1

Proof of Lemma 3.1. We split the proof into several steps.

STEP I. *Proof of item 1).* We first notice that $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ is a non-anticipative map. Now, let $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$, $h \in \mathbb{R} \setminus \{0\}$, and $i = 1, \dots, d$ (recall that $\mathbf{e}_1, \dots, \mathbf{e}_d$ denotes the standard orthonormal basis of \mathbb{R}^d), then we have

$$\begin{aligned} & \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}} + h \mathbf{e}_i 1_{[t, T]}) - \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})}{h} \\ &= \int_{\mathbb{R}^d} \|\hat{\mathbf{x}}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t, T]}\|_\infty \frac{\zeta(\mathbf{z} + h \mathbf{e}_i) - \zeta(\mathbf{z})}{h} d\mathbf{z} \\ & \xrightarrow{h \rightarrow 0} \int_{\mathbb{R}^d} \|\hat{\mathbf{x}}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t, T]}\|_\infty \partial_{z_i} \zeta(\mathbf{z}) d\mathbf{z}, \end{aligned}$$

where $\partial_{z_i} \zeta(\mathbf{z})$ denotes the partial derivative of ζ in the \mathbf{e}_i -direction at the point \mathbf{z} , which is given by $-z_i \zeta(\mathbf{z})$. This proves that $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ admits first-order vertical derivatives at every $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$. In a similar way we can prove that $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ also admits second-order vertical derivatives at every $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$.

STEP II. *Proof of item 2).* We begin noticing that

$$\begin{aligned} & |\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}} + h \mathbf{e}_i 1_{[t, T]}) - \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})| \\ &= \left| \int_{\mathbb{R}^d} \|\hat{\mathbf{x}}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - (\mathbf{z} - h \mathbf{e}_i) 1_{[t, T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} \right. \\ & \quad \left. - \int_{\mathbb{R}^d} \|\hat{\mathbf{x}}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t, T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} \right| \leq |h \mathbf{e}_i| = |h|, \end{aligned}$$

where we have used the fact that $\int_{\mathbb{R}^d} \zeta(\mathbf{z}) d\mathbf{z} = 1$. It is then easy to see that, for every $i = 1, \dots, d$, $\partial_{x_i}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ is bounded by the constant 1. Proceeding along the same lines as for $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$, we deduce that $|\partial_{x_i}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}} + h \mathbf{e}_j 1_{[t, T]}) - \partial_{x_i}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})|$ is bounded by $|h| \int_{\mathbb{R}^d} |\partial_{z_i} \zeta(\mathbf{z})| d\mathbf{z} = |h| \int_{\mathbb{R}^d} |z_i| \zeta(\mathbf{z}) d\mathbf{z} = |h| \sqrt{\frac{2}{\pi}}$. This allows to prove that, for every $i, j = 1, \dots, d$, $\partial_{x_i x_j}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ is bounded by $\sqrt{\frac{2}{\pi}}$.

STEP III. *Proof of item 3).* We begin noting that

$$\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) = \int_{\mathbb{R}^d} \|\hat{\mathbf{x}}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t, T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} \geq -C_\zeta,$$

with $C_\zeta := \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z}$, which proves the first part of item 3).

Now, let $(t, \mathbf{x}) \in \Lambda$. Using the fact that ζ is a radial function, we have (when $d = 1$, $(-\infty, 0] \times \mathbb{R}^0$ and $[0, +\infty) \times \mathbb{R}^0$ stand for $(-\infty, 0]$ and $[0, +\infty)$, respectively)

$$\int_{\mathbb{R}^d} \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t, T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z}$$

$$\begin{aligned}
&= \int_{[0,+\infty) \times \mathbb{R}^{d-1}} \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z}1_{[t,T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} \\
&\quad + \int_{(-\infty,0] \times \mathbb{R}^{d-1}} \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z}1_{[t,T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} \\
&= \int_{[0,+\infty) \times \mathbb{R}^{d-1}} \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z}1_{[t,T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} \\
&\quad + \int_{[0,+\infty) \times \mathbb{R}^{d-1}} \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) + \mathbf{z}1_{[t,T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z}. \tag{C.1}
\end{aligned}$$

Now, we observe that, for every $\mathbf{z} \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z}1_{[t,T]}\|_\infty \\
&= \max \left\{ \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t \wedge t_0)\|_\infty, \max_{t \leq s \leq T} |\mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0) - \mathbf{z}| \right\} \tag{C.2}
\end{aligned}$$

and similarly for $\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) + \mathbf{z}1_{[t,T]}\|_\infty$. Moreover, by the elementary inequality $|\mathbf{x} - \mathbf{z}| + |\mathbf{x} + \mathbf{z}| \geq 2|\mathbf{x}|$, valid for every $\mathbf{x}, \mathbf{z} \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\max_{t \leq s \leq T} |\mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0) - \mathbf{z}| + \max_{t \leq s \leq T} |\mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0) + \mathbf{z}| \\
&\geq \max_{t \leq s \leq T} \left\{ |\mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0) - \mathbf{z}| + |\mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0) + \mathbf{z}| \right\} \\
&\geq 2 \max_{t \leq s \leq T} |\mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0)|.
\end{aligned}$$

Then, using the elementary fact that if $a + b \geq 2c$ it holds that $\max\{\ell, a\} + \max\{\ell, b\} \geq 2 \max\{\ell, c\}$, for every $a, b, c, \ell \in \mathbb{R}$, we find

$$\begin{aligned}
&\max \left\{ \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t \wedge t_0)\|_\infty, \max_{t \leq s \leq T} |\mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0) - \mathbf{z}| \right\} \\
&\quad + \max \left\{ \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t \wedge t_0)\|_\infty, \max_{t \leq s \leq T} |\mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0) + \mathbf{z}| \right\} \\
&\geq 2 \max \left\{ \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t \wedge t_0)\|_\infty, \max_{t \leq s \leq T} |\mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0)| \right\}.
\end{aligned}$$

By (C.2) it follows that the last quantity coincides with $2 \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty$. Therefore, by (C.1) we obtain (also recalling that $C_\zeta := \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z}$)

$$\begin{aligned}
\kappa_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}) + C_\zeta &= \int_{\mathbb{R}^d} \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z}1_{[t,T]}\|_\infty \zeta(\mathbf{z}) d\mathbf{z} \\
&\geq 2 \int_{[0,+\infty) \times \mathbb{R}^{d-1}} \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty \zeta(\mathbf{z}) d\mathbf{z} \\
&= \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty,
\end{aligned}$$

which concludes the proof of item 3). Finally, the explicit expression of the constant C_ζ , reported in (3.3), follows from the fact that $C_\zeta = \mu_{\chi^2(d), \frac{1}{2}}$, where $\mu_{\chi^2(d), \frac{1}{2}}$ denotes the moment of order 1/2 of a χ^2 -distribution with d degrees of freedom.

STEP IV. *Proof of the second inequality in (3.4).* The second inequality in (3.4) follows easily from an application of the triangular inequality, namely noting that $\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z}1_{[t,T]}\|_\infty \leq \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty + |\mathbf{z}|$.

STEP V. *Proof of the first inequality in (3.4) for the case $\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty > 2C_\zeta$.* When $\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty > 2C_\zeta$, we have, by item 3),

$$\begin{aligned} \kappa_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}) &\geq \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty - C_\zeta \\ &= \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty \left(1 - \frac{C_\zeta}{\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty}\right) \\ &\geq \frac{1}{2} \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty \\ &\geq \frac{1}{2} (\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty^{d+1} \wedge \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty), \end{aligned}$$

which proves the first inequality in (3.4) with $\alpha_d := \frac{1}{2}$, for the case $\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty > 2C_\zeta$.

STEP VI. *Proof of the first inequality in (3.4) for the case $\|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty \leq 2C_\zeta$.*

STEP VI-1. Our aim is to prove that for every fixed d there exists some constant $\alpha_d > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^d} \max\{a, |\mathbf{y} - \mathbf{z}|\} \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} \\ \geq \alpha_d \min\{a^{d+1}, a\} + \alpha_d \min\{|\mathbf{y}|^{d+1}, |\mathbf{y}|\}, \quad \forall (a, \mathbf{y}) \in [0, 2C_\zeta] \times \mathbb{R}^d. \end{aligned} \quad (\text{C.3})$$

As a matter of fact, suppose for a moment that (C.3) holds true. Then, applying (C.3) with $a := \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t \wedge t_0)\|_\infty$ and $\mathbf{y}_s := \mathbf{x}(t) - \mathbf{x}_0(s \wedge t_0)$, for every $s \in [t, T]$, and taking the maximum over $s \in [t, T]$, we find (using (C.2))

$$\begin{aligned} \kappa_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}) &= \int_{\mathbb{R}^d} \max\left\{a, \max_{t \leq s \leq T} |\mathbf{y}_s - \mathbf{z}|\right\} \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} \\ &= \int_{\mathbb{R}^d} \max_{t \leq s \leq T} \left\{ \max\{a, |\mathbf{y}_s - \mathbf{z}|\} \right\} \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} \\ &\geq \max_{t \leq s \leq T} \left\{ \int_{\mathbb{R}^d} \max\{a, |\mathbf{y}_s - \mathbf{z}|\} \zeta(\mathbf{z}) d\mathbf{z} \right\} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} \\ &= \max_{t \leq s \leq T} \left\{ \int_{\mathbb{R}^d} \max\{a, |\mathbf{y}_s - \mathbf{z}|\} \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z} \right\} \\ &\geq \max_{t \leq s \leq T} \left\{ \alpha_d \min\{a^{d+1}, a\} + \alpha_d \min\{|\mathbf{y}_s|^{d+1}, |\mathbf{y}_s|\} \right\} \\ &= \alpha_d \min\{a^{d+1}, a\} + \alpha_d \max_{t \leq s \leq T} \left\{ \min\{|\mathbf{y}_s|^{d+1}, |\mathbf{y}_s|\} \right\} \\ &= \alpha_d \min\{a^{d+1}, a\} + \alpha_d \min \left\{ \max_{t \leq s \leq T} |\mathbf{y}_s|^{d+1}, \max_{t \leq s \leq T} |\mathbf{y}_s| \right\}. \end{aligned}$$

Hence, by the elementary inequality

$$\min\{a^{d+1}, a\} + \min\{b^{d+1}, b\} \geq \min\{\max\{a^{d+1}, b^{d+1}\}, \max\{a, b\}\}, \quad \forall a, b \geq 0,$$

we conclude that

$$\begin{aligned} \kappa_\infty^{(t_0, \mathbf{x}_0)}(t, \mathbf{x}) &\geq \alpha_d \min \left\{ \max \left\{ a^{d+1}, \max_{t \leq s \leq T} |\mathbf{y}_s|^{d+1} \right\}, \max \left\{ a, \max_{t \leq s \leq T} |\mathbf{y}_s| \right\} \right\} \\ &= \alpha_d \min \left\{ \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty^{d+1}, \|\mathbf{x}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0)\|_\infty \right\}, \end{aligned}$$

where the last equality follows from (C.2) with $\mathbf{z} = 0$. This yields the first inequality in (3.4). It remains to prove (C.3).

STEP VI-2. *Proof of (C.3).* For every positive integer d and $a \geq 0$, let $G_a: \mathbb{R}^d \rightarrow \mathbb{R}$ be given by

$$G_a(\mathbf{y}) := \int_{\mathbb{R}^d} \max\{a, |\mathbf{y} - \mathbf{z}|\} \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z}, \quad \forall \mathbf{y} \in \mathbb{R}^d. \quad (\text{C.4})$$

Moreover, let $F_d: [0, +\infty) \rightarrow \mathbb{R}$ be defined as (differently to the notation used for G_a , we emphasize the dependence of F_d on the dimension d ; we do this because of statement (C.7) below which changes with d)

$$F_d(a) := G_a(\mathbf{0}) = \int_{\mathbb{R}^d} \max\{a, |\mathbf{z}|\} \zeta(\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}^d} |\mathbf{z}| \zeta(\mathbf{z}) d\mathbf{z}, \quad \forall a \in [0, +\infty). \quad (\text{C.5})$$

Notice that G_a and F_d are convex functions on their domains.

Let us fix some notations. We denote by $\partial_{\mathbf{y}} G_a(\mathbf{y})$ and $\partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{y})$ (resp. $F'_d(a)$, $F''_d(a)$, \dots , $F_d^{(n)}(a)$) the gradient and Hessian (resp. first-order derivative, second-order derivative, \dots , n -th order derivative) of G_a at \mathbf{y} (resp. F_d at a). When $a = 0$, $F'_d(a)$, $F''_d(a)$, \dots , $F_d^{(n)}(a)$ are right-derivatives. We also denote by I the $d \times d$ identity matrix. Finally, given A and B in $\mathcal{S}(d)$ (the set of symmetric $d \times d$ matrices), the inequality $B \leq A$ means that the symmetric matrix $A - B$ is positive semi-definite.

Our aim is to prove the following: *for every d , there exist constants $\beta_d > 0$ and $L_d > 0$ such that*

$$\begin{aligned} \forall a \in [0, 2C_\zeta], G_a \in C^2(\mathbb{R}^d), \partial_{\mathbf{y}} G_a(\mathbf{0}) = 0, \partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{0}) \geq \beta_d I \text{ and} \\ \partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{y}) - \partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{0}) \geq -L_d |\mathbf{y}| I, \quad \forall \mathbf{y} \in \mathbb{R}^d \end{aligned} \quad (\text{C.6})$$

and

$$\begin{aligned} F_d \in C^{d+1}([0, +\infty)), F'_d(0) = \dots = F_d^{(d)}(0) = 0, F_d^{(d+1)}(0) \geq \beta_d \text{ and} \\ F_d^{(d+1)}(a) - F_d^{(d+1)}(0) \geq -L_d a, \quad \forall a \geq 0. \end{aligned} \quad (\text{C.7})$$

Suppose for a moment that (C.6) and (C.7) hold. Then, by (C.6) we show below that there exist some constants $\delta_d, \tilde{\delta}_d \in (0, 1]$ such that

$$G_a(\mathbf{y}) \geq G_a(\mathbf{0}) + \frac{1}{4} \beta_d |\mathbf{y}|^2, \quad \forall |\mathbf{y}| \leq \delta_d, \forall a \in [0, 2C_\zeta], \quad (\text{C.8})$$

$$F_d(a) \geq F_d(0) + \frac{1}{2(d+1)!} \beta_d a^{d+1}, \quad \forall a \in [0, \tilde{\delta}_d]. \quad (\text{C.9})$$

As a matter of fact, for every fixed $\mathbf{y} \in \mathbb{R}^d$, set $\varphi_a(\lambda) := G_a(\lambda \mathbf{y})$, for every $\lambda \in \mathbb{R}$. Since $\varphi_a \in C^2(\mathbb{R})$, the Taylor expression given by

$$\varphi_a(1) = \varphi_a(0) + \varphi'_a(0) + \frac{1}{2} \varphi''_a(0) + \int_0^1 (1-\lambda) (\varphi''_a(\lambda) - \varphi''_a(0)) d\lambda,$$

which written in terms of G_a becomes (denoting by $\langle \cdot, \cdot \rangle$ the scalar product in \mathbb{R}^d)

$$\begin{aligned} G_a(\mathbf{y}) &= G_a(\mathbf{0}) + \langle \partial_{\mathbf{y}} G_a(\mathbf{0}), \mathbf{y} \rangle + \frac{1}{2} \langle \partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{0}) \mathbf{y}, \mathbf{y} \rangle \\ &\quad + \int_0^1 (1-\lambda) \langle (\partial_{\mathbf{y}\mathbf{y}} G_a(\lambda \mathbf{y}) - \partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{0})) \mathbf{y}, \mathbf{y} \rangle d\lambda \\ &\geq G_a(\mathbf{0}) + \langle \partial_{\mathbf{y}} G_a(\mathbf{0}), \mathbf{y} \rangle + \frac{1}{2} \beta_d |\mathbf{y}|^2 - L_d |\mathbf{y}|^3 \int_0^1 \lambda (1-\lambda) d\lambda \\ &= G_a(\mathbf{0}) + \frac{1}{2} \beta_d |\mathbf{y}|^2 - \frac{1}{6} L_d |\mathbf{y}|^3. \end{aligned}$$

Hence

$$G_a(\mathbf{y}) \geq G_a(\mathbf{0}) + \frac{1}{4} \beta_d |\mathbf{y}|^2, \quad \forall |\mathbf{y}| \leq \delta_d, \text{ where } \delta_d := 1 \wedge \left(\frac{3}{2} \frac{\beta_d}{L_d} \right).$$

This proves (C.8). Similarly, we consider the following Taylor expression for F_d :

$$F_d(a) = \sum_{k=0}^{d+1} \frac{F_d^{(k)}(0)}{k!} a^k + \frac{1}{d!} \int_0^a (F_d^{(d+1)}(b) - F_d^{(d+1)}(0)) (a-b)^d db.$$

By (C.7) we obtain

$$\begin{aligned} F_d(a) &= F_d(0) + \frac{F_d^{(d+1)}(0)}{(d+1)!} a^{d+1} + \frac{1}{d!} \int_0^a (F_d^{(d+1)}(b) - F_d^{(d+1)}(0)) (a-b)^d db \\ &\geq F_d(0) + \frac{\beta_d}{(d+1)!} a^{d+1} - \frac{L_d}{d!} \int_0^a b (a-b)^d db \\ &= F_d(0) + \frac{\beta_d}{(d+1)!} a^{d+1} - \frac{L_d}{(d+2)!} a^{d+2}. \end{aligned}$$

Hence

$$F_d(a) \geq F_d(0) + \frac{1}{2} \frac{\beta_d}{(d+1)!} a^{d+1}, \quad \forall a \in [0, \tilde{\delta}_d], \text{ where } \tilde{\delta}_d := 1 \wedge \left(\frac{d+2}{2} \frac{\beta_d}{L_d} \right).$$

This proves (C.9). Now, we notice that from (C.8) we have

$$G_a(\mathbf{y}) \geq G_a(\mathbf{0}) + \frac{1}{4} \beta_d |\mathbf{y}|^{d+1}, \quad \forall |\mathbf{y}| \leq \delta_d, \forall a \in [0, 2C_\zeta].$$

Moreover, since G_a is a convex function, it follows that

$$\begin{aligned} G_a(\mathbf{y}) &\geq \min \left(G_a(\mathbf{0}) + \frac{1}{4} \beta_d |\mathbf{y}|^{d+1}, G_a(\mathbf{0}) + \frac{1}{4} \beta_d \delta_d^d |\mathbf{y}| \right) \\ &\geq \min \left(G_a(\mathbf{0}) + \frac{1}{4} \beta_d \delta_d^d |\mathbf{y}|^{d+1}, G_a(\mathbf{0}) + \frac{1}{4} \beta_d \delta_d^d |\mathbf{y}| \right) \\ &= G_a(\mathbf{0}) + \frac{1}{4} \beta_d \delta_d^d (|\mathbf{y}|^{d+1} \wedge |\mathbf{y}|), \quad \forall \mathbf{y} \in \mathbb{R}^d. \end{aligned} \quad (\text{C.10})$$

Proceeding along the same lines, we deduce by (C.9) that

$$F_d(a) \geq F_d(0) + \frac{1}{2} \frac{\beta_d}{(d+1)!} \tilde{\delta}_d^d (a^{d+1} \wedge a), \quad \forall a \geq 0. \quad (\text{C.11})$$

So, in particular, since $G_a(\mathbf{0}) = F_d(a)$ and $F_d(0) = 0$, we obtain, from (C.10) and (C.11),

$$G_a(\mathbf{y}) \geq \frac{1}{2} \frac{\beta_d}{(d+1)!} \tilde{\delta}_d^d (a^{d+1} \wedge a) + \frac{1}{2} \frac{\beta_d}{(d+1)!} \delta_d^d (|\mathbf{y}|^{d+1} \wedge |\mathbf{y}|), \quad \forall (a, \mathbf{y}) \in [0, 2C_\zeta] \times \mathbb{R}^d,$$

which proves (C.3) with $\alpha_d := \frac{1}{2} \frac{\beta_d}{(d+1)!} (\tilde{\delta}_d^d \wedge \delta_d^d)$ for the case $a \leq 2C_\zeta$. It remains to prove (C.6) and (C.7).

STEP VI-3. *Proof of (C.7).* From the definition (C.5) of F_d we see that F_d is continuous. Moreover, by direct calculation we find

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{F_d(a+h) - F_d(a)}{h} &= \int_{|\mathbf{z}| \leq a} \zeta(\mathbf{z}) d\mathbf{z}, \quad \forall a \geq 0, \\ \lim_{h \rightarrow 0^+} \frac{F_d(a-h) - F_d(a)}{-h} &= \int_{|\mathbf{z}| \leq a} \zeta(\mathbf{z}) d\mathbf{z}, \quad \forall a > 0. \end{aligned}$$

Hence, the first derivative of F_d exists everywhere and is given by $F_d'(a) = \int_{|\mathbf{z}| \leq a} \zeta(\mathbf{z}) d\mathbf{z}$, $\forall a \geq 0$. Notice that $F_d'(0) = 0$. We also see that F_d' is continuous on $[0, +\infty)$. Now, for every $r > 0$ let $S_{d-1}(r)$ denote the surface area of the boundary of the ball $\{\mathbf{z} \in \mathbb{R}^d : |\mathbf{z}| \leq r\}$, which is given by $S_{d-1}(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}$, where $\Gamma(\cdot)$ is the Gamma function. Then, recalling that ζ is a radial function and using d -dimensional spherical coordinates (see for instance Appendix C.3 in [37]), we get

$$F_d'(a) = \int_0^a \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{1}{2}r^2} S_{d-1}(r) dr, \quad \forall a \geq 0.$$

So, in particular, the second derivative of F_d exists everywhere and is given by

$$F_d''(a) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{1}{2}a^2} S_{d-1}(a) = \frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})} a^{d-1} e^{-\frac{1}{2}a^2}, \quad \forall a \geq 0.$$

We deduce that $F_d \in C^\infty([0, +\infty))$. We also observe that every derivative of F_d is bounded, so in particular $F_d^{(d+1)}$ is Lipschitz. As a consequence, there exists $L_d > 0$ such that

$$F_d^{(d+1)}(a) - F_d^{(d+1)}(0) \geq -L_d a, \quad \forall a \geq 0.$$

Finally, let us prove by induction on d that $F'_d(0) = \dots = F_d^{(d)}(0) = 0$ and $F_d^{(d+1)}(0) > 0$. For $d = 1$ we have, by direct calculation, $F_1(0) = F'_1(0) = 0$ and $F''_1(0) = 1/\sqrt{2\pi} > 0$. Let us now suppose that the claim holds true for F_d , for some $d \geq 1$, and let us prove it for F_{d+1} . By the explicit expressions of F_{d+1} and F'_{d+1} we see that $F_{d+1}(0) = F'_{d+1}(0) = 0$. Moreover

$$F''_{d+1}(a) = C_{d+1} a^d e^{-\frac{1}{2}a^2},$$

where $C_{d+1} := 2^{1-(d+1)/2}/\Gamma((d+1)/2) > 0$. So, in particular, $F''_{d+1}(0) = 0$. Now, we observe that

$$F'''_{d+1}(a) = d C_{d+1} a^{d-1} e^{-\frac{1}{2}a^2} - C_{d+1} a^{d+1} e^{-\frac{1}{2}a^2} = \frac{C_{d+1}}{C_d} (d - a^2) F''_d(a),$$

where $C_d := 2^{1-d/2}/\Gamma(d/2) > 0$. Therefore

$$F^{\text{iv}}_{d+1}(a) = \frac{C_{d+1}}{C_d} ((d - a^2) F'''_d(a) - 2a F''_d(a)).$$

Moreover, by the general Leibniz rule, we have

$$F_{d+1}^{(3+n)}(a) = \frac{C_{d+1}}{C_d} \sum_{k=0}^n \binom{n}{k} (d - a^2)^{(n-k)} F_d^{(2+k)}(a), \quad \text{for every } n \geq 2,$$

where $(d - a^2)^{(n-k)}$ denotes the $(n - k)$ -th derivative of the map $a \mapsto d - a^2$. Since $(d - a^2)^{(n-k)}$ is identically equal to zero whenever $n - k \geq 3$, it follows that

$$\begin{aligned} F_{d+1}^{(3+n)}(a) &= \frac{C_{d+1}}{C_d} \left(\binom{n}{n-2} (d - a^2)^{(2)} F_d^{(2+n-2)}(a) \right. \\ &\quad \left. + \binom{n}{n-1} (d - a^2)^{(1)} F_d^{(2+n-1)}(a) + \binom{n}{n} (d - a^2) F_d^{(2+n)}(a) \right) \\ &= \frac{C_{d+1}}{C_d} (-n(n-1) F_d^{(2+n-2)}(a) - 2na F_d^{(2+n-1)}(a) + (d - a^2) F_d^{(2+n)}(a)). \end{aligned}$$

In conclusion, we have

$$\begin{aligned} F'''_{d+1}(0) &= \frac{C_{d+1}}{C_d} d F''_d(0), \\ F^{\text{iv}}_{d+1}(0) &= \frac{C_{d+1}}{C_d} d F'''_d(0), \\ F_{d+1}^{(3+n)}(0) &= \frac{C_{d+1}}{C_d} (-n(n-1) F_d^{(2+n-2)}(0) + d F_d^{(2+n)}(0)), \quad \text{for every } n \geq 2. \end{aligned}$$

From the formulae above it is straightforward to see that the claim holds. This concludes the proof of (C.7).

STEP VI-4. *Proof of (C.6).* From the definition (C.4) of G_a we see that $G_a \in C^\infty(\mathbb{R}^d)$. Moreover, we have, for every $i, j = 1, \dots, d$,

$$\begin{aligned}\partial_{y_i} G_a(\mathbf{y}) &= - \int_{\mathbb{R}^d} \max\{a, |\mathbf{y} - \mathbf{z}|\} \partial_{z_i} \zeta(\mathbf{z}) d\mathbf{z} = \int_{\mathbb{R}^d} \max\{a, |\mathbf{y} - \mathbf{z}|\} z_i \zeta(\mathbf{z}) d\mathbf{z}, \\ \partial_{y_i y_j} G_a(\mathbf{y}) &= \int_{\mathbb{R}^d} \max\{a, |\mathbf{y} - \mathbf{z}|\} \partial_{z_i z_j} \zeta(\mathbf{z}) d\mathbf{z} = \int_{\mathbb{R}^d} \max\{a, |\mathbf{y} - \mathbf{z}|\} (z_i z_j - \delta_{ij}) \zeta(\mathbf{z}) d\mathbf{z},\end{aligned}$$

where δ_{ij} is the Kronecker delta. Since ζ is a radial function, we have $\zeta(\mathbf{z}) = \zeta(-\mathbf{z})$, for every $\mathbf{z} \in \mathbb{R}^d$, therefore $\partial_{y_i} G_a(\mathbf{0}) = 0$.

We now prove that for every fixed d there exists $L_d > 0$ such that, for every $a \geq 0$, we have

$$\partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{y}) - \partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{0}) \geq -L_d |\mathbf{y}| I, \quad \forall \mathbf{y} \in \mathbb{R}^d,$$

which can be equivalently written as

$$\sum_{i,j=1}^d (\partial_{y_i y_j} G_a(\mathbf{y}) - \partial_{y_i y_j} G_a(\mathbf{0})) w_i w_j \geq -L_d |\mathbf{y}| |\mathbf{w}|^2, \quad \forall \mathbf{y}, \mathbf{w} \in \mathbb{R}^d, \quad (\text{C.12})$$

where y_i (resp. w_i) denotes the i -th component of \mathbf{y} (resp. \mathbf{w}). We start noticing that, for every $i, j = 1, \dots, d$, we have (we use the elementary inequality $|\max\{a, b+c\} - \max\{a, c\}| \leq |b|$, valid for every $a, c \geq 0$ and $b \in \mathbb{R}$, with $b = |\mathbf{y} - \mathbf{z}| - |\mathbf{z}|$ and $c = |\mathbf{z}|$)

$$\begin{aligned}|\partial_{y_i y_j} G_a(\mathbf{y}) - \partial_{y_i y_j} G_a(\mathbf{0})| &\leq \int_{\mathbb{R}^d} |\max\{a, |\mathbf{y} - \mathbf{z}|\} - \max\{a, |\mathbf{z}|\}| |\partial_{z_i z_j} \zeta(\mathbf{z})| d\mathbf{z} \\ &= \int_{\mathbb{R}^d} |\max\{a, |\mathbf{y} - \mathbf{z}| - |\mathbf{z}| + |\mathbf{z}|\} - \max\{a, |\mathbf{z}|\}| |\partial_{z_i z_j} \zeta(\mathbf{z})| d\mathbf{z} \\ &\leq \int_{\mathbb{R}^d} |\mathbf{y}| |\partial_{z_i z_j} \zeta(\mathbf{z})| d\mathbf{z} = \frac{L_d}{d} |\mathbf{y}|,\end{aligned}$$

with $\frac{L_d}{d} = \int_{\mathbb{R}^d} |\partial_{z_i z_j} \zeta(\mathbf{z})| d\mathbf{z}$. Then, for every $\mathbf{w} \in \mathbb{R}^d$, we obtain

$$\left| \sum_{i,j=1}^d (\partial_{y_i y_j} G_a(\mathbf{y}) - \partial_{y_i y_j} G_a(\mathbf{0})) w_i w_j \right| \leq \frac{L_d}{d} |\mathbf{y}| \sum_{i,j=1}^d |w_i| |w_j| \leq L_d |\mathbf{y}| |\mathbf{w}|^2,$$

which proves (C.12).

Finally, we prove that for every fixed d there exists $\hat{\beta}_d > 0$ such that, for every $a \in [0, 2C_\zeta]$,

$$\partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{0}) \geq \hat{\beta}_d I. \quad (\text{C.13})$$

As a matter of fact, for every $\mathbf{w} \in \mathbb{R}^d$, we have

$$\langle \partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{0}) \mathbf{w}, \mathbf{w} \rangle = \sum_{i,j=1}^d \partial_{y_i y_j} G_a(\mathbf{0}) w_i w_j = \sum_{i,j=1}^d w_i w_j \int_{\mathbb{R}^d} \max\{a, |\mathbf{z}|\} (z_i z_j - \delta_{ij}) \zeta(\mathbf{z}) d\mathbf{z}$$

$$= \sum_{i=1}^d w_i^2 \int_{\mathbb{R}^d} \max\{a, |\mathbf{z}|\} (z_1^2 - 1) \zeta(\mathbf{z}) d\mathbf{z} - \sum_{i \neq j} w_i w_j \int_{\mathbb{R}^d} \max\{a, |\mathbf{z}|\} z_1 z_2 \zeta(\mathbf{z}) d\mathbf{z}.$$

Now, notice that $\int_{\mathbb{R}^d} \max\{a, |\mathbf{z}|\} z_1 z_2 \zeta(\mathbf{z}) d\mathbf{z} = 0$, for every $a \geq 0$. Hence

$$\begin{aligned} \langle \partial_{\mathbf{y}\mathbf{y}} G_a(\mathbf{0}) \mathbf{w}, \mathbf{w} \rangle &= |\mathbf{w}|^2 \int_{\mathbb{R}^d} \max\{a, |\mathbf{z}|\} (z_1^2 - 1) \zeta(\mathbf{z}) d\mathbf{z} \\ &= |\mathbf{w}|^2 \frac{1}{d} \sum_{i=1}^d \int_{\mathbb{R}^d} \max\{a, |\mathbf{z}|\} (z_i^2 - 1) \zeta(\mathbf{z}) d\mathbf{z} \\ &= |\mathbf{w}|^2 \frac{1}{d} \int_{\mathbb{R}^d} \max\{a, |\mathbf{z}|\} (|\mathbf{z}|^2 - d) \zeta(\mathbf{z}) d\mathbf{z}. \end{aligned}$$

Let $H: [0, +\infty) \rightarrow \mathbb{R}$ be defined as

$$H(a) := \int_{\mathbb{R}^d} \max\{a, |\mathbf{z}|\} (|\mathbf{z}|^2 - d) \zeta(\mathbf{z}) d\mathbf{z}, \quad \forall a \in [0, +\infty).$$

Notice that (C.13) follows if we prove the following (actually, it would be enough to require that $H(a) > 0$ for every $a \geq 0$ and H decreasing; (C.14) is a sufficient condition for this):

$$H(0) > 0, \quad \lim_{a \rightarrow +\infty} H(a) = 0, \quad H \text{ is a strictly decreasing function.} \quad (\text{C.14})$$

As a matter of fact, if (C.14) holds, then

$$\inf_{a \in [0, 2C_\zeta]} H(a) = H(2C_\zeta) > 0,$$

from which (C.13) follows with $\hat{\beta}_d = \frac{1}{d} H(2C_\zeta)$.

It remains to prove (C.14). Denoting by $\mu_{\chi^2(d), p}$ the moment of order $p > 0$ of a χ^2 -distribution with d degrees of freedom, and recalling that $\Gamma(\cdot)$ is the Gamma function, we have

$$H(0) = \mu_{\chi^2(d), \frac{3}{2}} - d \mu_{\chi^2(d), \frac{1}{2}} = 2^{\frac{3}{2}} \frac{\Gamma(\frac{d}{2} + \frac{3}{2})}{\Gamma(\frac{d}{2})} - d 2^{\frac{1}{2}} \frac{\Gamma(\frac{d}{2} + \frac{1}{2})}{\Gamma(\frac{d}{2})} = \mu_{\chi^2(d), \frac{1}{2}} > 0.$$

Concerning the function H , we also have that (we perform the change of variables $\mathbf{z} = a\mathbf{w}$ under the integral sign)

$$\lim_{a \rightarrow +\infty} H(a) = \lim_{a \rightarrow +\infty} \int_{\mathbb{R}^d} a^2 \max\{1, |\mathbf{w}|\} (a^2 |\mathbf{w}|^2 - d) \zeta(a\mathbf{w}) d\mathbf{w} = 0,$$

where the limit follows from an application of the Lebesgue dominated convergence theorem.

Now, proceeding as in Step VI-3 for the function F , we deduce that $H \in C^\infty([0, +\infty))$ and, for every $a \geq 0$,

$$H'(a) = \int_{|\mathbf{z}| \leq a} (|\mathbf{z}|^2 - d) \zeta(\mathbf{z}) d\mathbf{z}, \quad H''(a) = \frac{a^2 - d}{(2\pi)^{\frac{d}{2}}} e^{-\frac{1}{2}a^2} S_{d-1}(a),$$

where $S_{d-1}(a)$ denotes the surface area of the boundary of the ball $\{\mathbf{z} \in \mathbb{R}^d : |\mathbf{z}| \leq a\}$. Notice that

$$H'(0) = 0, \quad \lim_{a \rightarrow +\infty} H'(a) = \int_{\mathbb{R}^d} (|\mathbf{z}|^2 - d) \zeta(\mathbf{z}) d\mathbf{z} = \mu_{\chi^2(d),1} - d = 0.$$

Then, we deduce from the sign of H'' that $H'(a) < 0$, for every $a > 0$. This implies that H is a strictly decreasing function and concludes the proof. \square

C.2. Proof of Lemma 3.2

Proof of Lemma 3.2. We split the proof into three steps.

STEP I. *Proof of item 1).* We first notice that $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ is a non-anticipative map. Now, let $(t, \hat{\mathbf{x}}) \in \hat{\mathbf{A}}$ and $\delta > 0$, with $t < T$ and $\delta \leq T - t$, then we have

$$\begin{aligned} & \frac{\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t + \delta, \hat{\mathbf{x}}(\cdot \wedge t)) - \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})}{\delta} \\ &= \int_0^{+\infty} \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))} \frac{\eta(s - \delta) - \eta(s)}{\delta} ds \\ & \quad - \frac{1}{\delta} \int_0^\delta \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))} \eta(s - \delta) ds. \end{aligned}$$

Notice that (writing $\frac{\eta(s - \delta) - \eta(s)}{\delta} = -\int_{s - \delta}^s \eta'(\tilde{s}) d\tilde{s}$, using Fubini's theorem and the integrability of η' , namely $\int_0^{+\infty} |\eta'(s)| ds < +\infty$)

$$\begin{aligned} & \int_0^{+\infty} \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))} \frac{\eta(s - \delta) - \eta(s)}{\delta} ds \\ & \xrightarrow{\delta \rightarrow 0^+} - \int_0^{+\infty} \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))} \eta'(s) ds. \end{aligned}$$

We also have, using that $\frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}} \leq 1$ and $\eta(s - \delta) = (s - \delta) e^{-(s - \delta)} \geq -\delta e^\delta = \eta(-\delta)$, for every $0 \leq s \leq \delta$,

$$\left| \frac{1}{\delta} \int_0^\delta \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t + s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))} \eta(s - \delta) ds \right| \leq \delta e^\delta.$$

This implies that

$$\frac{1}{\delta} \int_0^\delta \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))} \eta(s - \delta) ds \xrightarrow{\delta \rightarrow 0^+} 0.$$

This proves that the horizontal derivative of $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ exists everywhere on $[0, T] \times D([0, T]; \mathbb{R}^d)$ and is given by

$$\partial_t^H \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) = - \int_0^{+\infty} \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))} \eta'(s) ds. \quad (\text{C.15})$$

Moreover, at $t = T$ we have

$$\begin{aligned} \partial_t^H \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(T, \hat{\mathbf{x}}) &= \lim_{t \rightarrow T^-} \partial_t^H \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) \\ &= - \lim_{t \rightarrow T^-} \int_0^{+\infty} \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))} \eta'(s) ds \\ &= - \int_0^{+\infty} \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(T, \hat{\mathbf{x}}(\cdot \wedge T-))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(T, \hat{\mathbf{x}}(\cdot \wedge T-))} \eta'(s) ds, \end{aligned} \quad (\text{C.16})$$

where $\hat{\mathbf{x}}(\cdot \wedge T-)$ is given by (A.1) with $t = T$. In conclusion, the horizontal derivative of $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ exists everywhere on $\hat{\Lambda}$.

Let us now consider the vertical derivatives of $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$. Given $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$, $h \in \mathbb{R} \setminus \{0\}$, and $i = 1, \dots, d$, we have (recalling that $\mathbf{e}_1, \dots, \mathbf{e}_d$ denotes the standard orthonormal basis of \mathbb{R}^d)

$$\begin{aligned} &\frac{\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}} + h \mathbf{e}_i 1_{[t, T]}) - \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})}{h} \\ &= \int_0^{+\infty} \frac{\frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t) + h \mathbf{e}_i 1_{[t, T]})}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t) + h \mathbf{e}_i 1_{[t, T]})} - \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}}{h} \eta(s) ds. \end{aligned}$$

By item 1) of Lemma 3.1, we have that

$$\begin{aligned} &\frac{\frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t) + h \mathbf{e}_i 1_{[t, T]})}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t) + h \mathbf{e}_i 1_{[t, T]})} - \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}}{h} \\ &\xrightarrow{h \rightarrow 0} \frac{(1 + C_\zeta) \partial_{x_i}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t)))^2}. \end{aligned}$$

Then, by the Lebesgue dominated convergence theorem, we obtain

$$\int_0^{+\infty} \frac{\frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t) + h \mathbf{e}_i 1_{[t, T]})}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t) + h \mathbf{e}_i 1_{[t, T]})} - \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}}{h} \eta(s) ds$$

$$\xrightarrow{h \rightarrow 0} \int_0^{+\infty} \frac{(1 + C_\zeta) \partial_{x_i}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t)))^2} \eta(s) ds.$$

This proves that $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ admits first-order vertical derivatives at every $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$, which are given by

$$\partial_{x_i}^V \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) = \int_0^{+\infty} \frac{(1 + C_\zeta) \partial_{x_i}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t)))^2} \eta(s) ds, \quad (\text{C.17})$$

for every $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$ and every $i = 1, \dots, d$. In a similar way we can prove that $\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}$ also admits second-order vertical derivatives at every $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$. In particular, it holds that

$$\begin{aligned} \partial_{x_i x_j}^V \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) &= \int_0^{+\infty} \frac{(1 + C_\zeta) \partial_{x_i x_j}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t)))^2} \eta(s) ds \\ &\quad - 2 \int_0^{+\infty} \frac{(1 + C_\zeta) \partial_{x_i}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t)) \partial_{x_j}^V \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t))}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+s) \wedge T, \hat{\mathbf{x}}(\cdot \wedge t)))^3} \eta(s) ds, \end{aligned} \quad (\text{C.18})$$

for every $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$ and every $i, j = 1, \dots, d$.

STEP II. *Proof of item 2).* We prove the continuity of the map $((t_0, \mathbf{x}_0), (t, \hat{\mathbf{x}})) \mapsto \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$ on $\Lambda \times \hat{\Lambda}$, which, in particular, implies the continuity of the map $(t, \hat{\mathbf{x}}) \mapsto \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$ on $\hat{\Lambda}$. In a similar way, we can prove the continuity on $\hat{\Lambda}$ of the pathwise derivatives $(t, \hat{\mathbf{x}}) \mapsto \partial_t^H \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$, $(t, \hat{\mathbf{x}}) \mapsto \partial_{x_i}^V \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$, $(t, \hat{\mathbf{x}}) \mapsto \partial_{x_i x_j}^V \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$, for every $i, j = 1, \dots, d$.

Let us prove that the map $((t_0, \mathbf{x}_0), (t, \hat{\mathbf{x}})) \mapsto \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$ is continuous on $\Lambda \times \hat{\Lambda}$. Fix $(t, \hat{\mathbf{x}}), (s, \hat{\mathbf{y}}) \in \hat{\Lambda}$ and $(t_0, \mathbf{x}_0), (s_0, \mathbf{y}_0) \in \Lambda$. We observe that, for every $\mathbf{z} \in \mathbb{R}^d$, we have

$$\begin{aligned} &\|\hat{\mathbf{x}}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z}1_{[t, T]}\|_\infty \\ &= \max \left(\sup_{0 \leq r < t} |\hat{\mathbf{x}}(r) - \mathbf{x}_0(r \wedge t_0)|, \max_{t \leq r \leq T} |\hat{\mathbf{x}}(t) - \mathbf{x}_0(r \wedge t_0) - \mathbf{z}| \right) \\ &= \max \left(\|\hat{\mathbf{x}}(\cdot \wedge t-) - \mathbf{x}_0(\cdot \wedge t \wedge t_0)\|_\infty, \|\hat{\mathbf{x}}(t) - \mathbf{x}_0(t \vee (r \wedge t_0)) - \mathbf{z}\|_\infty \right), \end{aligned}$$

where $\hat{\mathbf{x}}(\cdot \wedge t-)$ is given by (A.1). Similarly

$$\begin{aligned} &\|\hat{\mathbf{y}}(\cdot \wedge s) - \mathbf{y}_0(\cdot \wedge s_0) - \mathbf{z}1_{[s, T]}\|_\infty \\ &= \max \left(\|\hat{\mathbf{y}}(\cdot \wedge s-) - \mathbf{y}_0(\cdot \wedge s \wedge s_0)\|_\infty, \|\hat{\mathbf{y}}(s) - \mathbf{y}_0(s \vee (r \wedge s_0)) - \mathbf{z}\|_\infty \right). \end{aligned}$$

As a consequence, using the elementary inequality $\max\{a, b\} - \max\{c, d\} \leq \max\{a - c, b - d\}$, valid for every $a, b, c, d \in \mathbb{R}$, we obtain (recall that the function $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ is defined by

(3.2))

$$\begin{aligned}
& \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) - \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}}) \\
&= \int_{\mathbb{R}^d} \left(\|\hat{\mathbf{x}}(\cdot \wedge t) - \mathbf{x}_0(\cdot \wedge t_0) - \mathbf{z} 1_{[t, T]}\|_\infty - \|\hat{\mathbf{y}}(\cdot \wedge s) - \mathbf{y}_0(\cdot \wedge s_0) - \mathbf{z} 1_{[s, T]}\|_\infty \right) \zeta(\mathbf{z}) d\mathbf{z} \\
&= \int_{\mathbb{R}^d} \left(\max \left(\|\hat{\mathbf{x}}(\cdot \wedge t-) - \mathbf{x}_0(\cdot \wedge t \wedge t_0)\|_\infty, \|\hat{\mathbf{x}}(t) - \mathbf{x}_0(t \vee (r \wedge t_0)) - \mathbf{z}\|_\infty \right) \right. \\
&\quad \left. - \max \left(\|\hat{\mathbf{y}}(\cdot \wedge s-) - \mathbf{y}_0(\cdot \wedge s \wedge s_0)\|_\infty, \|\hat{\mathbf{y}}(s) - \mathbf{y}_0(s \vee (r \wedge s_0)) - \mathbf{z}\|_\infty \right) \right) \zeta(\mathbf{z}) d\mathbf{z} \\
&\leq \int_{\mathbb{R}^d} \left(\max \left(\|\hat{\mathbf{x}}(\cdot \wedge t-) - \mathbf{x}_0(\cdot \wedge t \wedge t_0)\|_\infty - \|\hat{\mathbf{y}}(\cdot \wedge s-) - \mathbf{y}_0(\cdot \wedge s \wedge s_0)\|_\infty, \|\hat{\mathbf{x}}(t) \right. \right. \\
&\quad \left. \left. - \mathbf{x}_0(t \vee (r \wedge t_0)) - \mathbf{z}\|_\infty - \mathbf{z} - \|\hat{\mathbf{y}}(s) - \mathbf{y}_0(s \vee (r \wedge s_0)) - \mathbf{z}\|_\infty \right) \right) \zeta(\mathbf{z}) d\mathbf{z}.
\end{aligned}$$

Now, notice that

$$\begin{aligned}
& \|\hat{\mathbf{x}}(\cdot \wedge t-) - \mathbf{x}_0(\cdot \wedge t \wedge t_0)\|_\infty - \|\hat{\mathbf{y}}(\cdot \wedge s-) - \mathbf{y}_0(\cdot \wedge s \wedge s_0)\|_\infty \\
&\leq \|\hat{\mathbf{x}}(\cdot \wedge t-) - \mathbf{x}_0(\cdot \wedge t \wedge t_0) - \hat{\mathbf{y}}(\cdot \wedge s-) + \mathbf{y}_0(\cdot \wedge s \wedge s_0)\|_\infty \\
&\leq \|\hat{\mathbf{x}}(\cdot \wedge t-) - \hat{\mathbf{y}}(\cdot \wedge s-)\|_\infty + \|\mathbf{x}_0(\cdot \wedge t \wedge t_0) - \mathbf{y}_0(\cdot \wedge s \wedge s_0)\|_\infty.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \|\hat{\mathbf{x}}(t) - \mathbf{x}_0(t \vee (r \wedge t_0)) - \mathbf{z}\|_\infty - \|\hat{\mathbf{y}}(s) - \mathbf{y}_0(s \vee (r \wedge s_0)) - \mathbf{z}\|_\infty \\
&\leq \|\hat{\mathbf{x}}(t) - \mathbf{x}_0(t \vee (\cdot \wedge t_0)) - \hat{\mathbf{y}}(s) + \mathbf{y}_0(s \vee (\cdot \wedge s_0))\|_\infty \\
&\leq |\hat{\mathbf{x}}(t) - \hat{\mathbf{y}}(s)| + \|\mathbf{x}_0(t \vee (\cdot \wedge t_0)) - \mathbf{y}_0(s \vee (\cdot \wedge s_0))\|_\infty.
\end{aligned}$$

In conclusion, we obtain

$$\begin{aligned}
& |\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) - \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}})| \leq \|\hat{\mathbf{x}}(\cdot \wedge t-) - \hat{\mathbf{y}}(\cdot \wedge s-)\|_\infty + |\hat{\mathbf{x}}(t) - \hat{\mathbf{y}}(s)| \\
&+ \|\mathbf{x}_0(\cdot \wedge t \wedge t_0) - \mathbf{y}_0(\cdot \wedge s \wedge s_0)\|_\infty + \|\mathbf{x}_0(t \vee (\cdot \wedge t_0)) - \mathbf{y}_0(s \vee (\cdot \wedge s_0))\|_\infty.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \left| \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})} - \frac{\hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}})}{1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}})} \right| \\
&= \frac{(1 + C_\zeta) |\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) - \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}})|}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}}))} \\
&\leq (1 + C_\zeta) \frac{\|\hat{\mathbf{x}}(\cdot \wedge t-) - \hat{\mathbf{y}}(\cdot \wedge s-)\|_\infty + |\hat{\mathbf{x}}(t) - \hat{\mathbf{y}}(s)|}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}}))} \\
&\quad + (1 + C_\zeta) \frac{\|\mathbf{x}_0(\cdot \wedge t \wedge t_0) - \mathbf{y}_0(\cdot \wedge s \wedge s_0)\|_\infty + \|\mathbf{x}_0(t \vee (\cdot \wedge t_0)) - \mathbf{y}_0(s \vee (\cdot \wedge s_0))\|_\infty}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}}))}.
\end{aligned}$$

By the latter inequality, we obtain (denoting $\hat{\mathbf{x}}^t(\cdot) := \hat{\mathbf{x}}(\cdot \wedge t)$ and $\hat{\mathbf{y}}^s(\cdot) := \hat{\mathbf{y}}(\cdot \wedge s)$)

$$\begin{aligned}
& |\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) - \hat{\chi}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}})| \tag{C.19} \\
& \leq \int_0^{+\infty} \left| \frac{\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t)}{1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t)} - \frac{\hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s)}{1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s)} \right| \eta(r) dr \\
& \leq \int_0^{+\infty} \frac{(1 + C_\zeta) \|\hat{\mathbf{x}}^t(\cdot \wedge ((t+r) \wedge T)-) - \hat{\mathbf{y}}^s(\cdot \wedge ((s+r) \wedge T)-)\|_\infty}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s))} \eta(r) dr \\
& + \int_0^{+\infty} \frac{(1 + C_\zeta) |\hat{\mathbf{x}}^t((t+r) \wedge T) - \hat{\mathbf{y}}^s((s+r) \wedge T)|}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s))} \eta(r) dr \\
& + \int_0^{+\infty} \frac{(1 + C_\zeta) \|\mathbf{x}_0(\cdot \wedge (t+r) \wedge t_0) - \mathbf{y}_0(\cdot \wedge (s+r) \wedge s_0)\|_\infty}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s))} \eta(r) dr \\
& + \int_0^{+\infty} \frac{(1 + C_\zeta) \|\mathbf{x}_0(((t+r) \wedge T) \vee (\cdot \wedge t_0)) - \mathbf{y}_0(((s+r) \wedge T) \vee (\cdot \wedge s_0))\|_\infty}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s))} \eta(r) dr,
\end{aligned}$$

where $\hat{\mathbf{x}}^t(\cdot \wedge ((t+r) \wedge T)-)$ and $\hat{\mathbf{y}}^s(\cdot \wedge ((s+r) \wedge T)-)$ are defined as in (A.1). Notice that, for every fixed $r \geq 0$,

$$|\hat{\mathbf{x}}^t((t+r) \wedge T) - \hat{\mathbf{y}}^s((s+r) \wedge T)| = |\hat{\mathbf{x}}(t) - \hat{\mathbf{y}}(s)|. \tag{C.20}$$

On the other hand, concerning the term $\|\hat{\mathbf{x}}^t(\cdot \wedge ((t+r) \wedge T)-) - \hat{\mathbf{y}}^s(\cdot \wedge ((s+r) \wedge T)-)\|_\infty$ we distinguish four cases, for every fixed $r > 0$:

- if both $t < T$ and $s < T$, then $\|\hat{\mathbf{x}}^t(\cdot \wedge ((t+r) \wedge T)-) - \hat{\mathbf{y}}^s(\cdot \wedge ((s+r) \wedge T)-)\|_\infty = \|\hat{\mathbf{x}}(\cdot \wedge t) - \hat{\mathbf{y}}(\cdot \wedge s)\|_\infty$;
- if $t = T$ and $s < T$, it holds that

$$\begin{aligned}
& \|\hat{\mathbf{x}}^t(\cdot \wedge ((t+r) \wedge T)-) - \hat{\mathbf{y}}^s(\cdot \wedge ((s+r) \wedge T)-)\|_\infty \\
& = \|\hat{\mathbf{x}}(\cdot \wedge T-) - \hat{\mathbf{y}}(\cdot \wedge s)\|_\infty \leq \|\hat{\mathbf{x}}(\cdot) - \hat{\mathbf{y}}(\cdot \wedge s)\|_\infty;
\end{aligned}$$

- similarly to the previous case, if $t < T$ and $s = T$, then $\|\hat{\mathbf{x}}^t(\cdot \wedge ((t+r) \wedge T)-) - \hat{\mathbf{y}}^s(\cdot \wedge ((s+r) \wedge T)-)\|_\infty \leq \|\hat{\mathbf{x}}(\cdot \wedge t) - \hat{\mathbf{y}}(\cdot)\|_\infty$;
- finally, if both $t = T$ and $s = T$, it holds that

$$\begin{aligned}
& \|\hat{\mathbf{x}}^t(\cdot \wedge ((t+r) \wedge T)-) - \hat{\mathbf{y}}^s(\cdot \wedge ((s+r) \wedge T)-)\|_\infty \\
& = \|\hat{\mathbf{x}}(\cdot \wedge T-) - \hat{\mathbf{y}}(\cdot \wedge T-)\|_\infty \leq \|\hat{\mathbf{x}}(\cdot) - \hat{\mathbf{y}}(\cdot)\|_\infty.
\end{aligned}$$

In conclusion, for every $r > 0$ and any $t, s \in [0, T]$, it holds that,

$$\|\hat{\mathbf{x}}^t(\cdot \wedge ((t+r) \wedge T)-) - \hat{\mathbf{y}}^s(\cdot \wedge ((s+r) \wedge T)-)\|_\infty \leq \|\hat{\mathbf{x}}(\cdot \wedge t) - \hat{\mathbf{y}}(\cdot \wedge s)\|_\infty, \tag{C.21}$$

where $\hat{\mathbf{x}}(\cdot \wedge t) = \hat{\mathbf{x}}(\cdot)$ if $t = T$ and, similarly, $\hat{\mathbf{y}}(\cdot \wedge s) = \hat{\mathbf{y}}(\cdot)$ if $s = T$.

Plugging (C.20) and (C.21) into (C.19), we conclude that

$$|\hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}}) - \hat{\chi}_\infty^{(s_0, \mathbf{y}_0)}(s, \hat{\mathbf{y}})|$$

$$\begin{aligned}
&\leq \int_0^{+\infty} \frac{(1 + C_\zeta) \|\hat{\mathbf{x}}(\cdot \wedge t) - \hat{\mathbf{y}}(\cdot \wedge s)\|_\infty}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s))} \eta(r) dr \\
&+ \int_0^{+\infty} \frac{(1 + C_\zeta) |\hat{\mathbf{x}}(t) - \hat{\mathbf{y}}(s)|}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s))} \eta(r) dr \\
&+ \int_0^{+\infty} \frac{(1 + C_\zeta) \|\mathbf{x}_0(\cdot \wedge (t+r) \wedge t_0) - \mathbf{y}_0(\cdot \wedge (s+r) \wedge s_0)\|_\infty}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s))} \eta(r) dr \\
&+ \int_0^{+\infty} \frac{(1 + C_\zeta) \|\mathbf{x}_0(((t+r) \wedge T) \vee (\cdot \wedge t_0)) - \mathbf{y}_0(((s+r) \wedge T) \vee (\cdot \wedge s_0))\|_\infty}{(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}((t+r) \wedge T, \hat{\mathbf{x}}^t))(1 + C_\zeta + \hat{\kappa}_\infty^{(s_0, \mathbf{y}_0)}((s+r) \wedge T, \hat{\mathbf{y}}^s))} \eta(r) dr.
\end{aligned}$$

This proves the continuity of the map $((t_0, \mathbf{x}_0), (t, \hat{\mathbf{x}})) \mapsto \hat{\chi}_\infty^{(t_0, \mathbf{x}_0)}(t, \hat{\mathbf{x}})$ on $\mathbf{\Lambda} \times \hat{\mathbf{\Lambda}}$.

STEP III. *Proof of items 3) and 4).* Regarding item 3), the bound for the horizontal derivative follows from formulae (C.15)-(C.16), from the fact that $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}/(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}) \leq 1$, and also from equality $\int_0^{+\infty} |\eta'(s)| ds = \frac{2}{e}$. On the other hand, the bounds concerning the vertical derivatives follow from formulae (C.17) and (C.18), from the bounds on the vertical derivatives of $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}$ given in item 2) of Lemma 3.1, and also from inequality $1/(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}) \leq 1$ (here we use that $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)} \geq -C_\zeta$, see item 3 of Lemma 3.1).

Finally, regarding item 4), the first inequality is a direct consequence of the two inequalities in (3.4). On the other hand, the second inequality follows from the second inequality in (3.4) together with the fact that $1/(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}) \leq 1$, and also from the inequality $\hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}/(1 + C_\zeta + \hat{\kappa}_\infty^{(t_0, \mathbf{x}_0)}) \leq 1$. \square

Appendix D: Cylindrical approximation

In the present Appendix we state two results already proved in [18, 19], namely Theorem 3.5 in [18] and Theorem 3.12 in [19], which correspond respectively to Lemma D.1 and Lemma D.2 below. Notice however that in [18, 19] the pathwise derivatives are defined in an alternative manner. For this reason, in order to help the reader, we prefer to report the proof of these two results in the present setting.

D.1. The deterministic calculus via regularization

We begin recalling some results from the deterministic calculus of regularization, as developed in Section 3.2 of [25] and Section 2.2 of [18], for which we refer for all the details. The only difference with respect to [25] and [18] being that here we consider \mathbb{R}^d -valued paths (with d not necessarily equal to 1), even if, as usual, we rely on the one-dimensional theory, as we work component by component.

Firstly, for every $t \geq 0$ and any function $\mathbf{f}: [0, t] \rightarrow \mathbb{R}^d$ we define the following extensions to the entire real line:

$$\mathbf{f}_{(0,t]}(s) := \begin{cases} \mathbf{0}, & s > t, \\ \mathbf{f}(s), & s \in [0, t], \\ \mathbf{f}(0), & s < 0, \end{cases} \quad \mathbf{f}_{[0,t]}(s) := \begin{cases} \mathbf{f}(t), & s > t, \\ \mathbf{f}(s), & s \in [0, t], \\ \mathbf{0}, & s < 0. \end{cases}$$

Definition D.1. Let $\mathbf{f}: [0, t] \rightarrow \mathbb{R}^d$ and $g: [0, t] \rightarrow \mathbb{R}$ be càdlàg functions. When the limit

$$\int_{[0,t]} g(s) d^- \mathbf{f}(s) := \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} g_{(0,t]}(s) \frac{\mathbf{f}_{[0,t]}(s + \varepsilon) - \mathbf{f}_{[0,t]}(s)}{\varepsilon} ds$$

exists and it is finite, we denote it by $\int_{[0,t]} g d^- \mathbf{f}$ and call it **forward integral of g with respect to \mathbf{f}** .

We recall from [18], Proposition 2.11, the following integration by parts formula, which will be used several times in this Appendix.

Proposition D.1. Let $\mathbf{f}: [0, t] \rightarrow \mathbb{R}^d$ and $g: [0, t] \rightarrow \mathbb{R}$ be càdlàg functions, with g being of bounded variation. We have the **integration by parts formula**

$$\int_{[0,t]} g(s) d^- \mathbf{f}(s) = g(t) \mathbf{f}(t) - \int_{(0,t]} \mathbf{f}(s) dg(s),$$

where $\int_{(0,t]} \mathbf{f}(s) dg(s)$ is a Lebesgue-Stieltjes integral on $(0, t]$.

D.2. Cylinder terminal condition ξ

Lemma D.1. Suppose that ξ is a cylinder (or tame) function, in the sense that it admits the representation

$$\xi(\mathbf{x}) = g\left(\int_{[0,T]} \psi_0(t) d^- \mathbf{x}(t), \dots, \int_{[0,T]} \psi_n(t) d^- \mathbf{x}(t)\right), \quad \mathbf{x} \in C([0, T]; \mathbb{R}^d),$$

for some non-negative integer n , where we have the following.

- $g: \mathbb{R}^{d(n+1)} \rightarrow \mathbb{R}$ is of class $C^2(\mathbb{R}^{d(n+1)})$ and, together with its first and second-order partial derivatives, satisfies a polynomial growth condition;
- $\psi_0, \dots, \psi_n: [0, T] \rightarrow \mathbb{R}$ are of class $C^2([0, T])$.

Then, the function v defined by (4.4) is in $\mathbf{C}^{1,2}(\Lambda)$ and is a classical (smooth) solution of equation (4.2).

Proof. Let $(t, \mathbf{x}) \in \mathbf{\Lambda}$ and consider $\mathbf{W}^{t, \mathbf{x}} = (\mathbf{W}_s^{t, \mathbf{x}})_{s \in [0, T]}$ given by (4.5). From the definition of v in (4.4), we have

$$\begin{aligned} v(t, \mathbf{x}) &= \mathbb{E} \left[g \left(\int_{[0, T]} \psi_0(s) d^- \mathbf{W}_s^{t, \mathbf{x}}, \dots, \int_{[0, T]} \psi_n(s) d^- \mathbf{W}_s^{t, \mathbf{x}} \right) \right] \\ &= \mathbb{E} \left[g \left(\int_{[0, t]} \psi_0(s) d^- \mathbf{x}(s) + \int_t^T \psi_0(s) d\mathbf{W}_s, \dots \right) \right], \end{aligned}$$

where the second equality follows from the fact that the forward integral coincides with the Itô integral when the integrator is the Brownian motion (or, more generally, a continuous semimartingale), see for instance Proposition 6, Section 3.3, in [85].

In order to prove that $v \in \mathbf{C}^{1,2}(\mathbf{\Lambda})$, we consider the following map $\hat{v}: \hat{\mathbf{\Lambda}} \rightarrow \mathbb{R}$ consistent with v :

$$\hat{v}(t, \hat{\mathbf{x}}) = \mathbb{E} \left[g \left(\int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s) + \int_t^T \psi_0(s) d\mathbf{W}_s, \dots \right) \right],$$

for all $(t, \hat{\mathbf{x}}) \in \hat{\mathbf{\Lambda}}$. Notice that

$$\hat{v}(t, \hat{\mathbf{x}}) = \hat{V} \left(t, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s), \dots \right),$$

where $\hat{V}: [0, T] \times \mathbb{R}^{d(n+1)} \rightarrow \mathbb{R}$ is given by

$$\hat{V}(t, \mathbf{z}) = \mathbb{E} \left[g \left(\mathbf{z}_0 + \int_t^T \psi_0(s) d\mathbf{W}_s, \dots, \mathbf{z}_n + \int_t^T \psi_n(s) d\mathbf{W}_s \right) \right],$$

for all $t \in [0, T]$ and $\mathbf{z} \in \mathbb{R}^{d(n+1)}$, with $\mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_n)$ and $\mathbf{z}_0, \dots, \mathbf{z}_n \in \mathbb{R}^d$. Let $\sigma: [0, T] \rightarrow \mathbb{R}^{d(n+1) \times d}$ be given by

$$\sigma(t) = \begin{bmatrix} \psi_0(t) I \\ \psi_1(t) I \\ \vdots \\ \psi_n(t) I \end{bmatrix} \quad (\text{D.1})$$

where I denotes the $d \times d$ identity matrix. It is well-known (see, for instance, Theorem 5.6.1 in [44]) that $\hat{V} \in C^{1,2}([0, T] \times \mathbb{R}^{d(n+1)})$ and satisfies (here $\partial_t \hat{V}(t, \mathbf{z})$ denotes the time derivative, while $\partial_{\mathbf{z}\mathbf{z}} \hat{V}(t, \mathbf{z})$ is the Hessian matrix of spatial derivatives)

$$\begin{cases} \partial_t \hat{V}(t, \mathbf{z}) + \frac{1}{2} \text{tr}[\sigma(t) \sigma^\top(t) \partial_{\mathbf{z}\mathbf{z}} \hat{V}(t, \mathbf{z})] = 0, & (t, \mathbf{z}) \in [0, T] \times \mathbb{R}^{d(n+1)}, \\ \hat{V}(T, \mathbf{z}) = g(\mathbf{z}), & \mathbf{z} \in \mathbb{R}^{d(n+1)}. \end{cases} \quad (\text{D.2})$$

Let us find the expression of the pathwise derivatives of \hat{v} in terms of \hat{V} . Concerning the horizontal derivative at $(t, \hat{\mathbf{x}}) \in \hat{\mathbf{\Lambda}}$, with $t < T$, we have

$$\partial_t^H \hat{v}(t, \hat{\mathbf{x}}) = \lim_{\delta \rightarrow 0^+} \frac{\hat{v}(t + \delta, \hat{\mathbf{x}}(\cdot \wedge t)) - \hat{v}(t, \hat{\mathbf{x}})}{\delta}$$

$$\begin{aligned}
&= \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \left\{ \hat{V} \left(t + \delta, \int_{[0, t+\delta]} \psi_0(s) d^- \hat{\mathbf{x}}(s \wedge t), \dots \right) - \hat{V} \left(t, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s), \dots \right) \right\} \\
&= \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \left\{ \hat{V} \left(t + \delta, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s), \dots \right) - \hat{V} \left(t, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s), \dots \right) \right\} \\
&= \partial_t \hat{V} \left(t, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s), \dots \right).
\end{aligned}$$

Moreover, at $t = T$ we have

$$\begin{aligned}
\partial_t^H \hat{v}(T, \hat{\mathbf{x}}) &= \lim_{t \rightarrow T^-} \partial_t^H \hat{v}(t, \hat{\mathbf{x}}) = \lim_{t \rightarrow T^-} \partial_t \hat{V} \left(t, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s), \dots \right) \\
&= \partial_t \hat{V} \left(T, \int_{[0, T]} \psi_0(s) d^- \hat{\mathbf{x}}(s) - \psi_0(T) (\hat{\mathbf{x}}(T) - \hat{\mathbf{x}}(T-)), \dots \right),
\end{aligned}$$

where the latter equality follows from the integration by parts formula of Proposition D.1. Concerning the first-order vertical derivatives at $(t, \hat{\mathbf{x}}) \in \hat{\Lambda}$, for every $i = 1, \dots, d$, we have

$$\begin{aligned}
\partial_{x_i}^V \hat{v}(t, \hat{\mathbf{x}}) &= \lim_{h \rightarrow 0} \frac{\hat{v}(t, \hat{\mathbf{x}} + h \mathbf{e}_i 1_{[t, T]}) - \hat{v}(t, \hat{\mathbf{x}})}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left\{ \hat{V} \left(t, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s) + \psi_0(t) h \mathbf{e}_i, \dots \right) \right. \\
&\quad \left. - \hat{V} \left(t, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s), \dots \right) \right\} \\
&= \left\langle \sigma_i(t), \partial_{\mathbf{z}} \hat{V} \left(t, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s), \dots \right) \right\rangle,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{d(n+1)}$, $\sigma_i(t)$ is the i -th column of the matrix $\sigma(t)$ in (D.1), and $\partial_{\mathbf{z}} \hat{V}$ is the gradient of spatial derivatives of \hat{V} .

Concerning the second-order vertical derivatives, it holds that

$$\partial_{\mathbf{xx}}^V \hat{v}(t, \hat{\mathbf{x}}) = \sigma^\top(t) \partial_{\mathbf{zz}} \hat{V} \left(t, \int_{[0, t]} \psi_0(s) d^- \hat{\mathbf{x}}(s), \dots \right) \sigma(t).$$

Since $\hat{V} \in C^{1,2}([0, T] \times \mathbb{R}^{d(n+1)})$, we deduce that $\hat{v} \in C^{1,2}(\hat{\Lambda})$, so that $v \in C^{1,2}(\Lambda)$. Finally, since \hat{V} is a classical (smooth) solution of equation (D.2), using the relations between the pathwise derivatives of \hat{v} (and hence of v) and the derivatives of \hat{V} , we deduce that v is a classical (smooth) solution of equation (4.2). \square

D.3. Cylindrical approximation

Lemma D.2. *Suppose that $\xi: C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}$ is continuous and satisfies the polynomial growth condition*

$$|\xi(\mathbf{x})| \leq M(1 + \|\mathbf{x}\|_\infty^p), \quad \text{for all } \mathbf{x} \in C([0, T]; \mathbb{R}^d),$$

for some positive constants M and p . Then, there exists a sequence $\{\xi_n\}_n$, with ξ_n being a map from $C([0, T]; \mathbb{R}^d)$ into \mathbb{R} , such that the following holds.

- I) $\{\xi_n\}_n$ converges pointwise to ξ as $n \rightarrow +\infty$.
- II) If ξ is bounded then ξ_n is bounded uniformly with respect to n .
- III) For every n , ξ_n is given by

$$\xi_n(\mathbf{x}) = g_n \left(\int_{[0, T]} \psi_0(t) d^- \mathbf{x}(t), \dots, \int_{[0, T]} \psi_n(t) d^- \mathbf{x}(t) \right),$$

where the properties below hold:

- i) for every n , $g_n: \mathbb{R}^{d(n+1)} \rightarrow \mathbb{R}$ is of class $C^\infty(\mathbb{R}^{d(n+1)})$, with partial derivatives of every order satisfying a polynomial growth condition; moreover, g_n satisfies

$$|g_n(\mathbf{z})| \leq M'(1 + |\mathbf{z}|^{p'}), \quad \text{for all } \mathbf{z} \in \mathbb{R}^{d(n+1)},$$

for some positive constants M' and p' , not depending on n ;

- ii) the functions $\psi_\ell: [0, T] \rightarrow \mathbb{R}$, $\ell \geq 0$, satisfy what follows:

- a) ψ_ℓ is of class $C^\infty([0, T])$;
- b) ψ_ℓ is uniformly bounded with respect to ℓ ;
- c) the first derivative of ψ_ℓ is bounded in $L^1([0, T])$, uniformly with respect to ℓ .

Proof. Step I. Let $\{e_\ell\}_{\ell \geq 0}$ be the following orthonormal basis of $L^2([0, T]; \mathbb{R})$:

$$e_0(t) = \frac{1}{\sqrt{T}}, \quad e_{2\ell-1}(t) = \sqrt{\frac{2}{T}} \sin\left(2\ell\pi \frac{t}{T}\right), \quad e_{2\ell}(t) = \sqrt{\frac{2}{T}} \cos\left(2\ell\pi \frac{t}{T}\right),$$

for all $\ell \geq 1$ and $t \in [0, T]$. Consider the linear operator $\Lambda: C([0, T]; \mathbb{R}^d) \rightarrow C([0, T]; \mathbb{R}^d)$ given by

$$(\Lambda \mathbf{x})(t) = \mathbf{x}(T) \frac{t}{T}, \quad \text{for all } t \in [0, T], \mathbf{x} \in C([0, T]; \mathbb{R}^d).$$

Observe that $(\mathbf{x} - \Lambda \mathbf{x})(0) = (\mathbf{x} - \Lambda \mathbf{x})(T) = 0$. Now, for every $n \geq 0$, define

$$\begin{aligned} s_n(\mathbf{x} - \Lambda \mathbf{x}) &= \sum_{\ell=0}^n (\mathbf{x} - \Lambda \mathbf{x})_\ell e_\ell, \\ \sigma_n(\mathbf{x} - \Lambda \mathbf{x}) &= \frac{s_0(\mathbf{x} - \Lambda \mathbf{x}) + \dots + s_n(\mathbf{x} - \Lambda \mathbf{x})}{n+1} = \sum_{\ell=0}^n \frac{n+1-\ell}{n+1} (\mathbf{x} - \Lambda \mathbf{x})_\ell e_\ell, \end{aligned}$$

where $(\mathbf{x} - \Lambda \mathbf{x})_\ell$ is given by

$$\int_0^T (\mathbf{x}(t) - (\Lambda \mathbf{x})(t)) e_\ell(t) dt = \int_0^T \mathbf{x}(t) e_\ell(t) dt - \mathbf{x}(T) \mathcal{E}_\ell(T) + \mathbf{x}(T) \frac{1}{T} \int_0^T \mathcal{E}_\ell(t) dt,$$

with \mathcal{E}_ℓ being a primitive of e_ℓ . In particular, we take

$$\mathcal{E}_0(t) = \frac{t}{\sqrt{T}} - \frac{\sqrt{T}}{2}, \quad \mathcal{E}_{2\ell-1}(t) = -\sqrt{\frac{T}{2}} \frac{1}{\ell\pi} \cos\left(2\ell\pi \frac{t}{T}\right), \quad \mathcal{E}_{2\ell}(t) = \sqrt{\frac{T}{2}} \frac{1}{\ell\pi} \sin\left(2\ell\pi \frac{t}{T}\right),$$

for all $\ell \geq 1$ and $t \in [0, T]$. Then

$$(\mathbf{x} - \Lambda \mathbf{x})_\ell = \int_0^T \mathbf{x}(t) e_\ell(t) dt - \mathbf{x}(T) \mathcal{E}_\ell(T) = - \int_{[0, T]} \mathcal{E}_\ell(t) d^- \mathbf{x}(t),$$

for all $\ell \geq 0$. By Fejér's theorem (see for instance Theorem III.3.4 in [94]), we have

$$\|\sigma_n(\mathbf{x} - \Lambda \mathbf{x}) - (\mathbf{x} - \Lambda \mathbf{x})\|_\infty \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \|\sigma_n(\mathbf{x} - \Lambda \mathbf{x})\|_\infty \leq \|\mathbf{x} - \Lambda \mathbf{x}\|_\infty,$$

for all $\mathbf{x} \in C([0, T]; \mathbb{R}^d)$. Consider the linear operator $T_n: C([0, T]; \mathbb{R}^d) \rightarrow C([0, T]; \mathbb{R}^d)$ given by $(e_{-1}(t) := \frac{t}{T})$, for all $t \in [0, T]$

$$\begin{aligned} T_n \mathbf{x} &= \Lambda \mathbf{x} + \sigma_n(\mathbf{x} - \Lambda \mathbf{x}) - (\sigma_n(\mathbf{x} - \Lambda \mathbf{x}))(0) \\ &= \mathbf{x}(T) e_{-1} + \sum_{\ell=1}^n \frac{n+1-\ell}{n+1} (\mathbf{x} - \Lambda \mathbf{x})_\ell (e_\ell - e_\ell(0)), \end{aligned}$$

for all $n \geq 0$, where for the latter equality we used the fact that e_0 is constant. Then, for any $\mathbf{x} \in C([0, T]; \mathbb{R}^d)$, $\|T_n \mathbf{x} - \mathbf{x}\|_\infty \rightarrow 0$, as n tends to infinity. Furthermore, there exists a positive constant C , independent of n , such that

$$\|T_n \mathbf{x}\|_\infty \leq C \|\mathbf{x}\|_\infty, \quad \text{for all } \mathbf{x} \in C([0, T]; \mathbb{R}^d), \quad n \geq 0. \quad (\text{D.3})$$

Then, we define $\tilde{\xi}_n(\mathbf{x}) := \xi(T_n \mathbf{x})$. Notice that

$$\begin{aligned} \tilde{\xi}_n(\mathbf{x}) &= \xi\left(\mathbf{x}(T) e_{-1} + \sum_{\ell=1}^n \frac{n+1-\ell}{n+1} (\mathbf{x} - \Lambda \mathbf{x})_\ell (e_\ell - e_\ell(0))\right) \\ &= \xi\left(\mathbf{x}(T) e_{-1} - \sum_{\ell=1}^n \frac{n+1-\ell}{n+1} \left(\int_{[0, T]} \mathcal{E}_\ell(t) d^- \mathbf{x}(t)\right) (e_\ell - e_\ell(0))\right) \\ &= \tilde{g}_n\left(\mathbf{x}(T), \int_{[0, T]} \mathcal{E}_1(t) d^- \mathbf{x}(t), \dots, \int_{[0, T]} \mathcal{E}_n(t) d^- \mathbf{x}(t)\right) \\ &= \tilde{g}_n\left(\int_{[0, T]} 1 d^- \mathbf{x}(t), \int_{[0, T]} \mathcal{E}_1(t) d^- \mathbf{x}(t), \dots, \int_{[0, T]} \mathcal{E}_n(t) d^- \mathbf{x}(t)\right), \end{aligned}$$

where the last inequality follows from the identity $\mathbf{x}(T) = \int_{[0, T]} 1 d^- \mathbf{x}(t)$, while \tilde{g}_n is a map from $\mathbb{R}^{d(n+1)}$ into \mathbb{R} given by

$$\tilde{g}_n(\mathbf{z}) := \xi\left(\mathbf{z}_0 e_{-1} - \sum_{\ell=1}^n \frac{n+1-\ell}{n+1} \mathbf{z}_\ell (e_\ell - e_\ell(0))\right), \quad (\text{D.4})$$

for all $\mathbf{z} \in \mathbb{R}^{d(n+1)}$, with $\mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_n)$ and $\mathbf{z}_0, \dots, \mathbf{z}_n \in \mathbb{R}^d$. From now on we denote

$$\psi_0(t) = 1, \quad \psi_\ell(t) = \mathcal{E}_\ell(t), \quad \ell \geq 1. \quad (\text{D.5})$$

Step II. We begin introducing the double sequence $\{g_{n,k}\}_{n \geq 0, k \geq 1}$, with $g_{n,k}: \mathbb{R}^{d(n+1)} \rightarrow \mathbb{R}$ given by

$$g_{n,k}(\mathbf{z}) = \int_{\mathbb{R}^{d(n+1)}} \tilde{g}_n(\mathbf{z} - \mathbf{w}) \zeta_{n,k}(\mathbf{w}) d\mathbf{w},$$

where $\zeta_{n,k}(\mathbf{z}) = k^{d(n+1)} \zeta_n(k\mathbf{z})$, for all $\mathbf{z} \in \mathbb{R}^{d(n+1)}$, with

$$\zeta_n(\mathbf{z}) = c_n \prod_{\ell=0}^n \exp\left(\frac{1}{\mathbf{z}_\ell^2 - 2^{-2\ell}}\right) 1_{\{|\mathbf{z}_\ell| < 2^{-\ell}\}}, \quad \text{for all } \mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_n) \in \mathbb{R}^{d(n+1)}.$$

The constant $c_n > 0$ is such that $\int_{\mathbb{R}^{d(n+1)}} \zeta_n(\mathbf{z}) d\mathbf{z} = 1$. We also introduce the double sequence $\{\xi_{n,k}\}_{n \geq 0, k \geq 1}$, with $\xi_{n,k}: C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}$ given by

$$\xi_{n,k}(\mathbf{x}) = g_{n,k}\left(\int_{[0,T]} \psi_0(t) d^-\mathbf{x}(t), \dots, \int_{[0,T]} \psi_n(t) d^-\mathbf{x}(t)\right), \quad \mathbf{x} \in C([0, T]; \mathbb{R}^d),$$

with ψ_0, \dots, ψ_n as in (D.5). Our aim is to apply Lemma D.1 in [19]. To this end, we need to prove the following items:

- a) $\xi_{n,k}$ is continuous;
- b) for every $\mathbf{x} \in C([0, T]; \mathbb{R}^d)$, $|\xi_{n,k}(\mathbf{x}) - \tilde{\xi}_n(\mathbf{x})| \rightarrow 0$ as $k \rightarrow +\infty$;
- c) $\{\xi_{n,k}\}_{n \geq 0, k \geq 1}$ is equicontinuous on compact sets.

Suppose for a moment that items a)-b)-c) hold true. Then, by Lemma D.1 in [19] we deduce the existence of a subsequence $\{\xi_{n,k_n}\}_n$ converging pointwise to ξ . Hence, we set $\xi_n := \xi_{n,k_n}$ and $g_n := g_{n,k_n}$. It is then easy to see that $\{\xi_n\}_n$ is the claimed sequence. It remains to prove a)-b)-c). As items a) and b) can be easily proved, we only report the proof of item c).

Step III. Let us prove item c). We begin noticing that $g_{n,k}$ can be rewritten as follows:

$$g_{n,k}(\mathbf{z}) = \int_{\mathbf{E}_n} \tilde{g}_n(\mathbf{z} - \mathbf{w}) \zeta_{n,k}(\mathbf{w}) d\mathbf{w},$$

where

$$\mathbf{E}_n := \{\mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_n) \in \mathbb{R}^{d(n+1)} : |\mathbf{z}_\ell| \leq 2^{-\ell}, \ell = 0, \dots, n\}.$$

Then, the claim follows if we prove that for any compact set $\mathbf{K} \subset C([0, T]; \mathbb{R}^d)$ there exists a modulus of continuity $\rho_{\mathbf{K}}$ such that, for every n ,

$$\left| \tilde{g}_n\left(\int_{[0,T]} \psi_0(t) d^-\mathbf{x}(t) + \mathbf{z}_0, \dots, \int_{[0,T]} \psi_n(t) d^-\mathbf{x}(t) + \mathbf{z}_n\right) - \tilde{g}_n\left(\int_{[0,T]} \psi_0(t) d^-\mathbf{x}'(t) + \mathbf{z}_0, \dots, \int_{[0,T]} \psi_n(t) d^-\mathbf{x}'(t) + \mathbf{z}_n\right) \right| \leq \rho_{\mathbf{K}}(\|\mathbf{x} - \mathbf{x}'\|_\infty), \quad (\text{D.6})$$

for all $\mathbf{x}, \mathbf{x}' \in \mathbf{K}$ and $\mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_n) \in \mathbf{E}_n$. Let us prove (D.6). Given a compact set $\mathbf{K} \subset C([0, T]; \mathbb{R}^d)$, we denote

$$\mathcal{K} := \left\{ \mathbf{x} \in C([0, T]; \mathbb{R}^d) : \mathbf{x} = T_n \bar{\mathbf{x}} + \mathbf{z}_0 e_{-1} - \sum_{\ell=1}^n \frac{n+1-\ell}{n+1} \mathbf{z}_\ell (e_\ell - e_\ell(0)), \right. \\ \left. \text{for some } \bar{\mathbf{x}} \in \mathbf{K}, \mathbf{z} \in \mathbf{E}_n, n \geq 0 \right\}.$$

Recalling (D.4), we see that if \mathcal{K} is relatively compact then (D.6) follows from the uniform continuity of ξ on compact sets (which in turn follows from the continuity of ξ). In order to prove that \mathcal{K} is relatively compact, we observe that $\mathcal{K} \subset \mathcal{K}_1 + \mathcal{K}_2 := \{\mathbf{x} \in C([0, T]; \mathbb{R}^d) : \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 \in \mathcal{K}_1, \mathbf{x}_2 \in \mathcal{K}_2\}$, where

$$\mathcal{K}_1 := \{\mathbf{x} \in C([0, T]; \mathbb{R}^d) : \mathbf{x} = T_n \bar{\mathbf{x}}, \text{ for some } \bar{\mathbf{x}} \in \mathbf{K}, n \geq 0\}, \\ \mathcal{K}_2 := \left\{ \mathbf{x} \in C([0, T]; \mathbb{R}^d) : \mathbf{x} = \mathbf{z}_0 e_{-1} - \sum_{\ell=1}^n \frac{n+1-\ell}{n+1} \mathbf{z}_\ell (e_\ell - e_\ell(0)), \text{ for some } n, \mathbf{z} \in \mathbf{E}_n \right\}.$$

If we prove that \mathcal{K}_1 and \mathcal{K}_2 are relatively compact, it follows that \mathcal{K} is also relatively compact.

- \mathcal{K}_1 is relatively compact. Let $\{\mathbf{x}_h\}_h$ be a sequence in \mathcal{K}_1 . Let us prove that, up to a subsequence, $\{\mathbf{x}_h\}_h$ converges. For each h , there exists $\bar{\mathbf{x}}_h \in \mathbf{K}$ and $n_h \geq 0$ such that $\mathbf{x}_h = T_{n_h} \bar{\mathbf{x}}_h$. Suppose that, up to a subsequence, n_h goes to infinity (the proof is simpler when n_h is bounded). Since $\{\bar{\mathbf{x}}_h\}_h \subset \mathbf{K}$, there exists $\bar{\mathbf{x}} \in \mathbf{K}$ such that $\{\bar{\mathbf{x}}_h\}_h$ converges, up to a subsequence, to $\bar{\mathbf{x}}$. Then

$$\|\mathbf{x}_h - \bar{\mathbf{x}}\|_\infty = \|T_{n_h} \bar{\mathbf{x}}_h - \bar{\mathbf{x}}\|_\infty \leq \|T_{n_h} \bar{\mathbf{x}}_h - T_{n_h} \bar{\mathbf{x}}\|_\infty + \|T_{n_h} \bar{\mathbf{x}} - \bar{\mathbf{x}}\|_\infty.$$

By (D.3) and $\|T_{n_h} \bar{\mathbf{x}} - \bar{\mathbf{x}}\|_\infty \rightarrow 0$, the claim follows.

- \mathcal{K}_2 is relatively compact. Let $\{\mathbf{x}_h\}_h$ be a sequence in \mathcal{K}_2 . In order to prove that, up to a subsequence, $\{\mathbf{x}_h\}_h$ is convergent, we start noticing that

$$\mathbf{x}_h = \mathbf{z}_{0,h} e_{-1} - \sum_{\ell=1}^{n_h} \frac{n_h+1-\ell}{n_h+1} \mathbf{z}_{\ell,h} (e_\ell - e_\ell(0)),$$

for some $n_h \geq 0$ and $\mathbf{z}_h = (\mathbf{z}_{0,h}, \dots, \mathbf{z}_{n_h,h}) \in \mathbf{E}_{n_h}$. Suppose that, up to a subsequence, n_h goes to infinity, otherwise the proof is simpler. Notice that, each sequence $\{\mathbf{z}_{\ell,h}\}_h$ converges, up to a subsequence, to some \mathbf{z}_ℓ , with $|\mathbf{z}_\ell| \leq 2^{-\ell}$. By Cantor's diagonal argument, there exists a subsequence of $\{\mathbf{x}_h\}_h$, which we still denote $\{\mathbf{x}_h\}_h$, such that every $\{\mathbf{z}_{\ell,h}\}_h$ converges to \mathbf{z}_ℓ . We construct this subsequence in such a way that, for every h , $|\mathbf{z}_{0,h} - \mathbf{z}_0| + \dots + |\mathbf{z}_{n_h,h} - \mathbf{z}_{n_h}| \leq 1/h$. It follows that $\|\mathbf{x}_h - \mathbf{x}\|_\infty \rightarrow 0$, where $\mathbf{x} = \mathbf{z}_0 e_{-1} - \sum_{\ell=1}^\infty \mathbf{z}_\ell (e_\ell - e_\ell(0))$.

□

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