

Effective models for non-perfectly conducting thin coaxial cables

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Abstract

Continuing past work on the modelling of coaxial cables, we investigate the question of the modeling of non-perfectly conducting thin coaxial cables. Starting from 3D Maxwell's equations, we derive, by asymptotic analysis with respect to the (small) transverse dimension of the cable, a simplified effective 1D model. This model involves a fractional time derivatives that accounts for the so-called skin effects in highly conducting regions.

Keywords: Maxwell's equations, Coaxial cables, Asymptotic analysis

Statement of the problem

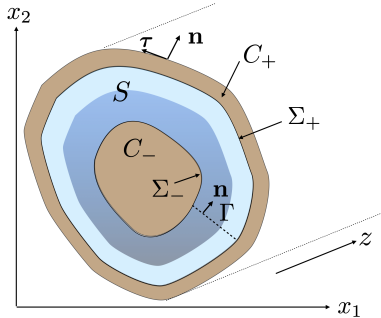


Figure 1: Section the coaxial cable. Σ_+ and Σ_- are the outer and inner boundary of S .

Denoting $\delta > 0$ a small parameter, we consider a family of (thin) domains $\Omega^\delta = \mathcal{G}_\delta(\Omega)$ where

$$\mathcal{G}_\delta : (x_1, x_2, z) \longrightarrow (\delta x_1, \delta x_2, z).$$

and Ω is the disjoint union of a conducting domain Ω_c and a dielectric one Ω_d ,

$$\Omega_c = C \times \mathbb{R}, \quad \Omega_d = S \times \mathbb{R},$$

where $C = C^+ \cup C^-$, C^+ corresponding to the outer metallic shield and C^- to the inner metallic wire and S is non-simply connected, see Figure 1. Accordingly, we have, with obvious notation

$$\Omega^\delta = \Omega_d^\delta \cup \Omega_c^\delta.$$

We are interested in the solution (E^δ, H^δ) of 3D Maxwell's equation in Ω^δ :

$$\begin{cases} \varepsilon^\delta \partial_t E^\delta + \sigma^\delta E^\delta - \mathbf{curl} H^\delta = \mathbf{j}^\delta, \\ \mu^\delta \partial_t H^\delta + \mathbf{curl} E^\delta = \mathbf{0}, \end{cases} \quad (1)$$

with zero initial data. In Ω_c^δ , $(\varepsilon^\delta, \mu^\delta)$ are constant equal to (ε_c, μ_c) and $\mathbf{j}^\delta = \mathbf{0}$. In Ω_d^δ , $(\varepsilon^\delta, \mu^\delta)$ do not depend on z and are obtained by a scaling in the transverse variable $\mathbf{x}_T = (x_1, x_2)$ of fixed distributions in the reference domain Ω_d , for instance

$$\varepsilon^\delta(\mathbf{x}_T, z) = \varepsilon(\mathbf{x}_T/\delta).$$

The source term \mathbf{j}^δ is defined similarly, moreover it is compactly supported, it has no longitudinal component and is divergence free. The conductivity is weak in the dielectric Ω_d^δ , but very high in Ω_c^δ . More precisely

$$\sigma^\delta(\mathbf{x}_T, z) = \begin{cases} \delta^{-4} \sigma_c & \text{in } \Omega_c^\delta, \\ \delta \sigma(\mathbf{x}_T/\delta), & \text{in } \Omega_d^\delta. \end{cases} \quad (2)$$

Note that the $O(\delta^{-4})$ magnitude of σ^δ in Ω_c^δ gives rise to a skin depth in $O(\delta^2)$, small with respect to δ .

Our approach consists in obtaining the formal behaviour of the solution for small δ . To do so, we propose two distinct asymptotic expansions of the solution Ω_d^δ and Ω_c^δ that we match using transmission conditions. We present below our main results.

Electromagnetic field in the dielectric

We introduce the following notations.

- ∇ for the 2D transverse gradient in \mathbf{x}_T , identified to a 3D vector with third component 0,
- $S_\Gamma := S \setminus \Gamma$ where Γ is a cut that makes S_Γ simply connected (see Figure 1),
- $[\cdot]_\Gamma$ for the jump across Γ in the direction \mathbf{n} ,
- $\tilde{\nabla}$ is the 2D transverse gradient in S_Γ ,
- $\partial_{\mathbf{n}}$ is the normal derivative, and $\partial_{\boldsymbol{\tau}} \psi = \tilde{\nabla} \psi \cdot \boldsymbol{\tau}$ the tangential derivative.

We obtain that, for small δ and all $\mathbf{x}_T \in S^\delta$,

$$\begin{aligned} E^\delta(\mathbf{x}_T, z, t) &\sim V^\delta(z, t) \nabla \varphi_e(\mathbf{x}_T/\delta) \\ &+ \delta \left(\int_0^t V^\delta(z, s) ds \right) \nabla \varphi_r(\mathbf{x}_T/\delta), \\ &+ \delta \partial_z V^\delta(z, t) (\varphi_e - \varphi_m)(\mathbf{x}_T/\delta) \mathbf{e}_z, \end{aligned}$$

$$\begin{aligned} H^\delta(\mathbf{x}_T, z, t) &\sim I^\delta(z, t) \nabla \psi_m(\mathbf{x}_T/\delta), \\ &+ \delta \left(\int_0^t \partial_t^{\frac{1}{2}} I^\delta(z, s) ds \right) \nabla \psi_r(\mathbf{x}_T/\delta), \\ &+ \delta \partial_z I^\delta(z, t) (\psi_e - \psi_m)(\mathbf{x}_T/\delta) \mathbf{e}_z, \end{aligned}$$

where $\mathbf{e}_z = (0, 0, 1)^t$. Moreover:

- i) The potential $\varphi_e \in H^1(S)$ satisfies,
 $\operatorname{div} \varepsilon \nabla \varphi_e = 0(S)$, $\varphi_e = 0(\Sigma_+)$, $\varphi_e = 1(\Sigma_-)$,
and the same for φ_m with μ^{-1} instead of ε .
- ii) The potential $\psi_m \in H^1(S_\Gamma)$ satisfies

$$\operatorname{div} \mu \nabla \psi_m = 0(S_\Gamma), \quad \partial_n \psi_m = 0(\partial S),$$

and $[\psi_m]_\Gamma = 1$, $[\partial_n \psi_m]_\Gamma = 0$. The same holds for ψ_e with ε^{-1} instead of μ . Moreover

$$\int_S \mu \psi_e = \int_S \mu \psi_m = 0.$$

- iii) The function $\varphi_r \in H_0^1(S)$ is the solution of

$$\operatorname{div} \varepsilon \nabla \varphi_r = -\operatorname{div} \sigma \nabla \varphi_e.$$

- iv) The function $\psi_r \in H^1(S)$ satisfies

$$\operatorname{div} \mu \nabla \psi_r = 0(S), \quad \mu \partial_n \psi_r = -\sqrt{\frac{\mu_c}{\sigma_c}} \partial_\tau^2 \psi_m(\partial S).$$

- v) The electric potential $V^\delta(z, t)$ and current $I^\delta(z, t)$ are 1D unknowns governed by generalized telegrapher's equations:

$$\begin{cases} C \partial_t V^\delta + \delta G V^\delta + \partial_z I^\delta = j, \\ L \partial_t I^\delta + \delta R \partial_t^{\frac{1}{2}} I^\delta + \partial_z V^\delta = 0, \end{cases} \quad (3)$$

where $j(z, t)$ is an effective source term,

$$j(z, t) = \int_S \mathbf{j}(\mathbf{x}_T, z, t) \cdot \nabla \varphi_e(\mathbf{x}_T), \quad (4)$$

and $\partial_t^{\frac{1}{2}}$ is the square root derivative in the sense of Caputo

$$\partial_t^{\frac{1}{2}} u(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial_\tau u(\tau)}{\sqrt{t-\tau}} d\tau.$$

As in [1], the capacity C , inductance L and conductance G are given by:

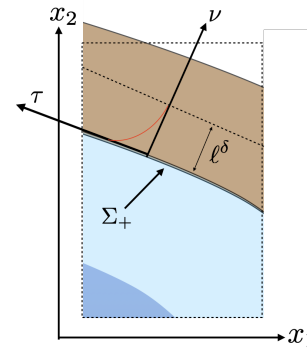
$$C = \int_S \varepsilon |\nabla \varphi_e|^2, \quad L = \int_S \mu |\nabla \psi_m|^2, \quad G = \int_S \sigma |\nabla \varphi_e|^2.$$

Moreover, we obtain an explicit expression for the resistance R , which takes into account skin effects:

$$R = \int_{\partial S} \sqrt{\frac{\mu_c}{\sigma_c}} |\partial_\tau \psi_m|^2. \quad (5)$$

This generalizes formulas of the literature (see [2], chapter 13) already derived in very simple cases.

Electric field in the outer conductor



In the rescaled conducting domain C_+ the electromagnetic fields are described using tangential and normal coordinates (τ, ν) . The penetration depth ℓ^δ of the fields is in $O(\delta)$.

L^+ being the length of Σ_+ , one shows that there exists a 3D field

$$E^+ : [0, L^+] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$$

such that in $C_+^\delta \times \mathbb{R}$ and for small δ ,

$$E^\delta(\mathbf{x}_T(\tau, \nu), z, t) \sim \delta^2 E^+(\tau/\delta, \nu/\delta^2, z, t),$$

(and a similar property holds for the magnetic field). The important fact is that the component E_z^+ is solution of the 1D heat equation

$$\mu_c \sigma_c \partial_t E_z^+ - \partial_\nu^2 E_z^+ = 0, \quad (6)$$

and thus satisfies, at the boundary $\nu = 0$:

$$\partial_\nu E_z^+ + \sqrt{\mu_c \sigma_c} \partial_t^{\frac{1}{2}} E_z^+ = 0.$$

The above equation is used when writing transmission conditions across $\Sigma_+^\delta \times \mathbb{R}$.

This explains the appearance of $\partial_t^{\frac{1}{2}}$ in the effective model (3).

References

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