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Alessandro Socionovo, Yacine Chitour, Frédéric Jean, Roberto Monti, Ludovic Rifford, et al.. Not all sub-Riemannian minimizing geodesics are smooth. 2025. hal-04885315

HAL Id: hal-04885315

<https://ensta-paris.hal.science/hal-04885315v1>

Preprint submitted on 14 Jan 2025

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Not all sub-Riemannian minimizing geodesics are smooth

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January 14, 2025

Abstract

A longstanding open question in sub-Riemannian geometry is the following: are sub-Riemannian length minimizers smooth? We give a negative answer to this question, exhibiting an example of a C^2 but not C^3 length-minimizer of a real-analytic (even polynomial) sub-Riemannian structure.

1 Introduction

Let M be a smooth, connected manifold of dimension $n \geq 3$, equipped with a sub-Riemannian structure (Δ, g) . This structure consists of a bracket generating distribution Δ of rank $m \leq n$ on M , that is, a smooth subbundle of TM of dimension m generated locally by m smooth vector fields X^1, \dots, X^m satisfying the Hörmander condition

$$\text{Lie}\{X^1, \dots, X^m\}(x) = T_x M \quad \forall x \in \mathcal{V},$$

and a smooth metric g on Δ . By the Chow-Rashevsky Theorem, M is horizontally path-connected with respect to Δ . In other words, for any two points $x, y \in M$, there exists a horizontal path $\gamma : [0, T] \rightarrow M$ connecting them, i.e., an absolutely continuous curve $\gamma : [0, T] \rightarrow M$ satisfying

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \quad \text{for almost every } t \in [0, T], \quad \gamma(0) = x, \quad \gamma(T) = y.$$

The sub-Riemannian distance d_{SR} associated with (Δ, g) is defined as the infimum of the lengths of horizontal paths connecting two points: for every $x, y \in M$,

$$d_{SR}(x, y) := \inf \left\{ \text{length}^g(\gamma) \mid \gamma : [0, T] \rightarrow M \text{ horizontal s.t. } \gamma(0) = x, \gamma(T) = y \right\},$$

where the length of a horizontal path, computed using the norm $|\cdot|_g$ induced by the metric g , is given by

$$\text{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|_g dt.$$

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Sub-Riemannian geometry explores the metric and geometric properties of the resulting metric space (M, d_{SR}) . In the special case where $m = n$, the framework reduces to the Riemannian case, where all absolutely continuous curves are horizontal. A distinguished feature of sub-Riemannian geometry, when $m < n$, is the presence of singular horizontal paths. These paths are central to one of the most challenging problems in the field: understanding the regularity of the horizontal curves minimizing the sub-Riemannian distance d_{SR} . The aim of this paper is to demonstrate that these curves are not necessarily smooth.

The Hopf-Rinow Theorem remains valid in the sub-Riemannian setting. For further details on the notions and results of sub-Riemannian geometry mentioned in the introduction, we refer the reader to Bellaïche's monograph [3], or to the books by Montgomery [12], Agrachev, Barilari and Boscain [1], and the fifth author [17]. If the metric space (M, d_{SR}) is complete, then for any points $x, y \in M$, there exists a horizontal path $\gamma : [0, T] \rightarrow M$ between x and y that minimizes length, *i.e.*, $d_{SR}(x, y) = \text{length}^g(\gamma)$. When reparametrized with constant speed, such a path is referred to as a *minimizing geodesic*. It minimizes the energy $\int_0^T |\dot{\gamma}(t)|_g^2 dt$ between its endpoints in fixed time T . By the Pontryagin Maximum Principle, a minimizing geodesic is either the projection of a so-called normal extremal or strictly singular. In the former case, it is smooth because it is the projection of a solution of a smooth Hamiltonian system associated with (Δ, g) in T^*M . In the latter case, which cannot be ruled out due to a famous example by Montgomery [11], its regularity remains uncertain. To date, results on the regularity of strictly singular minimizing geodesics are known in only a few cases. Building on a previous result by Leonardi and Monti [9], Hakavuori and Le Donne [7] demonstrated that minimizing geodesics cannot exhibit corner-type singularities. In [2], Barilari, Chitour, Jean, Prandi, and Sigalotti used this result to prove that minimizing geodesics for rank-2 sub-Riemannian structures of step up to 4 are of class C^1 . Then Monti, Pigati and Vittone proved in [14] the everywhere existence of a tangent line in the tangent cone to minimizing geodesics, a step toward their C^1 regularity. In the case where M and Δ are real-analytic, Sussmann [20] (see also [4]) showed that any strictly singular minimizing geodesic $\gamma : [0, T] \rightarrow M$ is smooth (and analytic whenever g is analytic) on a open dense subset of $[0, T]$. In this context, Belotto da Silva, Figalli, Parusiński and Rifford [5] proved that minimizing geodesics for rank-2 sub-Riemannian structures in dimension 3 are semianalytic. Additionally, Le Donne, Paddeu, and Socionovo [8] obtained the same result in any dimension under the assumption that the distribution has rank 2, is equiregular, and is metabelian. In these last two cases, the above-mentioned result of Hakavuori and Le Donne allows one to show that the minimizing geodesics are indeed of class $C^{1,\alpha}$ for some $\alpha \in (0, 1]$. We present in this paper a counterexample to smoothness of minimizing geodesics that has two main motivations. On the one hand, it is motivated by the result from [5], which follows from a precise description of the orbits of the singular line field given by the trace of the distribution Δ on the Martinet surface Σ_Δ . On the other hand, it also has its origins in [9], where it is explained how to construct examples of singular curves with any kind of singularity, and in the study of (non-)minimality of half-parabolic type curves in [13].

We consider the sub-Riemannian structure (Δ, g) in \mathbb{R}^3 with coordinates (x_1, x_2, x_3) , generated by an orthonormal family of vector fields $\{X^1, X^2\}$ defined as

$$X^1 = \partial_1 \quad \text{and} \quad X^2 = \partial_2 + P(x)^2 \partial_3,$$

where

$$P(x) = x_1^2 - x_2^m \quad \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

and m is an odd integer satisfying $m \geq 5$. Besides the motivations described above, the counterexample took this particular form after a study of several types of possible examples in [18], its structure (in particular with the square of P) being inspired by the Liu–Sussmann example [10].

The Martinet surface of this distribution is given by

$$\Sigma_\Delta := \left\{ x \in \mathbb{R}^3 \mid [X^1, X^2](x) \in \Delta(x) \right\} = \left\{ Q = 0 \right\} \quad \text{with} \quad Q(x) = \partial_1 P^2(x) = 4x_1 (x_1^2 - x_2^m).$$


As any horizontal curve contained in Σ_Δ is singular, the curve $\bar{\gamma} : [0, \infty) \rightarrow \Sigma_\Delta$ given by

$$\bar{\gamma}(t) = (t^{\bar{m}}, t, 0) \quad \forall t \geq 0, \quad \text{with} \quad \bar{m} := \frac{m}{2}, \quad (1.1)$$

is a singular horizontal path of the distribution Δ . Moreover it is not smooth whenever \bar{m} is not an integer. We show the following result.

Theorem 1.1. *For every odd integer $m \geq 5$ and for any sufficiently small $\epsilon > 0$, the curve $\bar{\gamma}|_{[0, \epsilon]}$ is the unique horizontal path minimizing the distance between $\bar{\gamma}(0)$ and $\bar{\gamma}(\epsilon)$ with respect to (Δ, g) . Furthermore, its arc length reparametrization is of class $C^{\bar{m}-1/2}$ but not $C^{\bar{m}+1/2}$.*

The example with the lowest regularity provided by Theorem 1.1 is of class C^2 but not C^3 , and it is achieved for $m = 5$. In [18], it is shown that this result is sharp in the sense that, for every $\epsilon > 0$, the curve $\bar{\gamma}|_{[0, \epsilon]}$ is not a minimizer as soon as it is C^1 but not C^2 . Theorem 1.1 disproves the claim that sub-Riemannian minimizing geodesics are always of class C^∞ but leaves open the question of C^1 or C^2 regularity. The remainder of the paper is dedicated to the proof of Theorem 1.1. We outline the general structure of the proof in Section 2, referring to subsequent sections for the required technical results.

Research funding. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 101034255. 

2 Proof of Theorem 1.1

The purpose of this section is to outline the proof of Theorem 1.1, presenting its structure and referring to the subsequent sections for the detailed demonstration of the required results.

The proof of Theorem 1.1 proceeds by contradiction. We fix $\epsilon > 0$ and we assume that there exists a horizontal path

$$\gamma_\epsilon : [0, \text{length}^g(\gamma_\epsilon)] \rightarrow \mathbb{R}^3,$$

parametrized by arc length (with respect to g), which is minimizing from $\bar{\gamma}(0)$ to $\bar{\gamma}(\epsilon)$ but is not identical to $\bar{\gamma}_\epsilon := \bar{\gamma}|_{[0, \epsilon]}$, up to reparametrization. Then, we have

$$d_{SR}(\bar{\gamma}(0), \bar{\gamma}(\epsilon)) = \text{length}^g(\gamma_\epsilon) \leq \text{length}^g(\bar{\gamma}_\epsilon). \quad (2.1)$$

After a careful study of such a curve γ_ϵ , we will derive a contradiction if $\epsilon > 0$ is sufficiently small. The proof consists in several steps that we now describe. The first step is a straightforward consequence of the fact that γ_ϵ is necessarily a regular horizontal path.

Step 1: Projection of the minimization problem to the plane (x_1, x_2) .

Being a minimizing horizontal path between two points $y, z \in \mathbb{R}^3$ with respect to (Δ, g) is equivalent to having a projection onto the (x_1, x_2) -plane that minimizes the Euclidean length among all curves joining (y_1, y_2) to (z_1, z_2) and along which the integral of $P^2 dx_2$ is equal to $z_3 - y_3$. Thus, the projection of γ_ϵ onto the (x_1, x_2) -plane, denoted by ω_ϵ , has Euclidean length $L(\omega_\epsilon) = \text{length}^g(\gamma_\epsilon)$, is parametrized by (Euclidean) arc length, and minimizes the Euclidean length $L(\zeta)$ among all Lipschitz curves $\zeta : [0, \tau] \rightarrow \mathbb{R}^2$ satisfying the following conditions:

$$\zeta(0) = A_0 := (0, 0), \quad \zeta(\tau) = A_\epsilon := (\epsilon^{\bar{m}}, \epsilon), \quad \text{and} \quad \int_\zeta P(x)^2 dx_2 = 0. \quad (2.2)$$

Furthermore, since γ_ϵ is not identical to $\bar{\gamma}_\epsilon$, it cannot be contained in Σ_Δ and then is not a singular curve. Thus it must correspond to the projection of a normal extremal. Consequently (see [1, 9, 17]), the curve $\omega_\epsilon : [0, L(\omega_\epsilon)] \rightarrow \mathbb{R}^2$ is associated with a function $\theta_\epsilon : [0, L(\omega_\epsilon)] \rightarrow \mathbb{R}$, where $\theta_\epsilon(0) \in (-\pi, \pi]$, and a constant $\lambda_\epsilon \in \mathbb{R}$, such that the following system holds:

$$\dot{\omega}_\epsilon(t) = \begin{pmatrix} \cos \theta_\epsilon(t) \\ \sin \theta_\epsilon(t) \end{pmatrix} \quad \text{and} \quad \dot{\theta}_\epsilon(t) = \lambda_\epsilon Q(\omega_\epsilon(t)) \quad \forall t \in [0, L(\omega_\epsilon)]. \quad (2.3)$$

In particular, the functions ω_ϵ and θ_ϵ are analytic. By construction, $L(\omega_\epsilon) = \text{length}^g(\gamma_\epsilon)$ is no greater than $L(\bar{\omega}_\epsilon) = \text{length}^g(\bar{\gamma}_\epsilon)$, where $\bar{\omega}_\epsilon$ is the projection of $\bar{\gamma}_\epsilon$.

Before delving into the study of ω_ϵ , the next step is to address a problem of calculus of variations with constraints which will be instrumental in proving the main result of Step 3 and in reaching a contradiction in Step 4.

Step 2: A problem of calculus of variations with constraints relying on P -sublevel sets. The following result concerns the length of curves remaining in the region where $P \leq \rho$. The proof is provided in Appendix A.

Proposition 2.1. *Given $\rho > 0$, the following properties hold.*

- (i) *For every $K > 0$, there exist $C(K) > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, if $\rho \in (0, K\epsilon^{3\bar{m}-1})$, then every Lipschitz curve $\zeta : [0, \tau] \rightarrow \mathbb{R}^2$ satisfying*

$$\zeta(0) = A_0, \quad \zeta(\tau) = A_\epsilon, \quad \text{and} \quad \zeta_1(t) > 0, \quad P(\zeta(t)) \leq \rho \quad \forall t \in [0, \tau], \quad (2.4)$$

admits the following lower bound on its length,

$$L(\zeta) \geq L(\bar{\omega}_\epsilon) - C(K)\rho^{1-\frac{1}{\bar{m}}}. \quad (2.5)$$

- (ii) *Define the functions f_ρ and Γ_ρ on $[0, +\infty)$ by*

$$f_\rho(t) := (t^{\bar{m}} + \rho)^{\frac{1}{2}} \quad \text{and} \quad \Gamma_\rho(t) := (f_\rho(t), t) \quad \forall t \geq 0,$$

and for any interval $I \subset [0, +\infty)$, set $\Gamma_\rho(I) := \{\Gamma_\rho(t) \mid t \in I\}$. Then, for all $s \geq t \geq 0$,

$$L(\Gamma_\rho([t, s])) - L([\Gamma_\rho(t), \Gamma_\rho(s)]) \leq \frac{\bar{m}^2}{2} (\bar{m} - 1) s^{\bar{m}-3} (s - t)^2. \quad (2.6)$$

The next step consists in conducting a detailed analysis of the curve ω_ϵ to describe its shape as precisely as possible. To simplify notation, we now omit the ϵ subscript and write ω , $\bar{\omega}$, θ , and λ instead of ω_ϵ , $\bar{\omega}_\epsilon$, θ_ϵ , and λ_ϵ , respectively.

Step 3: Anatomy of ω .

As we shall show, the curve ω cannot be injective and must therefore admit at least one loop. We define a loop of ω as any curve ℓ corresponding to the restriction of ω to an interval $J_\ell = [s_\ell^-, s_\ell^+] \subset [0, L(\omega)]$, where $s_\ell^+ \neq s_\ell^-$, such that $\omega(s_\ell^-) = \omega(s_\ell^+)$. The proof of the following result occupies the entire Section 3.

Proposition 2.2. *There are constants $\epsilon_0 > 0$, $c > 0$, and $C > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, the following properties hold.*

- (i) *For every $t \in (0, L(\omega))$, $\omega_1(t) > 0$ and $|\omega_2(t)| \leq 2\epsilon$.*

- (ii) *$\beta := \max_{t \in [0, L(\omega)]} |P(\omega(t))| \leq C\epsilon^{3\bar{m}-1}$.*

(iii) $\lambda < 0$.

(iv) $P(\omega(t)) > 0$ for all $t \in (0, L(\omega))$.

(v) ω has a unique loop ℓ , it satisfies

$$\omega_2(s_\ell^-) \geq c\epsilon, \quad c\beta\epsilon^{-\bar{m}} \leq L(\ell) \leq C\beta^{1-1/m}, \quad \text{and} \quad \max_{t \in J_\ell} |P(\omega(t))| = \beta.$$

(vi) $|\lambda|\beta^2 \geq c$.

(vii) If $t_* \in [0, L(\omega)] \setminus J_\ell$ is a local maximum of $t \mapsto P(\omega(t))$, then $|\lambda|P(\omega(t_*))^{1+1/\bar{m}} \leq C$.

(viii) $\int_{[0, L(\omega)]} |\dot{\theta}(t)| dt \leq 6\pi$.

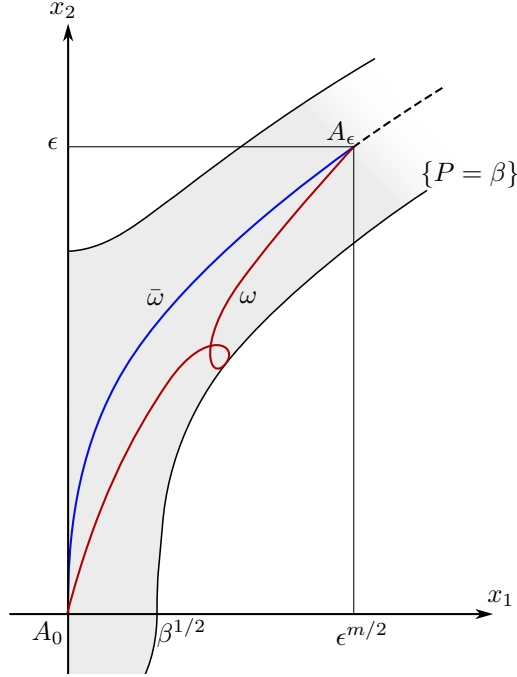


Figure 1: A drawing of $\bar{\omega}$ and ω

Assertion (iii) is stated for clarity but will not be used in the next step. Together with (iv) and (2.3), it allows one to show that the signed curvature of ω is negative, helping to visualize the curve ω . The proof of the above proposition is challenging and requires ruling out the possibility that the curve has multiple loops and crosses from one side of $\{P = 0, x_1 \geq 0\}$ to the other. The proof follows from Stokes theorem, the isoperimetric inequalities and geometric considerations.

We are now ready to demonstrate how the assertions of Proposition 2.2 can be combined with Proposition 2.1 to reach a contradiction.

Step 4: The Contradiction.

We consider the simple Lipschitz curve $\nu : [0, s_\ell^- + L(\omega) - s_\ell^+] \rightarrow [0, +\infty) \times \mathbb{R}$, which connects A_0 to A_ϵ , defined as the concatenation

$$\nu := \omega_{|[0, s_\ell^-]} * \omega_{|[s_\ell^+, L(\omega)]}.$$

We observe that

$$L(\bar{\omega}) \geq L(\omega) = L(\nu) + L(\ell). \quad (2.7)$$

Next, we set

$$\rho := (\beta\epsilon^{-1})^{\frac{m}{m-1}}$$

and we note that, by Proposition 2.2 (ii), $\rho \leq \epsilon^{3\bar{m}-1}$ for sufficiently small $\epsilon \in (0, \epsilon_0)$. We now distinguish two cases: either $P \circ \nu \leq \rho$, or this condition does not hold.

Case 1: $P \circ \nu \leq \rho$

As $\rho \leq \epsilon^{3\bar{m}-1}$, formula (2.5) in Proposition 2.1 gives $L(\nu) \geq L(\bar{\omega}) - C(1)\rho^{1-1/m}$. Combining this with (2.7) yields

$$L(\ell) \leq L(\bar{\omega}) - L(\nu) \leq C(1)\rho^{1-\frac{1}{m}} = C(1)\beta\epsilon^{-1}.$$

Using the lower bound for $L(\ell)$ from Proposition 2.2 (v), we then have $c\epsilon^{-\bar{m}} \leq C(1)\epsilon^{-1}$, which leads to a contradiction for sufficiently small $\epsilon \in (0, \epsilon_0)$, as $m \geq 5$.

Case 2: The condition $P \circ \nu \leq \rho$ is not satisfied

First, we note that if P reaches a local maximum at $t_* \in [0, L(\omega)] \setminus J_\ell$ with $P(\omega(t_*)) \geq \rho$, then assertion (vii) of Proposition 2.2 implies

$$|\lambda|\beta^2\beta^{\frac{4-m}{m-1}}\epsilon^{-\frac{m+2}{m-1}} = |\lambda|(\beta\epsilon^{-1})^{\frac{m+2}{m-1}} = |\lambda|\rho^{1+\frac{1}{m}} \leq |\lambda|P(\omega(t_*))^{1+\frac{1}{m}} \leq C,$$

which, by Proposition 2.2 (vi), is impossible for sufficiently small $\epsilon \in (0, \epsilon_0)$, since $m \geq 5$. Therefore, as the condition $P \circ \nu \leq \rho$ is not satisfied, we have $P(\omega(s_\ell^-)) = P(\omega(s_\ell^+)) > \rho$. This allows us to define $t^-, t^+ \in [0, L(\omega)]$ by

$$t^- := \max\{t \in [0, s^-] \mid P(\omega(t)) = \rho\} \quad \text{and} \quad t^+ := \min\{t \in [s^+, L(\omega)] \mid P(\omega(t)) = \rho\},$$

and, additionally, for sufficiently small $\epsilon \in (0, \epsilon_0)$, we have

$$P(\omega(t)) > \rho \quad \forall t \in (t^-, s_\ell^-] \cup [s_\ell^+, t^+) \quad \text{and} \quad P(\omega(t)) < \rho \quad \forall t \in [0, t^-] \cup (t^+, L(\omega)]. \quad (2.8)$$

Then, we claim that

$$0 < \omega_2(t^+) - \omega_2(t^-) \leq t^+ - t^- \leq 2\beta^{\frac{1}{2}}\epsilon^{1-\bar{m}} \quad \text{and} \quad 0 < \omega_2(t^-) < \omega_2(t^+) \leq 2\epsilon, \quad (2.9)$$

provided that $\epsilon \in (0, \epsilon_0)$ is sufficiently small. To prove the left inequalities, we consider the set $\mathcal{S} := \{0 \leq P \leq \rho, x_1 \geq 0\}$ whose boundary is the union of: the vertical segment $S := [A_0, (0, -\rho^{1/m})]$, the curve $\mathcal{C}_1 := \{P = 0, x_1 \geq 0\}$, and the curve $\mathcal{C}_2 := \{P = \rho, x_1 \geq 0\}$. The smooth curve $\omega^- := \omega_{[0, t^-]}$ connects \mathcal{C}_1 to \mathcal{C}_2 and does not intersect S for $t \in (0, t^-)$ (by Proposition 2.2 (i)). As a result, the support $\text{spt}(\omega^-)$ of ω^- divides the set \mathcal{S} into two connected components: \mathcal{S}_1 , which is bounded by S , the segment of \mathcal{C}_2 from $(0, -\rho^{1/m})$ to $\omega(t^-)$, and $\text{spt}(\omega^-)$; and \mathcal{S}_2 , which is the complement of \mathcal{S}_1 within $\mathcal{S} \setminus \text{spt}(\omega^-)$. Since $\omega(L(\omega)) = A_\epsilon \notin \mathcal{S}_1$, $P(\omega(t^+)) = \rho$, and the curves ω^- and $\omega_{[t^+, L(\omega)]}$ do not intersect (because ℓ is the unique loop of ω , as stated in Proposition 2.2 (v)), it follows that $\omega(t^+) \in \mathcal{S}_2 \cap \mathcal{C}_2$, which gives $\omega_2(t^+) > \omega_2(t^-)$. The inequality $\omega_2(t^+) - \omega_2(t^-) \leq t^+ - t^-$ holds because $|\dot{\omega}(t)| = 1$ for all $t \in [0, L(\omega)]$, as ω is parametrized by arc length. To prove the next inequality, we suppose for contradiction that $t^+ - t^- > 2\alpha$ with $\alpha := \beta^{1/2}\epsilon^{1-\bar{m}}$. Hence, we have either $s_\ell^- - t^- > \alpha$ or $t^+ - s_\ell^+ > \alpha$. Assume $s_\ell^- - t^- > \alpha$; the other case follows similarly. Set $\bar{s} := s_\ell^- - \alpha > t^-$. Since $\omega_2(s_\ell^-) \geq c\epsilon$ by Proposition 2.2 (v), and ω_2 is 1-Lipschitz, we have $\omega_2(t) \geq c\epsilon - \alpha$ for all $t \in [\bar{s}, s_\ell^-]$. Thus, since $\alpha = o(\epsilon)$ by Proposition 2.2 (ii), for sufficiently small $\epsilon \in (0, \epsilon_0)$, we have $\omega_2(t) \geq c\epsilon/2$

for all $t \in [\bar{s}, s_\ell^-]$. As a consequence, since $P(\omega(t)) > \rho > 0$ on $(\bar{s}, s_\ell^-]$ (by (2.8)), we have $\omega_1(t) \geq c'\epsilon^{\bar{m}}$ for all $t \in (\bar{s}, s_\ell^-]$, where $c' := (c/2)^{1/2}$. By applying (2.3), (2.8), and assertion (viii) of Proposition 2.2, we obtain

$$6\pi \geq \int_{\bar{s}}^{s_\ell^-} |\dot{\theta}(t)| dt = \int_{\bar{s}}^{s_\ell^-} 4|\lambda|\omega_1(t)P(\omega(t)) dt \geq 4\epsilon^{\frac{3}{2}}|\lambda|c'\epsilon^{\bar{m}}\rho = 4c'|\lambda|\beta^{\frac{m}{m-1}}\epsilon^{\frac{m+3}{2}-\frac{m}{m-1}}.$$

By assertions (ii) and (vi) of Proposition 2.2, we know that $\beta \leq C\epsilon^{3\bar{m}-1}$ and $|\lambda|\beta^2 \geq c$. Hence, as $m/(m-1) < 2$, the above inequality shows that the quantity

$$\beta^{\frac{m}{m-1}-2}\epsilon^{\frac{m+3}{2}-\frac{m}{m-1}} \geq \epsilon^{(3\bar{m}-1)(\frac{m}{m-1}-2)}\epsilon^{\frac{m+3}{2}-\frac{m}{m-1}} = \epsilon^{\frac{-2m^2+8m-7}{2(m-1)}}$$

is bounded from above for all $\epsilon > 0$ sufficiently small. Since the exponent of ϵ is negative, this leads to a contradiction. The left inequalities in (2.9) are thus established. The right inequalities follow for sufficiently small $\epsilon \in (0, \epsilon_0)$, from the fact that $\omega_2(s_\ell^-) \geq c\epsilon$ (Proposition 2.2 (v)), the 1-Lipschitz continuity of ω_2 , the upper bound on $t^+ - t^-$ just derived, and the upper bound for ω_2 from Proposition 2.2 (i).

We now set $s^- := \omega_2(t^-) > 0$, and $s^+ := \omega_2(t^+) \leq 2\epsilon$, and consider the concatenated curve

$$\bar{\nu} := \omega|_{[0, t^-]} * \Gamma_\rho([s^-, s^+]) * \omega|_{[t^+, L(\omega)]}.$$

From the right property in (2.8) and inequality (2.5) in Proposition 2.1 (since $\rho \leq \epsilon^{3\bar{m}-1}$), it follows that $L(\bar{\nu}) \geq L(\bar{\omega}) - C(1)\rho^{1-1/m}$. Combining this with (2.7), we deduce

$$L(\ell) \leq L(\bar{\omega}) - L(\nu) \leq L(\bar{\nu}) - L(\nu) + C(1)\rho^{1-\frac{1}{m}}. \quad (2.10)$$

Using (2.6) from Proposition 2.1 and the left inequalities in (2.9), we have

$$\begin{aligned} L(\bar{\nu}) - L(\nu) &\leq L(\Gamma_\rho([s^-, s^+])) - L(\nu|_{[t^-, s_\ell^-]} * \nu|_{[s_\ell^+, t^+]}) \\ &\leq L(\Gamma_\rho([s^-, s^+])) - L([\Gamma_\rho(s^-), \Gamma_\rho(s^+)]) \\ &\leq \frac{\bar{m}^2}{2}(\bar{m}-1)(s^+)^{m-3}(s^+ - s^-)^2 \leq 2^{m-5}m^2(m-2)\beta\epsilon^{-1}. \end{aligned}$$

Using the lower bound for $L(\ell)$ from Proposition 2.2 (v) and (2.10), we infer that $\epsilon^{-\bar{m}} \leq C'\epsilon^{-1}$ for some constant $C' > 0$. Since $m \geq 5$, this leads to a contradiction for sufficiently small $\epsilon \in (0, \epsilon_0)$. This ends the proof of Theorem 1.1.

3 Proof of Proposition 2.2

Recall that $\omega : I_\omega \rightarrow \mathbb{R}^2$, where $I_\omega := [0, L(\omega)]$, is a fixed curve satisfying (2.2) and (2.3), and that $\bar{\omega} := \bar{\omega}_\epsilon$ represents the projection of $\bar{\gamma}_\epsilon := \bar{\gamma}|_{[0, \epsilon]}$ onto the (x_1, x_2) -plane. By assumption, we have $L(\omega) \leq L(\bar{\omega})$. Recall that $m \geq 5$ is an odd integer, $\bar{m} := m/2$, and that a loop of ω is a curve ℓ defined as the restriction of ω to some interval $J_\ell = [s_\ell^-, s_\ell^+] \subset I_\omega$, with $s_\ell^+ \neq s_\ell^-$, such that $\omega(s_\ell^-) = \omega(s_\ell^+)$. We always denote by $J_\ell = [s_\ell^-, s_\ell^+] \subset I_\omega$ the interval associated with a loop ℓ of ω , and we define $\text{int}(J_\ell) = (s_\ell^-, s_\ell^+)$. Additionally, a loop ℓ is said to be simple if ω is injective on $[s_\ell^-, s_\ell^+)$.

Before beginning the proof of Proposition 2.2, we provide, in the next two sections, preliminary reminders about Stokes theorem, the isoperimetric inequality, and Gauss-Bonnet formula for curves in the plane.

3.1 Reminders on Stokes' theorem and the isoperimetric inequality

A Lipschitz curve is a curve $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ that is Lipschitz continuous. The support of η , denoted by $\text{spt}(\eta)$, is defined as $\text{spt}(\eta) := \eta([0, \tau])$. We say that η is closed if it satisfies $\eta(0) = \eta(\tau)$. For a Lipschitz closed curve η , the winding number (or index) of a point $x \in \mathbb{R}^2 \setminus \text{spt}(\eta)$ with respect to η is denoted by $\text{ind}(x, \eta) \in \mathbb{Z}$. By convention, $\text{ind}(0, \eta) = +1$ when $\eta(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. For each $k \in \mathbb{Z}$, we set

$$\mathcal{E}_k(\eta) := \left\{ x \in \mathbb{R}^2 \setminus \text{spt}(\eta) \mid \text{ind}(x, \eta) = k \right\},$$

and we define $\mathcal{E} \subset \mathbb{R}^2 \setminus \text{spt}(\eta)$ as the bounded open set

$$\mathcal{E}(\eta) := \bigcup_{k \in \mathbb{Z}, k \neq 0} \mathcal{E}_k(\eta).$$

Recalling that Q is defined by $Q(x) = 4x_1 P(x) = 4x_1(x_1^2 - x_2^2)$, we define the weighted area of η as

$$A(\eta) := \sum_{k \in \mathbb{Z}} k \mathcal{L}_Q(\mathcal{E}_k(\eta)),$$

where

$$\mathcal{L}_Q(\mathcal{E}_k(\eta)) := \int_{\mathcal{E}_k(\eta)} Q(x) dx \quad \forall k \in \mathbb{Z}.$$

In the following result, assertion (i) follows from Stokes' theorem and (ii) is a consequence of Radó's inequality in the plane. For two Lipschitz curves $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ and $\eta' : [0, \tau'] \rightarrow \mathbb{R}^2$ such that $\eta(\tau) = \eta'(0)$, we use the notation $\eta * \eta'$ to denote their concatenation on $[0, \tau + \tau']$. The reversed curve $\tilde{\omega} : [0, \epsilon] \rightarrow \mathbb{R}^2$ is defined by $\tilde{\omega}(t) := \tilde{\omega}(\epsilon - t)$ for $t \in [0, \epsilon]$.

Lemma 3.1. *Let $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ be a Lipschitz curve.*

- (i) *Assume that $\eta(0), \eta(\tau) \in \text{spt}(\tilde{\omega})$. Let $\tilde{\omega}^\eta$ denote the segment of $\tilde{\omega}$ connecting $\eta(\tau)$ to $\eta(0)$. Then, the concatenation $\eta * \tilde{\omega}^\eta$ forms a closed curve and we have:*

$$A(\eta * \tilde{\omega}^\eta) = \int_\eta P(x)^2 dx_2.$$

In particular, for $\eta = \omega$ we have

$$A(\omega * \tilde{\omega}) = \int_\omega P(x)^2 dx_2 = 0.$$

- (ii) *If η is closed, the weighted area $A(\eta)$ satisfies the inequality*

$$4\pi|A(\eta)| \leq \sup_{x \in \mathcal{E}(\eta)} |Q(x)| L(\eta)^2.$$

Proof. Let $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ be a Lipschitz curve. If $\eta(0), \eta(\tau) \in \text{spt}(\tilde{\omega})$, then the concatenation $\eta * \tilde{\omega}^\eta$ forms a closed curve. By Stokes' theorem and noting that P vanishes along $\tilde{\omega}^\eta$, we have

$$A(\eta * \tilde{\omega}^\eta) = \sum_{k \in \mathbb{Z}, k \neq 0} k \int_{\mathcal{E}_k(\eta * \tilde{\omega}^\eta)} \frac{\partial P^2}{\partial x_1}(x) dx = \int_{\eta * \tilde{\omega}^\eta} P^2 dx_2 = \int_\eta P^2 dx_2,$$

which proves the first part of (i). For η closed, denoting by \mathcal{L} the Lebesgue measure in the plane and using the definition of $A(\eta)$, we have

$$|A(\eta)| \leq \sum_{k \in \mathbb{Z}} |k| |\mathcal{L}_Q(\mathcal{E}_k(\eta))| \leq \sup_{x \in \mathcal{E}(\eta)} |Q(x)| \sum_{k \in \mathbb{Z}} |k| |\mathcal{L}(\mathcal{E}_k(\eta))| \leq \sup_{x \in \mathcal{E}(\eta)} |Q(x)| \frac{L(\eta)^2}{4\pi},$$

where the last inequality follows from Radó's isoperimetric inequality in the plane (see [16] and formula (1.9) in [15]), which proves (ii). \square

3.2 Curves with curvature of constant sign in the plane

We begin by recalling the definition of the signed curvature of a smooth curve. Let $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ be a smooth curve parametrized by arc length. For every $t \in [0, \tau]$, the signed curvature of η at t is the unique value $\kappa(t) \in \mathbb{R}$ such that

$$\ddot{\eta}(t) = \kappa(t)n(t),$$

where $n(t)$ is obtained by rotating the nonzero vector $\dot{\eta}(t)$ counterclockwise by an angle $\pi/2$. If $\alpha : [0, \tau] \rightarrow \mathbb{R}$ is a smooth function satisfying $\dot{\eta}(t) = (\cos \alpha(t), \sin \alpha(t))$ for all $t \in [0, \tau]$ (such a function always exists), then the signed curvature is given by

$$\kappa(t) = \dot{\alpha}(t) \quad \forall t \in [0, \tau]. \quad (3.1)$$

Next, we define a piecewise smooth continuous curve $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ as a continuous curve for which there exist times $0 = \tau_0 < \tau_1 < \dots < \tau_N = \tau$ such that each restriction $\eta|_{[\tau_i, \tau_{i+1}]}$, for $i = 0, \dots, N-1$, is smooth. For each $i = 0, \dots, N-1$, we denote by $\dot{\eta}(\tau_i^-)$ (resp. $\dot{\eta}(\tau_i^+)$) the left (resp. right) derivative of η at τ_i , with the conventions $\dot{\eta}(\tau_0^-) := \dot{\eta}(\tau_N)$ and $\dot{\eta}(\tau_N^+) := \dot{\eta}(\tau_0)$. We then consider such a curve η , assuming that it is closed, parametrized by arc length, and simple, meaning that the restriction $\eta|_{[0, \tau]}$ is injective. By the Jordan curve theorem, η divides the plane into two open sets: a bounded domain $\mathcal{D}(\eta)$ and its complement. In this setting, we have

$$\mathcal{D}(\eta) = \mathcal{E}_k(\eta) = \mathcal{E}(\eta), \quad (3.2)$$

where $k = 1$ if η is positively oriented and $k = -1$ otherwise. Recall that if v, w are two nonzero vectors in \mathbb{R}^2 , the oriented angle from v to w , denoted by $\text{ang}(v, w)$, is defined to lie in the interval $(-\pi, \pi)$, with a positive sign if (v, w) forms an oriented basis of \mathbb{R}^2 , and a negative sign otherwise. For any nonzero vectors $v, w \in \mathbb{R}^2$, let $\text{Tan}^1(v, w)$ (resp. $\text{Tan}^{-1}(v, w)$) denote the open set of nonzero vectors $u \in \mathbb{R}^2$ such that $\text{ang}(v, u) \in (\text{ang}(v, w), \pi)$ (resp. $\text{ang}(v, u) \in (-\pi, \text{ang}(v, w))$). The relation (3.2) implies the following result.

Lemma 3.2. *Let $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ be a piecewise smooth continuous curve which is closed, simple, parametrized by arc length, and satisfies (3.2) with $k = \pm 1$. Then, for any $t \in [0, \tau]$ and every $u \in \text{Tan}^k(\dot{\eta}(t^-), \dot{\eta}(t^+))$, we have*

$$\eta(t) + sk u \in \mathcal{E}(\eta) \quad \forall s > 0 \text{ small.}$$

Let us now assume that η is positively oriented. For each $i = 0, \dots, N-1$, if $\dot{\eta}(\tau_i^-) \neq \dot{\eta}(\tau_i^+)$, we define the discontinuity of the curvature at τ_i as the oriented angle $\delta_i = \text{ang}(\dot{\eta}(\tau_i^-), \dot{\eta}(\tau_i^+))$. However, if $\dot{\eta}(\tau_i^-) = -\dot{\eta}(\tau_i^+)$, meaning that η has a cusp at $\eta(\tau_i)$, we follow the convention in [19, Chapter 6]: if the cusp points toward $\mathcal{D}(\eta)$, we set $\delta_i := -\pi$, otherwise, we set $\delta_i = \pi$. In any case, the discontinuity of the curvature δ_i belongs to $[-\pi, \pi]$. The Gauss-Bonnet formula then states

$$2\pi = \sum_{i=0}^{N-1} \int_{\tau_i}^{\tau_{i+1}} \kappa(t) dt + \sum_{i=0}^{N-1} \delta_i. \quad (3.3)$$

From this, we can easily deduce the following result.

Lemma 3.3. *Let $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ be a piecewise smooth continuous curve associated with times $0 = \tau_0 < \tau_1 < \dots < \tau_N = \tau$, which is closed, simple, and parametrized by arc length. Let $\sigma = \pm 1$ be fixed.*

- (i) *If for every $i = 0, \dots, N-1$, the smooth signed curvature κ of the segment $\eta|_{[\tau_i, \tau_{i+1}]}$ satisfies $\sigma\kappa \geq 0$, and the discontinuity of the curvature δ_i at τ_i satisfies $\sigma\delta_i \in [0, \pi]$, then the set $\mathcal{D}(\eta)$ is convex, $\mathcal{D}(\eta) = \mathcal{E}_\sigma(\eta) = \mathcal{E}(\eta)$, and for every $i = 0, \dots, N-1$, $\sigma\delta_i = \sigma \text{ang}(\dot{\eta}(\tau_i^-), \dot{\eta}(\tau_i^+)) \in [0, \pi]$.*

(ii) If there exist indices $i_1 \neq i_2$ in $\{0, \dots, N-1\}$ such that

$$\sum_{i=0}^{N-1} \int_{\tau_i}^{\tau_{i+1}} \sigma \kappa(t) dt \geq 0 \quad \text{and} \quad \sigma \delta_i \in [0, \pi] \quad \forall i \in \{0, \dots, N-1\} \setminus \{i_1, i_2\},$$

then $\mathcal{D}(\eta) = \mathcal{E}_\sigma(\eta) = \mathcal{E}(\eta)$.

Proof. To prove (i), we observe that under the given assumption, the set $\mathcal{D}(\eta)$ admits a local supporting line at each point of its boundary $\partial\mathcal{D}(\eta)$. Specifically, for every $x \in \partial\mathcal{D}(\eta)$, there exist a neighborhood U of x and a closed half-plane P , bounded by a line L , such that $x \in L$ and $U \cap \mathcal{D}(\eta) \subset P$. By Tietze's theorem [21], this implies that $\mathcal{D}(\eta)$ is convex and that the oriented angles $\sigma\delta_0, \dots, \sigma\delta_{N-1}$ lie within $[0, \pi]$. Moreover, if $\sigma = 1$ then η is positively oriented, which, by (3.2), implies $\mathcal{D}(\eta) = \mathcal{E}_1(\eta) = \mathcal{E}(\eta)$. Conversely, if $\sigma = -1$, then η is negatively oriented, leading to $\mathcal{D}(\eta) = \mathcal{E}_{-1}(\eta) = \mathcal{E}(\eta)$ (by (3.2)).

To prove (ii), using (3.2), it suffices to show that η is positively oriented when $\sigma = 1$. Assume $\sigma = 1$ and suppose, for the sake of contradiction, that η is negatively oriented. Define the curve $\tilde{\eta} : [0, \tau] \rightarrow \mathbb{R}^2$ by $\tilde{\eta}(t) := \eta(\tau - t)$ for all $t \in [0, \tau]$. Let $\tilde{\delta}_0, \dots, \tilde{\delta}_{N-1} \in [-\pi, \pi]$ denote the discontinuities of the signed curvature at $t = 0, t = \tau - \tau_{N-1}, \dots, t = \tau - \tau_1$. The curve $\tilde{\eta}$ is positively oriented by construction. Furthermore, by assumption, the integral of the signed curvature $\tilde{\kappa}$ of $\tilde{\eta}$ over the set $[0, \tau - \tau_{N-1}] \cup \dots \cup [\tau - \tau_1, \tau]$ is nonpositive, and the discontinuities satisfy $\tilde{\delta}_i \in [-\pi, 0]$ for all $i \in \{0, \dots, N-1\} \setminus \{i_1, i_2\}$. By the Gauss-Bonnet formula, we have

$$2\pi = \sum_{i=0}^{N-1} \int_{\tau_i}^{\tau_{i+1}} \tilde{\kappa}(t) dt + \sum_{i=0, i \neq i_1, i_2}^{N-1} \tilde{\delta}_i + \tilde{\delta}_{i_1} + \tilde{\delta}_{i_2} \leq \tilde{\delta}_{i_1} + \tilde{\delta}_{i_2} \leq 2\pi.$$

Hence, equality holds, which forces $\tilde{\kappa} \equiv 0$ and $\tilde{\delta}_i = 0$ for all $i \neq i_1, i_2$. Since η is simple, such a configuration cannot arise. We obtain a contradiction. \square

The following result, derived from the Gauss-Bonnet formula, will also be instrumental in the proof of Proposition 2.2.

Lemma 3.4. *Let $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ be a smooth curve parametrized by arc length, and let $\sigma = \pm 1$ such that the signed curvature κ of η satisfies $\sigma\kappa \geq 0$. If $\eta|_{(0, \tau)}$ is injective and $\eta|_{(0, \tau)}$ does not cross the line $\eta(0) + \mathbb{R}\dot{\eta}(0)$, then the point $\eta(\tau)$ does not belong to the open ray $\eta(0) + (0, +\infty)\dot{\eta}(0)$.*

Proof. We may assume without loss of generality that $\eta(0) = (0, 0)$, $\dot{\eta}(0) = (1, 0)$, and $\eta_2 \geq 0$ on $[0, \tau]$. Note that under these assumptions, we necessarily have $\kappa \geq 0$. Moreover, since $\eta|_{(0, 0)}$ does not cross the line $\eta(0) + \mathbb{R}\dot{\eta}(0)$, we have $\int_0^\tau \kappa(t) dt > 0$. Suppose, for the sake of contradiction that $\eta(\tau) \in (0, +\infty)(1, 0)$. Consider the curve $\tilde{\eta}$, defined as the concatenation of the line segment $[\eta(0), \eta(\tau)]$ with the curve $\eta|_{[0, \tau]}$ traversed backward. This curves is positively oriented. Applying the Gauss-Bonnet formula, we obtain

$$2\pi = - \int_0^\tau \kappa(t) dt + \delta_0 + \pi,$$

where $\delta_0 \in [-\pi, \pi]$ represents the discontinuity of curvature of $\tilde{\eta}$ at the point $\eta(\tau)$. Since $\int_0^\tau \kappa(t) dt > 0$, this leads to a contradiction. \square

3.3 Preliminary observations on the curve ω

The curve $\omega : I_\omega = [0, L(\omega)] \rightarrow \mathbb{R}^2$ is an analytic curve parametrized by arc length, joining $A_0 = \bar{\omega}(0)$ to $A_\epsilon = \bar{\omega}(\epsilon)$. Its length $L(\omega)$ satisfies

$$L(\omega) \leq L(\bar{\omega}) = \int_0^\epsilon |\dot{\bar{\omega}}_\epsilon(t)| dt = \int_0^\epsilon (1 + \bar{m}^2 t^{m-2})^{\frac{1}{2}} dt = \epsilon + \frac{\bar{m}^2 \epsilon^{m-1}}{2(m-1)} + o(\epsilon^{m-1}), \quad (3.4)$$

for small $\epsilon > 0$. The following lemma provides several results that will be used repeatedly throughout the proof of Proposition 3.3. Notably, the property stated in (v) is reminiscent of an estimates used by Liu and Sussmann in [10, Lemma 1 p. 14] to establish the minimality of the (smooth) singular horizontal path they considered.

Lemma 3.5. *There are constants $\epsilon_0, C > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, the following hold.*

- (i) For every $t \in (0, L(\omega)]$, $\omega_1(t) > 0$.
- (ii) $\theta(0) \in (-\pi/2, \pi/2)$.
- (iii) $\max_{t \in [0, L(\omega)]} P(\omega(t)) > 0$.
- (iv) For every $t \in I_\omega$, $\omega_1(t) \leq 2\epsilon^{\bar{m}}$ and $|\omega_2(t)| \leq 2\epsilon$.
- (v) $\beta := \max_{t \in I_\omega} |P(\omega(t))| \leq C \epsilon^{3\bar{m}-1} = o(\epsilon^m)$ as $\epsilon \rightarrow 0$.
- (vi) The function $t \in I_\omega \mapsto \omega(t)$ is not injective.
- (vii) $\lambda \neq 0$.
- (viii) Any loop ℓ of ω satisfies $L(\ell) \leq C\beta^{1-\frac{1}{\bar{m}}} = o(\epsilon^{\bar{m}})$ as $\epsilon \rightarrow 0$.

Proof. To prove (i), assume that there is an interval $J = [t_0, t_1] \subset I_\omega$ such that $\omega_1(t_0) = \omega_1(t_1) = 0$ and $\omega_1(t) \leq 0$ for all $t \in J$. Then the curve $\hat{\omega} : I_\omega \rightarrow \mathbb{R}^2$, defined as

$$\hat{\omega}(t) := \begin{cases} \omega(t) & \text{if } t \notin J \\ (-\omega_1(t), \omega_2(t)) & \text{if } t \in J \end{cases}$$

has the same length as ω . Furthermore, it satisfies

$$\int_{\hat{\omega}} P^2 dx_2 = \int_0^{L(\omega)} P(\hat{\omega}(t))^2 \dot{\omega}_2(t) dt = \int_0^{L(\omega)} (\hat{\omega}_1(t)^2 - \hat{\omega}_2(t)^m)^2 \dot{\omega}_2(t) dt = \int_\omega P^2 dx_2 = 0.$$

As a consequence, if $\omega(t) < 0$ for some $t \in I_\omega$ then we can define a non-analytic curve that minimizes the length from $\bar{\omega}(0)$ to $\bar{\omega}(\epsilon)$, which is a contradiction. If $\omega_1(t) = 0$ for some $t \in (0, L(\omega))$, then $\cos \theta(t) = 0$. In that case, as well as when $t = 0$ and $\cos \theta(0) = 0$, the uniqueness of the solution to (2.3) with fixed initial conditions implies that the curve $s \mapsto \omega(t+s)$ must coincide with the straight line $s \mapsto (0, \omega_2(t) + (s-t)L(\omega) \sin \theta(t))$. This leads to a contradiction, thereby completing the proofs of (i) and (ii). By (ii) and (2.3), we have $P(\omega(t)) > 0$ for $t > 0$ small, which proves (iii).

To prove (iv), it is sufficient to show that there is no $t \in I_\omega$ such that $\omega_2(t) = 2\epsilon$ or $\omega_1(t) = 2\epsilon^{\bar{m}}$. Suppose there exists $t \in I_\omega$ such that $\omega_2(t) = 2\epsilon$. Then we have

$$L(\omega) \geq L([\omega(0), \omega(t)]) + L([\omega(t), \omega(\epsilon)]) = \sqrt{\omega_1(t)^2 + 4\epsilon^2} + \sqrt{(\epsilon^{\bar{m}} - \omega_1(t))^2 + \epsilon^2} \geq 3\epsilon,$$

which contradicts (3.4) for sufficiently small $\epsilon > 0$. Suppose now that there exists $t \in I_\omega$ such that $\omega_1(t) = 2\epsilon^{\bar{m}}$, then as above we have

$$L(\omega) \geq \sqrt{4\epsilon^m + \omega_2(t)^2} + \sqrt{\epsilon^m + (\epsilon - \omega_2(t))^2} \geq \epsilon \sqrt{1 + \epsilon^{m-2}},$$

where the last inequality follows from the fact that the function $z \in \mathbb{R} \mapsto \sqrt{4\epsilon^m + z^2} + \sqrt{\epsilon^m + (\epsilon - z)^2}$ has a minimum at $z = 2\epsilon/3$. The inequality contradicts (3.4) for sufficiently small $\epsilon > 0$.

To prove (v), we start by computing

$$J := \int_0^{L(\omega)} P(\omega(t))^2 dt.$$

Since ω joins $\bar{\omega}(0)$ to $\bar{\omega}(\epsilon)$ and $\int_{\omega} P^2 dx_2 = 0$, we have (noting that $1 - \dot{\omega}_2 \geq 1 - |\dot{\omega}| \geq 0$)

$$J = \int_0^{L(\omega)} (1 - \dot{\omega}_2(t)) P(\omega(t))^2 dt \leq \beta^2 \int_0^{L(\omega)} (1 - \dot{\omega}_2(t)) dt = \beta^2 (L(\omega) - \epsilon). \quad (3.5)$$

Furthermore, the derivative of $t \in I_{\omega} \mapsto P(t) := P(\omega(t))$ satisfies (using (iv))

$$|\dot{P}(t)| = |2\omega_1(t)\dot{\omega}_1(t) - m\omega_2^{m-1}(t)\dot{\omega}_2(t)| \leq 2|\omega_1(t)| + m|\omega_2(t)|^{m-1} \leq 4\epsilon^{\bar{m}} + m(2\epsilon)^{m-1} \leq 5\epsilon^{\bar{m}},$$

provided $\epsilon > 0$ is sufficiently small. We infer that (where \mathcal{L}^1 denotes the Lebesgue measure)

$$J \geq \int_{\{|P| \geq \beta/2\}} P(t)^2 dt \geq \frac{\beta^2}{4} \mathcal{L}^1(\{|P| \geq \beta/2\}) \geq \frac{\beta^2}{4} \frac{\beta}{5\epsilon^{\bar{m}}} = \frac{\beta^3}{20\epsilon^{\bar{m}}}.$$

The inequality in (v) follows directly from (3.5) and the inequality $L(\omega) - \epsilon \leq m^2\epsilon^{m-1}$, as established in (3.4) for sufficiently small $\epsilon > 0$. We deduce that $\beta = o(\epsilon^m)$ since $m \geq 5$.

To prove (vi), suppose, for contradiction, that ω is injective. Then, by Lemma 3.1 (i), we have $A(\omega * \bar{\omega}) = 0$. Since both ω and $\bar{\omega}$ are injective, the winding number of any point with respect to the closed curve $\eta := \omega * \bar{\omega}$ is ± 1 or 0 . Consequently, the bounded open set $\mathcal{E}(\omega * \bar{\omega})$ is the union of the two disjoint sets $\mathcal{E}_{-1}(\eta)$ and $\mathcal{E}_1(\eta)$. From assertion (i), the function $Q(\omega(t))$ has the same sign as $P(\omega(t))$. Furthermore, since $P(\bar{\omega}) \equiv 0$, the connected components of $\mathcal{E}_{-1}(\eta)$ are contained in $\{Q < 0\}$, while the connected components of $\mathcal{E}_1(\eta)$ are contained in $\{Q > 0\}$. As ω and $\bar{\omega}$ are not identical, one of the sets $\mathcal{E}_{-1}(\eta)$ or $\mathcal{E}_1(\eta)$ must be non-empty. This implies that $A(\eta) > 0$, contradicting the earlier result that $A(\omega * \bar{\omega}) = 0$. The contradiction proves that ω cannot be injective.

To prove (vii), we observe that if $\lambda = 0$, then by (2.3), ω is a straight line, and thus injective. However, this contradicts assertion (vi).

To prove (viii), we observe that the curve $\nu := \omega|_{[0, s_{\epsilon}^-]} * \omega|_{[s_{\epsilon}^+, L(\omega)]}$ satisfies (2.4). Applying (2.5) in Proposition 2.1 with $\rho = \beta$, we obtain

$$L(\bar{\omega}) = L(\ell) + L(\nu) \geq L(\ell) + L(\nu_{\epsilon}^{\rho}) \geq L(\ell) + L(\bar{\omega}) - C\rho^{1 - \frac{1}{m}}.$$

Rearranging terms gives the desired result. The relation $L(\ell) = o(\epsilon^{\bar{m}})$ follows from $m \geq 5$. \square

The next result follows from the upper bound for β given in Lemma 3.5, combined with inequality (3.4).

Lemma 3.6. *For every $K > 0$, there exist constants $\epsilon_0(K), C(K) > 0$ such that for any $\epsilon \in (0, \epsilon_0(K))$ and $t \in I_{\omega}$, the following holds:*

$$\omega_2(t) \geq K\epsilon \implies \begin{cases} \omega_1(s) & \geq C(K)\epsilon^{\bar{m}} \\ \omega_2(s) & \geq C(K)\epsilon \end{cases} \quad \forall s \in [t, L(\omega)].$$

Proof. Let $K > 0$ and $t \in I_{\omega}$ be such that $\omega_2(t) \geq K\epsilon$. Define $C(K) := K/2$, and assume there exists $s \in [t, L(\omega)]$ such that $\omega_2(s) < C(K)\epsilon$. Since $\omega_2(0) = 0$ and $\omega_2(L(\omega)) = \epsilon$, we deduce that

$$L(\omega) \geq L(\omega|_{[0, t]}) + L(\omega|_{[s, L(\omega)]}) \geq \omega_2(t) + \epsilon - \omega_2(s) \geq \epsilon + C(K)\epsilon.$$

This inequality contradicts (3.4) for sufficiently small $\epsilon > 0$. Hence, for sufficiently small $\epsilon > 0$, we must have $\omega_2(s) \geq C(K)\epsilon$ for all $s \in [t, L(\omega)]$. As a result, using the bound for β from Lemma 3.5 (v), for all $s \in [t, L(\omega)]$, we obtain:

$$\omega_1(s)^2 = \omega_2(s)^m + P(\omega(s)) \geq \omega_2(s)^m - |P(\omega(s))| \geq C(K)^m \epsilon^m - \beta \geq C(K)^m \epsilon^m - C\epsilon^{3\bar{m}-1}.$$

We conclude by noting that $3\bar{m} - 1 > m$. \square

3.4 Some obstructions to the minimality of ω

In the following result, we present a series of obstructions that arise from the minimality of ω . Each of these scenarios is ruled out because, if any were to occur, we could use Stokes' theorem and/or the isoperimetric inequality, as stated in Lemma 3.1, to construct a new curve satisfying (2.2) that is either shorter than ω or has the same length but fails to be analytic or identical to $\bar{\omega}$.

Lemma 3.7. *By taking $\epsilon_0 > 0$ from Lemma 3.5 smaller if necessary, none of the following situations can occur for any $\epsilon \in (0, \epsilon_0)$:*

- (i) *There exist a loop ℓ of ω and a Lipschitz closed curve $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ that intersects the curve $\omega|_{[0, s_\ell^-]} * \omega|_{[s_\ell^+, L(\omega)]}$ such that $L(\eta) \leq L(\ell)$ and $|A(\ell)| \leq |A(\eta)|$.*
- (ii) *There exist $t_1 < t_2 \in I_\omega$ and a Lipschitz closed curve $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ that intersects the curve $\omega|_{[0, t_1]} * [\omega(t_1), \omega(t_2)] * \omega|_{[t_2, L(\omega)]}$ such that $L(\eta) \leq L(\omega|_{[t_1, t_2]}) - L([\omega(t_1), \omega(t_2)])$ and $|A(\omega|_{[t_1, t_2]} * [\omega(t_2), \omega(t_1)])| < |A(\eta)|$.*
- (iii) *There exist a simple loop ℓ of ω , $t \in I_\omega \setminus \text{int}(J_\ell)$, $\sigma = \pm 1$, and $s > 0$ such that $\sigma(P \circ \ell) \geq 0$ and $\omega(t) + \sigma(0, s) \in \text{spt}(\ell)$.*
- (iv) *There exist $t_1 < t_2$ in I_ω such that $P(\omega(t_1)) = P(\omega(t_2)) = 0$, $(P \circ \omega)|_{(t_1, t_2)} < 0$, and ω is injective on $[t_1, t_2]$.*
- (v) *There exist a loop ℓ of ω and $t^* \in I_\omega \setminus \text{int}(J_\ell)$ such that $\max_{t \in J_\ell} |Q(\omega(t))| \leq Q(\omega(t^*))$.*
- (vi) *There exist two loops ℓ_1 and ℓ_2 of ω such that $\text{int}(J_{\ell_1}) \cap \text{int}(J_{\ell_2}) = \emptyset$ and $Q \circ \ell_1, Q \circ \ell_2 \geq 0$.*

Moreover, for every $K > 0$, there exists $\epsilon_0(K) > 0$ such that for every $\epsilon \in (0, \epsilon_0(K))$, the following situations cannot occur:

- (vii) *There exist a loop ℓ of ω and $t^* \in I_\omega \setminus \text{int}(J_\ell)$ such that $\max_{t \in J_\ell} |Q(\omega(t))| \leq -Q(t^*)$ and $\omega_2(t^*) \geq K\epsilon$.*
- (viii) *There exist two loops ℓ_1 and ℓ_2 of ω such that $\text{int}(J_{\ell_1}) \cap \text{int}(J_{\ell_2}) = \emptyset$ and $\omega_2(s_{\ell_1}^-), \omega_2(s_{\ell_2}^-) \geq K\epsilon$.*

Proof. Assertion (i) follows from assertion (ii) with $t_1 = s_\ell^-$ and $t_2 = s_\ell^+$. To prove (ii), we consider $t_1 < t_2 \in I_\omega$ and a Lipschitz closed curve $\eta : [0, \tau] \rightarrow \mathbb{R}^2$ satisfying the assumptions. By reversing the orientation of η if necessary, we may assume that there exists $t^* \in [t_2, L(\omega)]$ such that $\eta(0) = \omega(t^*)$, and that $A(\eta)$ and $A(\alpha)$, with $\alpha := \omega|_{[t_1, t_2]} * [\omega(t_2), \omega(t_1)]$, have the same sign. Define, for any $r > 0$, the curve $\eta^r : [0, \tau] \rightarrow \mathbb{R}^2$ by $\eta^r(t) := \eta(0) + r(\eta(t) - \eta(0))$. Next, choose $r \in (0, 1]$ such that $A(\eta^r) = A(\alpha)$. By construction, $L(\eta^r) \leq L(\eta)$, and the concatenated path

$$\zeta := \omega|_{[0, t_1]} * [\omega(t_1), \omega(t_2)] * \omega|_{[t_2, t^*]} * \eta^r * \omega|_{[t^*, L(\omega)]}$$

connects $\omega(0) = \bar{\omega}(0)$ to $\omega(\epsilon) = \bar{\omega}(\epsilon)$. Furthermore, by Lemma 3.1 (i), it satisfies

$$\int_\zeta P^2 dx_2 = \int_\omega P^2 dx_2 - \int_\alpha P^2 dx_2 + \int_{\eta^r} P^2 dx_2 = A(\eta^r) - A(\alpha) = 0.$$

Using the assumption $L(\eta) \leq L(\omega|_{[t_1, t_2]}) - L([\omega(t_1), \omega(t_2)])$, we compute

$$L(\zeta) - L(\omega) = L([\omega(t_1), \omega(t_2)]) + L(\eta^r) - L(\omega|_{[t_1, t_2]}) \leq L(\eta^r) - L(\eta) \leq 0.$$

Thus, ζ minimizes the length among all curves satisfying (2.2). Since ζ is not analytic and not identical to $\bar{\omega}$, this is a contradiction.

To prove (iii), let ℓ be a simple loop of ω , $t \in I_\omega \setminus J_\ell$, $\sigma = \pm 1$, and $s > 0$ such that $\sigma P \geq 0$ over ℓ and $\omega(t) + \sigma(0, s) \in \text{spt}(\ell)$. We treat only the case $\sigma = 1$, the proof for $\sigma = -1$ follows in the same manner. By reversing the orientation of ℓ if necessary, we may assume that it is positively oriented. Since ℓ is simple, the weighted area of the loops ℓ and $\eta := \ell - (0, s)$ are given by the integrals of $Q(x) = 4x_1 P(x)$ over $\mathcal{E}_1(\ell)$ and $\mathcal{E}_1(\eta)$, respectively. Observe that for every $x_1 > 0$, the function $x_2 \mapsto Q(x_1, x_2) = 4x_1(x_1^2 - x_2^m)$ is decreasing on \mathbb{R} . Consequently, since $Q \geq 0$ over ℓ , we have

$$|A(\ell)| = \int_{\mathcal{E}_1(\ell)} Q(x) dx < \int_{\mathcal{E}_1(\eta)} Q(x_1, x_2 - (0, s)) dx = |A(\eta)|.$$

By construction, $\omega(t) \in \text{spt}(\eta)$ and $L(\eta) = L(\ell)$. Therefore, the result follows from (i).

To prove (iv), we need to perform a reflection with respect to the set

$$S := (\{0\} \times (-\infty, 0)) \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid P(x_1, x_2) = 0, x_1 \geq 0\}.$$

Since $m \geq 5$, S is a 1-dimensional submanifold of \mathbb{R}^2 of class at least C^2 . Consequently, the signed distance function d_S to S , assumed to be positive on the open set $S^+ = (0, +\infty) \times (0, -\infty) \cup \{P > 0, x_2 \geq 0\}$, is of class at least C^2 and solution to the eikonal equation $|\nabla d_S| = 1$ in a neighborhood of S (see [6, Lemma 14.16]). Thus, there is a ball \mathcal{B} centered at $\bar{\omega}(0) = (0, 0)$ such that for every $x \in S \cap \mathcal{B}$, $\nabla d_S(x)$ is the unit normal vector at x , pointing toward S^+ . Moreover, for any small $s \in \mathbb{R}$, the point $x_s := x + s \nabla d_S(x)$ belongs to \mathcal{B} and satisfies $d_S(x_s) = s$ and $\nabla d_S(x_s) = \nabla d_S(x)$. In particular, for distinct points x, y in $S \cap \mathcal{B}$, the lines $\{x_s, s \in \mathbb{R}\}$ and $\{y_s, s \in \mathbb{R}\}$, which correspond to the orbits of the vector field ∇d_S , do not intersect as long as both x_s and y_s remain within \mathcal{B} . For each $r \in (0, 1]$, we define the reflection map $R_r : \mathcal{B} \rightarrow \mathbb{R}^2$ with respect to S as

$$R_r(x) := x - (1 + r) d_S(x) \nabla d_S(x) \quad \forall x \in \mathcal{B}.$$

By shrinking \mathcal{B} if necessary, R_r is well-defined and constitutes a diffeomorphism onto its image that reverses orientation. Furthermore, since the closure \bar{S}^+ of S^+ is convex, the projection π onto \bar{S}^+ is well-defined, and we have $R_r(x) := x + (1 + r)(\pi(x) - x)$ for any $x \in \mathbb{R}^2 \setminus S^+$, which shows that R_r is 1-Lipschitz on $\mathbb{R}^2 \setminus S^+$. Now, consider $t_1 < t_2$ in I_ω satisfying the assumptions of (iv). If $\epsilon > 0$ is sufficiently small, the Lipschitz curve $\omega_R : [t_1, t_2] \rightarrow \mathbb{R}^2$ defined by $\omega_R := R_1 \circ \omega|_{[t_1, t_2]}$ is well-defined, and its support is contained in $\mathbb{R}^2 \setminus S^+$. Moreover, by the above discussion, the mapping $\psi : (s, t) \mapsto \bar{\omega}(t) + s \bar{\nu}(t)$, where $\bar{\nu}(t) := |\dot{\bar{\omega}}(t)| \nabla d_S(\bar{\omega}(t)) = (1, -\bar{m}t^{\bar{m}-1})$, provides a diffeomorphism that enables the computation of the weighted area of $\omega * \hat{\omega}$. Here, $\hat{\omega}$ denotes the segment of ω joining $\omega(t_2)$ to $\omega(t_1)$. The Jacobian determinant of ψ at (s, t) is equal to $|\dot{\bar{\omega}}(t)|^2 + s \langle \dot{\bar{\omega}}(t), \dot{\bar{\nu}}(t) \rangle$. Considering the winding number $k = \pm 1$ of a point in the open set \mathcal{E} enclosed by the simple closed curve $\omega|_{[t_1, t_2]} * \hat{\omega}$, we have

$$A(\omega|_{[t_1, t_2]} * \hat{\omega}) = k \int_{\mathcal{E}} Q(x) dx = k \int_0^{+\infty} \int_{N_t} Q(\psi(-s, t)) (|\dot{\bar{\omega}}(t)|^2 - s \langle \dot{\bar{\omega}}(t), \dot{\bar{\nu}}(t) \rangle) ds dt,$$

where, for each $t \in [0, +\infty)$, the interval $N_t \subset [0, +\infty)$ represents the set of $s \geq 0$ such that $\Psi(-s, t) \in \mathcal{E}$. The reflection map R_1 reverses the orientation and, by construction, the open set \mathcal{E}_R , enclosed by the simple closed curve $\omega_R * \hat{\omega}$, coincides with the set of $\psi(s, t)$ for $t \in [0, +\infty)$ and $s \in N_t$. Thus,

$$A(\omega_R * \hat{\omega}) = -k \int_{\mathcal{E}_R} Q(x) dx = -k \int_0^{+\infty} \int_{N_t} Q(\psi(s, t)) (|\dot{\bar{\omega}}(t)|^2 + s \langle \dot{\bar{\omega}}(t), \dot{\bar{\nu}}(t) \rangle) ds dt.$$

Since Q is positive on \mathcal{E}_R and negative on \mathcal{E} , we conclude that $A(\omega_R * \hat{\omega})$ and $A(\omega|_{[t_1, t_2]} * \hat{\omega})$ have the same sign and satisfy

$$|A(\omega_R * \hat{\omega})| - |A(\omega|_{[t_1, t_2]} * \hat{\omega})| = \int_0^{+\infty} \int_{N_t} \Delta(s, t) |\dot{\bar{\omega}}(t)|^2 ds dt, \quad (3.6)$$

where $\Delta(s, t)$ is defined as

$$\Delta(s, t) := Q^+(s, t) + Q^-(s, t) + s a(t) (Q^+(s, t) - Q^-(s, t)).$$

with $Q^\sigma(t, s) := Q(\psi(\sigma s, t))$ for $\sigma = \pm 1$ and $a(t) := s \langle \dot{\bar{\omega}}(t), \dot{\bar{\nu}}(t) \rangle / |\dot{\bar{\omega}}(t)|^2$. We claim that $\Delta(s, t) > 0$ for all (s, t) satisfying $t \in [0, +\infty)$ and $s \in N_t$. On the one hand, we have

$$a(t) = \frac{\langle \dot{\bar{\omega}}(t), \dot{\bar{\nu}}(t) \rangle}{|\dot{\bar{\omega}}(t)|^2} = \frac{-\bar{m}(\bar{m}-1)t^{\bar{m}-2}}{1 + \bar{m}^2 t^{\bar{m}-2}} = -\bar{m}(\bar{m}-1)t^{\bar{m}-2} + o(t^{\bar{m}-2}).$$

On the other hand, recalling that $Q^\sigma = 4(\bar{\omega}_1 + s\bar{\nu}_1)((\bar{\omega}_1 + s\bar{\nu}_1)^2 - (\bar{\omega}_2 + s\bar{\nu}_2^m))$ (where we omit the dependence on t), the Newton binomial formula gives

$$Q^\sigma = 4 \sum_{i=1}^{m+1} \sigma^i c_i s^i \quad \text{with} \quad c_i = \binom{3}{i} \bar{\nu}_1^i \bar{\omega}_1^{3-i} - \binom{m}{i} \bar{\omega}_1 \bar{\nu}_2^i \bar{\omega}_2^{m-i} - \binom{m}{i-1} \bar{\nu}_1 \bar{\nu}_2^{i-1} \bar{\omega}_2^{m-i+1},$$

with the convention that the binomial coefficient $\binom{k}{l}$ is zero when $l > k$. Thus, by setting $M := (m+1)/2$, we obtain

$$\Delta(s, t) = 8 \sum_{k=1}^M (c_{2k} + a c_{2k-1}) s^{2k} = 8 \sum_{k=1}^2 (c_{2k} + a c_{2k-1}) s^{2k} + 8 \sum_{k=3}^M (c_{2k} + a c_{2k-1}) s^{2k}.$$

Expanding c_i as a function of t as $t \rightarrow 0$, we have

$$c_i = \binom{3}{i} t^{\bar{m}(3-i)} (1 + o(1)) - \binom{m}{i-1} (-\bar{m})^{i-1} t^{(\bar{m}-1)(i-1)} t^{m-i+1} (1 + o(1)),$$

because the remaining terms are negligible. Note that for $i \geq 4$, the first term in the above expression vanishes. For $k \geq 3$, we verify that $c_{2k} + a c_{2k-1} > 0$ for $t > 0$ small, since $a < 0$, $c_{2k} > 0$ and $c_{2k-1} < 0$. Moreover, using the Taylor expansion of $a(t)$ derived earlier and recalling that $m \geq 5$, we obtain

$$c_2 + a c_1 = 3t^{\bar{m}} + o(t^{\bar{m}}) \quad \text{and} \quad c_4 + a c_3 = -\bar{m}(\bar{m}-1)t^{\bar{m}-2} + o(t^{\bar{m}-2}).$$

In conclusion, for all t, s with t small, we have

$$\begin{aligned} \Delta(s, t) &\geq 8(3t^{\bar{m}} + o(t^{\bar{m}}))s^2 + 8(-\bar{m}(\bar{m}-1)t^{\bar{m}-2} + o(t^{\bar{m}-2}))s^4 \\ &= 8s^2(3t^{\bar{m}} - s^2\bar{m}(\bar{m}-1)t^{\bar{m}-2} + o(t^{\bar{m}-2})). \end{aligned}$$

Since $\Psi(-s, t) \in \mathcal{E}$ for all (s, t) with $t \in [0, +\infty)$ and $s \in N_t$, \mathcal{E} is contained in the open half plane $\{x_1 > 0\}$, and $\bar{\omega}_1(t) \geq 0$ for all t (by Lemma 3.5 (i)), it follows that $\bar{\omega}_1(t) - s\bar{\nu}_1(t) \geq 0$, which shows $s \leq t^{\bar{m}}$. Therefore, $\Delta(s, t) > 0$ for all (s, t) satisfying $t \in [0, +\infty)$ and $s \in N_t$, and as a consequence, by (3.6), we have $|A(\omega_R * \hat{\omega})| > |A(\omega_{|[t_1, t_2]} * \hat{\omega})|$. To complete the argument, for each $r \in (0, 1]$, define the Lipschitz curve $\omega_R^r : [t_1, t_2] \rightarrow \mathbb{R}^2$ by $\omega_R^r := R_r \circ \omega_{|[t_1, t_2]}$, which provides a continuous deformation of ω_R . Since $|A(\omega_R * \hat{\omega})| > |A(\omega_{|[t_1, t_2]} * \hat{\omega})| > 0$, where $A(\omega_R * \hat{\omega})$ and $A(\omega_{|[t_1, t_2]} * \hat{\omega})$ have the same sign, and since $A(\omega_R^r * \hat{\omega})$ varies continuously with r and tends to 0 as $r \downarrow 0$, there is $\bar{r} > 0$ such that

$$A(\omega_R^{\bar{r}} * \hat{\omega}) = A(\omega_{|[t_1, t_2]} * \hat{\omega}).$$

Consider the concatenation $\zeta := \omega_{|[0, t_1]} * \omega_R^{\bar{r}} * \omega_{|[t_2, L(\omega)]}$. By the 1-Lipschitz property of $R_{\bar{r}}$, we have $L(\omega_R^{\bar{r}}) \leq L(\omega_{|[t_1, t_2]})$. Moreover, by Lemma 3.1 (i), we can write

$$\begin{aligned} \int_{\zeta} P^2 dx_2 &= \int_{\omega_{|[0, t_1]}} P^2 dx_2 + \int_{\omega_R^{\bar{r}}} P^2 dx_2 + \int_{\omega_{|[t_2, L(\omega)]}} P^2 dx_2 \\ &= \int_{\omega} P^2 dx_2 + \int_{\omega_R^{\bar{r}}} P^2 dx_2 - \int_{\omega} P^2 dx_2 = A(\omega_R^{\bar{r}} * \hat{\omega}) - A(\omega_{|[t_1, t_2]} * \hat{\omega}) = 0. \end{aligned}$$

Thus, the curve ζ minimizes the length among all curves satisfying (2.2). However, it is not analytic nor identical to $\bar{\omega}$, leading to a contradiction.

To prove (v), suppose that ℓ is a loop of ω and $t^* \in I_\omega \setminus (s_\ell^-, s_\ell^+)$ such that $\max_{t \in J_\ell} |Q(\omega(t))| \leq Q(\omega(t^*))$. Since the loop encloses a set with non-empty interior, we have $Q^* := Q(\omega(t^*)) > 0$. For every $x_2 \in \mathbb{R}$, the function $x_1 \geq 0 \mapsto Q(x_1, x_2)$ is convex. For $x_2 \geq 0$ it vanishes at $x_1 = x_2^{\bar{m}}$ with a nonnegative derivative, and for every $x_2 < 0$, it vanishes at $x_1 = 0$ with a nonnegative derivative. Thus, the set $\{Q = Q^*, x_1 \geq 0\}$ forms a curve, and by the implicit function theorem, there is a smooth function $\varphi : \mathbb{R} \rightarrow (0, +\infty)$ such that

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, Q(x_1, x_2) = Q^*\} = \{(\varphi(x_2), x_2) \mid x_2 \in \mathbb{R}\}.$$

Noting that $\varphi'(x_2) = -\frac{\partial Q}{\partial x_2}(\varphi(x_2), x_2) / \frac{\partial Q}{\partial x_1}(\varphi(x_2), x_2)$, we compute for any for every $x_2 \in \mathbb{R}$:

$$\varphi'(x_2) = \frac{m\varphi(x_2)x_2^{m-1}}{3\varphi(x_2)^2 - x_2^m} \quad \text{and} \quad \varphi''(x_2) = \frac{m(m-1)\varphi(x_2)x_2^{m-2}}{3\varphi(x_2)^2 - x_2^m} - \frac{2m^2\varphi(x_2)x_2^{3m-2}}{(3\varphi(x_2)^2 - x_2^m)^3}.$$

If $x_2 \geq 0$, since $\varphi(x_2) > x_2^{\bar{m}}$ and $P(\varphi(x_2), x_2) = Q^*/(4\varphi(x_2)) > 0$, we obtain

$$\varphi''(x_2) \leq \frac{m(m-1)\varphi(x_2)x_2^{m-2}}{2\varphi(x_2)^2 + P(\varphi(x_2), x_2)} \leq \frac{m(m-1)}{2} \frac{x_2^{m-2}}{\varphi(x_2)} \leq \bar{m}(m-1)x_2^{\bar{m}-2}.$$

If $x_2 < 0$, since for every $a > 0$, the function $z \geq 0 \mapsto za^3/(3z^2 + a)^3$ attains its maximum at $z = \sqrt{a/15}$ with value $15^{5/2}\sqrt{a}/18^3$ and $\varphi(x_2) > 0$, we have

$$0 \leq \varphi''(x_2) \leq -\frac{2m^2\varphi(x_2)x_2^{3m-2}}{(3\varphi(x_2)^2 - x_2^m)^3} = \frac{2m^2}{x_2^2} \frac{\varphi(x_2)(-x_2^m)^3}{(3\varphi(x_2)^2 + (-x_2)^m)^3} \leq \frac{2m^2 15^{\frac{5}{2}}}{18^3} |x_2|^{\bar{m}-2}.$$

Therefore, there is a constant $K > 0$ such that we have for every $\bar{x}_2 \in [-1, 1]$, we have

$$\varphi(x_2) \leq \varphi(\bar{x}_2) + \varphi'(\bar{x}_2)(x_2 - \bar{x}_2) + K(x_2 - \bar{x}_2)^2 \quad \forall x_2 \in \mathbb{R}.$$

This shows that for sufficiently small $\epsilon > 0$, there is a disc \mathcal{D} of radius $\rho := L(\ell)/(2\pi)$, whose boundary is a circle passing through $\omega(t^*)$, and which is contained in $\{Q \geq Q(\omega(t^*))\}$. Therefore, let η be a parametrization of the circle with length $L(\ell)$. By Lemma 3.1 (ii), we have

$$|A(\ell)| \leq \frac{1}{4\pi} \sup_{x \in \mathcal{E}(\ell)} |Q(x)| L(\ell)^2 = Q(\omega(t^*)) \pi \rho^2 \leq \int_{\mathcal{D}} Q(x) dx = A(\eta),$$

which contradicts obstruction (i).

To prove (vi), we assume that $\max_{t \in I_{\ell_1}} |Q(\omega(t))| \geq \max_{t \in I_{\ell_2}} |Q(\omega(t))|$ and consider $t^* \in I_{\ell_1}$ such that $Q(\omega(t^*)) = \max_{t \in I_{\ell_1}} |Q(\omega(t))|$. The result follows directly from (v).

To prove (vii) we proceed similarly to (v). Let $K > 0$, ℓ a loop of ω , and $t^* \in I_\omega \setminus J_\ell$ be such that $\max_{t \in J_\ell} |Q(\omega(t))| \leq q^*$, with $q^* := -Q(\omega(t^*)) > 0$, and $\omega_2(t^*) \geq K\epsilon$. For every $x_2 > 0$, the function $x_1 \geq 0 \mapsto Q(x_1, x_2)$ is convex, vanishes at $x_1 = 0$ and $x_1 = x_2^{\bar{m}}$, and attains its minimum on the interval $[0, x_2^{\bar{m}}]$ at $x_1 = h(x_2) := x_2^{\bar{m}}/\sqrt{3}$, with the value $-8x_2^{3\bar{m}}/(3\sqrt{3})$. By the implicit function theorem, there exist smooth functions $\varphi_-, \varphi_+ : (x_2(q^*), +\infty) \rightarrow (0, +\infty)$, with $x_2(q^*)$ defined such that $-8x_2(q^*)^{3\bar{m}}/(3\sqrt{3}) = -q^*$, satisfying $0 < \varphi_- < h < \varphi_+ < x_2^{\bar{m}}$, for which we have

$$\begin{aligned} & \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 > x_2(q^*), Q(x_1, x_2) = -q^*\} \\ & = \{(\varphi_-(x_2), x_2) \mid x_2 > x_2(q^*)\} \cup \{(\varphi_+(x_2), x_2) \mid x_2 > x_2(q^*)\} \end{aligned}$$

and

$$\left\{x \in \mathbb{R}^2 \mid x_1 \geq 0, Q(x) < -q^*\right\} = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid \varphi_-(x_2) < x_1 < \varphi_+(x_2), x_2 > x_2(q^*)\right\}.$$

By Lemma 3.5 (iv) and (v), we have

$$q^* = 4\omega_1(t^*) |P(\omega(t^*))| \leq 8\epsilon^{\bar{m}}\beta \leq 8C\epsilon^{2m-1}. \quad (3.7)$$

Suppose $x^* = (x_1^*, x_2^*) := \omega(t^*)$ belongs to the graph of φ_- . Then,

$$P(x^*) = (x_1^*)^2 - (x_2^*)^m = \varphi_-(x_2^*)^2 - (x_2^*)^m < h(x_2^*)^2 - (x_2^*)^m = -\frac{2(x_2^*)^m}{3}.$$

Using the assumption $x_2^* \geq K\epsilon$ and $x_1^* \geq C(K)\epsilon^{\bar{m}}$ (from Lemma 3.6), it follows that $q^* \geq c\epsilon^{3\bar{m}}$ for some constant $c > 0$, which contradicts (3.7) for sufficiently small $\epsilon > 0$. Consequently, we may assume that $x^* = \omega(t^*)$ lies on the graph of φ_+ . As in the proof of (v), the first and second derivatives of φ_+ are given by

$$\varphi'_+(x_2) = \frac{m\varphi_+(x_2)x_2^{m-1}}{3\varphi_+(x_2)^2 - x_2^m} \quad \text{and} \quad \varphi''_+(x_2) = \frac{m(m-1)\varphi_+(x_2)x_2^{m-2}}{3\varphi_+(x_2)^2 - x_2^m} - \frac{2m^2\varphi_+(x_2)x_2^{3m-2}}{(3\varphi_+(x_2)^2 - x_2^m)^3}.$$

The denominator $3(\varphi_+^2 - h^2)$ is positive. Moreover, for every $x_2 \geq K\epsilon/2$, we have $\varphi_+(x_2) > h(x_2) \geq h(K\epsilon/2)$, implying, by (3.7), that

$$\varphi_+(x_2)^2 - 2h^2(x_2) = \left(x_2^m - \frac{q^*}{4\varphi_+(x_2)}\right) - \frac{2x_2^m}{3} = \frac{x_2^m}{3} - \frac{q^*}{4\varphi_+(x_2)} > 0, \quad (3.8)$$

for sufficiently small $\epsilon > 0$. Consequently, for $x_2 \in [K\epsilon/2, 2\epsilon]$, we deduce

$$\varphi''_+(x_2) \geq -\frac{2m^2\varphi_+(x_2)x_2^{3m-2}}{(3\varphi_+(x_2)^2 - x_2^m)^3} \geq -\frac{2m^2x_2^{\bar{m}}x_2^{3m-2}}{(3h(x_2)^2)^3} = -2m^2x_2^{\bar{m}-2}.$$

Let \mathcal{D} denote the open disc of radius $\rho := L(\ell)/(2\pi)$, which touches the graph of φ_+ from below at $\omega(t^*)$. We claim that \mathcal{D} lies within the set $\{|Q| \geq |Q(\omega(t^*))| = q^*\}$ for sufficiently small $\epsilon > 0$. If not, there exists $x_2 \in [x_2^* - L(\ell), x_2^* + L(\ell)]$ such that $h(x_2) \geq \varphi_+(x_2^*) - L(\ell)$. Since h is 1-Lipschitz for small ϵ , we infer $\varphi_+(x_2^*) - h(x_2^*) \leq 2L(\ell)$. This contradicts the inequalities

$$\varphi_+(x_2^*) - h(x_2^*) = \frac{\varphi_+(x_2^*)^2 - h(x_2^*)^2}{\varphi_+(x_2^*) + h(x_2^*)} \geq \frac{h(x_2^*)^2}{\varphi_+(x_2^*) + h(x_2^*)} \geq 2h(x_2^*) = \frac{2}{\sqrt{3}}(x_2^*)^{\bar{m}} \geq \frac{2K^{\bar{m}}}{\sqrt{3}}\epsilon^{\bar{m}}$$

and

$$L(\ell) \leq C\beta^{1-\frac{1}{m}} \leq C^2\epsilon^{(3\bar{m}-1)(m-1)/m} = o(\epsilon^{\bar{m}}).$$

following from (3.8) and Lemma 3.5 (v), (viii). We conclude as in (v).

To prove (viii), we assume that $\max_{t \in I_{\ell_1}} |Q(\omega(t))| \geq \max_{t \in I_{\ell_2}} |Q(\omega(t))|$. We consider $t^* \in I_{\ell_1}$ such that $|Q(\omega(t^*))| = \max_{t \in I_{\ell_1}} |Q(\omega(t))|$ and observe that, by Lemma 3.6, we may assume that $\omega_2(t^*) \geq K\epsilon$. If $Q(\omega(t^*)) \geq 0$, the result follows from (v); otherwise, it follows from (vii). \square

3.5 Intersections of ω with $\{P = 0\}$ and loops

By analyticity, the sets $\text{spt}(\omega)$ and $\{P = 0\}$ intersect finitely many times. Consequently, we define $\tau_0 = 0 < \tau_1 < \dots < \tau_N = L(\omega)$ such that

$$(P \circ \omega)^{-1}(\{0\}) = \left\{\tau_0 = 0, \dots, \tau_N = L(\omega)\right\}.$$

We then set

$$I_i := [\tau_i, \tau_{i+1}] \quad \forall i \in \mathcal{I} := \{0, \dots, N-1\}.$$

The intervals (τ_i, τ_{i+1}) for $i \in \mathcal{I}$ are the maximal intervals where the function $t \mapsto P(\omega(t))$ is either strictly positive or strictly negative. By Lemma 3.5 (i), the same property holds for $t \mapsto Q(\omega(t))$. We denote by \mathcal{I}^+ (resp. \mathcal{I}^-) the set of $i \in \mathcal{I}$ such that $P \circ \omega|_{(\tau_i, \tau_{i+1})} > 0$ (resp. $P \circ \omega|_{(\tau_i, \tau_{i+1})} < 0$). For every $i \in \mathcal{I}^+$ (resp. $i \in \mathcal{I}^-$), we define $\sigma_i = 1$ (resp. $\sigma_i = -1$). Finally, for each $i \in \mathcal{I}$, we refer to the first loop of $\omega|_{I_i}$, if it exists, as a loop ℓ associated with a subinterval $J_\ell = [s_\ell^-, s_\ell^+] \subset I_i$ such that $\omega|_{[s_\ell^-, s_\ell^+]}$ is injective. Such a loop is necessarily simple.

The following result consolidates several properties essential for completing the proof of Proposition 2.2. Assertions (i)-(iv) are based on our preliminary results and the Gauss-Bonnet formula. Assertions (v), (vi) and (xi) are direct consequences of the obstructions described in Lemma 3.7. Lastly, assertions (vii)-(x) are established through a combination of curvature arguments, incorporating the Gauss-Bonnet formula, and the obstructions from Lemma 3.7.

Lemma 3.8. *By taking $\epsilon_0 > 0$ from Lemma 3.5 smaller if necessary, for any $\epsilon \in (0, \epsilon_0)$ and every $i \in \mathcal{I}$, then following properties hold.*

- (i) *The signed curvature κ of the restriction $\omega|_{I_i}$ satisfies $\sigma_i \lambda \kappa \geq 0$.*
- (ii) *If $\omega|_{I_i}$ admits a first loop ℓ then, by setting $\sigma := \sigma_i \text{sgn}(\lambda)$, the set $\mathcal{E}(\ell)$ is strictly convex, $\mathcal{D}(\ell) = \mathcal{E}(\ell) = \mathcal{E}_\sigma(\ell)$, $\sigma \text{ang}(\dot{\omega}(s_\ell^+), \dot{\omega}(s_\ell^-)) \in (0, \pi)$, and $\int_{t \in J_\ell} \sigma \dot{\theta}(t) dt \in (\pi, 2\pi)$.*
- (iii) *$0 \in \mathcal{I}^+$ and $\omega_2(\tau_{i+1}) > \omega_2(\tau_0) = 0$ for all $i \in \mathcal{I}$.*
- (iv) *If $i \in \mathcal{I}^+$ and $i+1 \in \mathcal{I}$, then $i+1 \in \mathcal{I}^-$.*
- (v) *If $i \in \mathcal{I}^-$, then $\omega|_{I_i}$ admits a first loop ℓ associated with an interval $J_\ell \subset (\tau_i, \tau_{i+1})$.*
- (vi) *There is at most one index $i \in \mathcal{I}^+$ such that $\omega|_{I_i}$ is not injective.*
- (vii) *If $i \in \mathcal{I}^+$, then $\omega|_{I_i}$ admits at most one loop ℓ .*
- (viii) *If $\omega|_{I_i}$ admits a first loop ℓ , then for every $t \in [\tau_i, s_\ell^-]$, $\theta(t) \neq \sigma_i \pi/2 \pmod{2\pi}$.*
- (ix) *If $\omega|_{I_i}$ admits a unique loop, then $\lambda \cdot (\omega_2(\tau_{i+1}) - \omega_2(\tau_i)) \leq 0$.*

Moreover, for every $K > 0$, there exists $\epsilon_0(K) > 0$ such that for every $\epsilon \in (0, \epsilon_0(K))$ and every $i \in \mathcal{I}$, the following hold.

- (x) *If $i \in \mathcal{I}^-$ and $\omega|_{I_i}$ admits a first loop ℓ such that $\omega_2(s_\ell^-) \geq K\epsilon$ then ℓ is the unique loop of $\omega|_{I_i}$.*
- (xi) *There is at most one index $i \in \mathcal{I}^-$ such that $\omega|_{I_i}$ admits a loop ℓ such that $\omega_2(s_\ell^-) \geq K\epsilon$.*

Proof. Assertion (i) follows from the construction, (2.3), and (3.1). To prove (ii), suppose that $\omega|_{I_i}$ admits a first loop ℓ on an interval $[s_\ell^-, s_\ell^+]$. Reversing the orientation if necessary, we may assume that ℓ is positively oriented. Let δ be the discontinuity of the signed curvature of ℓ at s_ℓ^- . Since $\dot{\ell}(s_\ell^-) \neq \dot{\ell}(s_\ell^+)$ (because ω is solution of (2.3)), it follows that $\delta \neq 0$. If $\delta < 0$, then the point $\eta(s_\ell^- - s)$ would lie in $\mathcal{D}(\ell)$ for sufficiently small $s > 0$ (note that Lemma 3.5 (i)-(ii) ensure that $s_\ell^- > 0$). However, $\mathcal{D}(\ell)$ does not intersect the set $\{P = 0\}$, which contains $\omega(0) = A_0$. Since $\omega|_{[0, s_\ell^+]}$ is injective, this leads to a contradiction. We conclude the proof by applying Lemma 3.3 (i), (3.3), and (3.1), and by observing that the signed curvature of ℓ is nonzero on its smooth part.

Assertion (iii) follows directly from Lemma 3.5 (i)-(ii). To prove (iv), suppose for contradiction that both i and $i+1$ belongs to \mathcal{I}^+ . In this case, the curve $t \mapsto \omega(t)$ must remain within the convex set $\{P \geq 0, x_1 \geq 0\}$ for t near τ_{i+1} and be tangent to the set $\{P = 0\}$ at

$\omega(\tau_{i+1})$. If $\omega(\tau_{i+1}) = 0$, then we must have $\theta(\tau_{i+1}) = \pm\pi/2 \pmod{2\pi}$, which is prohibited (see the proof of Lemma 3.5 (ii)). Otherwise, by Lemma 3.5 (i), $\omega(\tau_{i+1})$ belongs to $\{P \geq 0, x_1 > 0\}$, a curve with nonzero curvature. It follows that $\dot{\theta}(\tau_{i+1}) \neq 0$. However, because $P(\omega(\tau_{i+1})) = 0$, equation (2.3) implies that $\dot{\theta}(\tau_{i+1}) = 0$, leading to a contradiction.

Assertion (v) follows directly from obstruction (iv) in Lemma 3.7. Similarly, assertions (vi) and (xi) follow respectively from obstructions (vi) and (viii) in Lemma 3.7.

To prove (vii), we suppose for contradiction that $\omega|_{I_i}$ admits a first loop ℓ on an interval $J_\ell = [s_\ell^-, s_\ell^+] \subset I_i$. By obstructions (vi) of Lemma 3.7, the curve $\omega|_{[s_\ell^+, \tau_{i+1}]}$ is injective. Therefore, it remains to show that the curves $\omega|_{[\tau_i, s_\ell^+]}$ and $\omega|_{(s_\ell^+, \tau_{i+1}]}$ do not intersect. Assume, for the sake of contradiction, that there is \bar{t} in (s_ℓ^+, τ_{i+1}) such that $\omega(\bar{t}) \in \text{spt}(\omega|_{[\tau_i, s_\ell^+]})$ and $\omega(t) \notin \text{spt}(\omega|_{[\tau_i, s_\ell^+]})$ for all $t \in (s_\ell^+, \bar{t})$. Note that, by analyticity, such a \bar{t} must exist if the curves $\omega|_{[\tau_i, s_\ell^+]}$ and $\omega|_{(s_\ell^+, \tau_{i+1}]}$ intersect. Now, consider $\bar{s} \in [\tau_i, s_\ell^+]$ such that $\omega(\bar{s}) = \omega(\bar{t})$. Observe that $\bar{s} \neq s_\ell^-$, because otherwise $\omega(\bar{s}) = \omega(s_\ell^-) = \omega(s_\ell^+)$, which contradicts the injectivity of $\omega|_{[s_\ell^+, \tau_{i+1}]}$. We distinguish two cases: $\bar{s} < s_\ell^-$ and $\bar{s} > s_\ell^-$.

Case $\bar{s} < s_\ell^-$. Define $\bar{\ell} = \omega|_{[\bar{s}, s_\ell^-]} * \omega|_{[s_\ell^+, \bar{t}]}$. By construction, $\bar{\ell}$ is a piecewise smooth continuous curve which is closed, simple, parametrized by arc length, and whose signed curvature has the same sign $\sigma = \pm 1$ as that of ℓ on its smooth segments. Since both ℓ and $\bar{\ell}$ have at most two singularities, Lemma 3.3 (ii) gives $\mathcal{D}(\bar{\ell}) = \mathcal{E}_\sigma(\bar{\ell}) = \mathcal{E}(\bar{\ell})$ and $\mathcal{D}(\ell) = \mathcal{E}_\sigma(\ell) = \mathcal{E}(\ell)$. If $\sigma = 1$, then both ℓ and $\bar{\ell}$ are positively oriented. By assertion (ii), $\mathcal{E}(\ell)$ is a convex set, and $\text{ang}(\dot{\omega}(s_\ell^+), \dot{\omega}(s_\ell^-)) \in (0, \pi) \in (0, \pi)$. Consequently, for every nonzero vector u such that $\text{ang}(\dot{\omega}(s_\ell^+), u) \in (\text{ang}(\dot{\omega}(s_\ell^+), \dot{\omega}(s_\ell^-)), \pi)$, we have $\text{ang}(\dot{\omega}(s_\ell^-), u) \in (\text{ang}(\dot{\omega}(s_\ell^-), \dot{\omega}(s_\ell^+)), \pi)$. By Lemma 3.2, it follows that $\mathcal{E}(\ell) \subset \mathcal{E}(\bar{\ell})$. Therefore, there exists a translation of the loop ℓ in the directions $(0, \pm 1)$ that intersects $\text{spt}(\bar{\ell})$. Lemma 3.7 (iii) provides an obstruction. The case $\sigma = -1$ follows analogously.

Case $\bar{s} > s_\ell^-$. Define $\bar{\ell} = \omega|_{[\bar{s}, \bar{t}]}$. Since $\omega|_{[s_\ell^+, \tau_{i+1}]}$ is injective, $\bar{\ell}$ is a piecewise smooth continuous curve which is closed, simple, parametrized by arc length, and whose signed curvature has the same sign $\sigma = \pm 1$ as that of ℓ on its smooth segments. Noting that both ℓ and $\bar{\ell}$ have only one singularity, Lemma 3.3 (ii) implies $\text{spt}(\omega|_{(s_\ell^-, \bar{s})}) \subset \mathcal{E}(\bar{\ell})$. Consequently, there exists a translation of the loop ℓ in the directions $(0, \pm 1)$ that intersects $\text{spt}(\bar{\ell})$. Lemma 3.7 (iii) then provides an obstruction.

Assertion (x) follows in the same manner by noting that the assumption $\omega_2(s_\ell^-) \geq K\epsilon$, together with Lemma 3.6 and obstruction (viii) of Lemma 3.7, implies that the curve $\omega|_{[s_\ell^+, \tau_{i+1}]}$ is injective.

To prove (viii), we assume for the sake of contradiction that there is $\bar{t} \in [\tau_i, s_\ell^-]$ such that $\theta(\bar{t}) = \sigma_i \pi/2 \pmod{2\pi}$. We address the case $\sigma_i = 1$, with the other case left to the reader. Since the curve ω points toward the set $\{P \geq 0\}$ at $\omega(\tau_i)$ and remains within this set on the interval J_ℓ , we must have $\bar{t} > \tau_i$ and $P(\omega(\bar{t})) > 0$. Let $\bar{h} > 0$ be the infimum of $h > 0$ such that $x(h) := \omega(\bar{t}) + h(0, 1)$ lies in $\text{spt}(\omega|_{[\tau_i, \bar{t}]}) \cup \{P = 0\}$. Note that $\bar{h} > 0$ because $P(\omega(\bar{t})) > 0$ and $\omega|_{[\tau_i, \bar{t}]}$ is injective. If $x(\bar{h})$ belongs to $\text{spt}(\omega|_{[\tau_i, \bar{t}]})$, then consider the unique $\tau \in [\tau_i, \bar{t}]$ such that $x(\bar{h}) = \omega(\tau)$, and define the curve $\eta := \omega|_{[\tau, \bar{t}]} * [\omega(\bar{t}), \omega(\tau)]$. Otherwise, define $\eta := \omega|_{[\tau_i, \bar{t}]} * [\omega(\bar{t}), x(\bar{h})] * P|_{[x(\bar{h}), \omega(\tau_i)]}$, where $P|_{[x(\bar{h}), \omega(\tau_i)]}$ denotes the segment of the curve $\{P = 0, x_1 \geq 0\}$ connecting $x(\bar{h})$ to $\omega(\tau_i)$. In both cases, the curve η is closed and simple. Since $\theta(\bar{t}) = \sigma_i \pi/2 \pmod{2\pi}$ and $\dot{\theta}(\bar{t}) \neq 0$, we have $\omega(\bar{t} + s) \in \mathcal{D}(\eta)$ for $s > 0$ small. Moreover, the restriction $\omega|_{[\tau_i, s_\ell^+]}$ is injective, does not intersect $\{P = 0\}$ and, by Lemma 3.4, $\omega|_{[\bar{t}, s_\ell^+]}$ does not intersect the segment $[\omega(\bar{t}), x(\bar{h})]$. Consequently, the curve $\omega|_{(\bar{t}, s_\ell^+)}$, and hence the loop ℓ , is contained in $\mathcal{D}(\eta)$. Therefore, a translation of the loop ℓ in the directions $(0, -1)$ must intersect $\text{spt}(\eta)$, specifically $\text{spt}(\omega|_{[\tau_i, \bar{t}]})$. This leads to a contradiction, as stated in obstruction (iii) of Lemma 3.7.

To prove (ix), we assume that $\omega|_{I_i}$ contains a unique loop ℓ , and we define the closed, simple

curve

$$\eta := \omega_{|[\tau_i, s_\ell^-]} * \omega_{|[s_\ell^+, \tau_{i+1}]} * P_{[\omega(\tau_{i+1}), \omega(\tau_i)]},$$

where $P_{[\omega(\tau_{i+1}), \omega(\tau_i)]}$ denotes the segment of the curve $\{P = 0, x_1 \geq 0\}$ connecting $\omega(\tau_{i+1})$ to $\omega(\tau_i)$. We consider the case $i \in \mathcal{I}^+$ and $\lambda > 0$. In this setting, by assertion (i), the signed curvature of $\omega_{|[\tau_i, s_\ell^-]}$ and $\omega_{|[s_\ell^+, \tau_{i+1}]}$ is nonnegative. Suppose, for the sake of contradiction, that $\omega_2(\tau_{i+1}) > \omega_2(\tau_i)$. In this case, the segment $P_{[\omega(\tau_{i+1}), \omega(\tau_i)]}$ also has nonnegative signed curvature. Thus, by Lemma 3.3 (ii), we have $\mathcal{D}(\eta) = \mathcal{E}_\sigma(\eta) = \mathcal{E}(\eta)$. Referring to assertion (ii), we deduce that for every nonzero vector u such that $\text{ang}(\dot{\omega}(s_\ell^+), u) \in (\text{ang}(\dot{\omega}(s_\ell^+), \dot{\omega}(s_\ell^-)), \pi)$, we have $\text{ang}(\dot{\omega}(s_\ell^-), u) \in (\text{ang}(\dot{\omega}(s_\ell^-), \dot{\omega}(s_\ell^+)), \pi)$. By Lemma 3.2, we conclude that $\mathcal{E}(\ell) \subset \mathcal{E}(\eta)$. This contradicts obstruction (iii) of Lemma 3.7. The other cases can be proven following the same reasoning. \square

3.6 A closer look at the first loop of ω

The results established in Lemma 3.8 are not yet sufficient to prove Proposition 2.2. To complete the proof, we must verify the necessary conditions stated in assertions (x) and (xi). These conditions will be derived from an analysis of the first loop of ω . By Lemma 3.5 (vi), the curve ω is not injective. Therefore, we consider its first loop, defined on an interval $J_0 := [s_0^-, s_0^+]$, where $\omega(s_0^-) = \omega(s_0^+)$, and $s_0^+ \in I_\omega$ is the smallest $s \in I_\omega$ such that $\omega_{|[0, s]}$ is injective but $\omega_{|[0, s]}$ is not. We denote this loop by ℓ_0 and set $L_0 := L(\ell_0)$. By obstruction (iv) of Lemma 3.7, $\omega_{|I_1}$ is not injective. Consequently, either $\omega_{|I_0}$ is not injective, in which case $J_0 \subset I_0$, or $\omega_{|I_0}$ is injective and $J_0 \subset I_1$. In summary, we have

$$J_0 \subset I_0 \quad \text{or} \quad J_0 \subset I_1. \quad (3.9)$$

The objective of the present section is to prove the following result.

Proposition 3.9. *There is $c > 0$ such that, by taking $\epsilon_0 > 0$ from Lemma 3.5 smaller if necessary, for any $\epsilon \in (0, \epsilon_0)$, the following holds:*

$$\omega_1(s) \geq c\epsilon^{\bar{m}} \quad \text{and} \quad \omega_2(s) \geq c\epsilon \quad \forall s \in [s_0^-, L(\omega)]. \quad (3.10)$$

The proof of Proposition 3.9 will follow from several lemmas. Before starting, we define β_0 , $t_0 \in I_0$ and $x_0 > 0$, $y_0 \in \mathbb{R}$, and $\delta_0 > 0$ by

$$\beta_0 := \max_{t \in J_0} |P(\omega(t))| = |P(\omega(t_0))| > 0, \quad x_0 := \omega_1(t_0), \quad y_0 := \omega_2(t_0), \quad \delta_0 := \frac{\beta_0}{\max\{x_0, |y_0|^{\bar{m}}\}}.$$

The first lemma is the following.

Lemma 3.10. *For every $K > 0$, there are $\epsilon_0(K), c(K) > 0$ such that there holds*

$$L_0 \geq K\delta_0 \quad \implies \quad y_0 \geq c(K)\epsilon.$$

Proof. Let $K > 0$ be fixed. Consider the point $p := (\epsilon^{\bar{m}} + L_0/4, \epsilon)$. We note that the length of the segment $[\omega(\epsilon), p]$ is equal to $L_0/4$ and

$$Q(p) = 4 \left(\epsilon^{\bar{m}} + \frac{L_0}{4} \right) \left(\left(\epsilon^{\bar{m}} + \frac{L_0}{4} \right)^2 - \epsilon^m \right) = 2L_0 \left(\epsilon^{\bar{m}} + \frac{L_0}{4} \right) \left(\epsilon^{\bar{m}} + \frac{L_0}{8} \right) \geq 2\epsilon^m L_0.$$

From the study of the level set $\{Q = Q(p), x_1 \geq 0\}$ conducted in the proof of assertion (v) of Lemma 3.7, there exists a disc $D \subset \mathbb{R}^2$ such that $Q(x) \geq Q(p)$ for all $x \in D$, and whose boundary is a circle ν such that $\nu(0) = p$ and $L(\nu) = L_0/2$. Let η be the curve given by the

concatenation of $[\bar{\omega}(\epsilon), p]$, ν , and $[p, \bar{\omega}(\epsilon)]$. By construction $L(\eta) = L_0$, so by Lemma 3.7 (i), we have necessarily, since ω is optimal,

$$|A(\ell_0)| > |A(\eta)|. \quad (3.11)$$

Assume that $L_0 \geq K\delta_0$. We estimate $|A(\ell_0)|$ from above and $|A(\eta)|$ from below. By Lemma 3.1 (ii), we have

$$|A(\ell_0)| \leq \frac{1}{\pi} \sup_{x \in \mathcal{E}(\ell_0)} |x_1 P(x)| L_0^2 \leq \frac{\beta_0 L_0^2}{\pi} \sup_{x \in \mathcal{E}(\ell_0)} x_1 \leq \frac{(x_0 + L_0) \beta_0 L_0^2}{\pi},$$

where we used that $\sup_{x_1 \in \mathcal{E}(\ell_0)}(x_1) = \max_{t \in I_0} \omega_1(t) \leq x_0 + L_0$, since $|\dot{\omega}_1| \leq 1$. Moreover, using the above lower bound for $Q(p)$, $\mathcal{L}(D) = L_0^2/(16\pi)$, and the assumption, we have

$$|A(\eta)| = |\mathcal{L}_Q(\mathcal{E}(\eta))| \geq \mathcal{L}(\mathcal{E}(\eta)) Q(p) \geq \frac{\epsilon^m L_0^3}{8\pi} \geq \frac{\epsilon^m L_0^2 K \delta_0}{8\pi} = \frac{\epsilon^m L_0^2 K \beta_0}{8\pi \max\{x_0, |y_0|^{\bar{m}}\}}.$$

Plugging the bounds on $|A(\ell_0)|$ and $|A(\eta)|$ in (3.11), we obtain

$$(x_0 + L_0) \max\{x_0 + L_0, |y_0|^{\bar{m}}\} \geq (x_0 + L_0) \max\{x_0, |y_0|^{\bar{m}}\} \geq \frac{K}{8} \epsilon^m.$$

If $x_0 + L_0 \leq |y_0|^{\bar{m}}$, the above inequality implies $|y_0|^{\bar{m}} \geq K\epsilon^m/8$, which proves the result. Alternatively, if $x_0 + L_0 > |y_0|^{\bar{m}}$, the inequality gives $x_0 + L_0 \geq c'\epsilon^m$ with $c' := \sqrt{K}/8$. By Lemma 3.5 (viii) and (v), we know that $L_0 \leq C\beta^{1-\frac{1}{m}} \leq C^2\epsilon^{(3\bar{m}-1)(1-\frac{1}{m})}$, where $(3\bar{m}-1)(1-\frac{1}{m}) > \bar{m}$ (because $m \geq 5$). Thus, for sufficiently small $\epsilon > 0$, we have $x_0 \geq c'\epsilon^m/2$. Since $\beta_0 \leq \beta = o(\epsilon^m)$ (by Lemma 3.5 (v)), it follows that

$$y_0^m = x_0^2 - \beta_0 \geq \frac{c'}{2} \epsilon^m + o(\epsilon^m),$$

which concludes the proof. \square

Our objective is now to demonstrate that the assumption of Lemma 3.10 holds for some constant $K > 0$ depending only on m . To this end, we begin with the following preparatory lemma.

Lemma 3.11. *There is $c > 0$ such that, by taking $\epsilon_0 > 0$ from Lemma 3.5 smaller if necessary, for any $\epsilon \in (0, \epsilon_0)$, the following holds:*

$$x_0 \geq c\beta_0^{\frac{1}{2}} \quad \text{and} \quad y_0 \geq -\beta_0^{\frac{1}{2}}. \quad (3.12)$$

Proof. We consider separately the cases where $J_0 \subset I_0$ and $J_0 \subset I_1$ (see (3.9)).

Case $J_0 \subset I_0$: Since $\beta_0 = P(\omega(t_0)) > 0$ in this case, we have $x_0^2 = \beta_0 + y_0^m$, which shows that the inequality for x_0 in (3.12) follows from the inequality for y_0 , with $c = 1/2$ and for sufficiently small $\epsilon > 0$. To prove that $y_0 \geq -\beta_0^{1/2}$, we proceed by contradiction and assume $y_0 < -\beta_0^{1/2}$. If $\omega_2(t) \geq 0$ for some $t \in J_0$ then $L_0 \geq 2|y_0| \geq 2\beta_0^{1/2}$ and thus (using that $\beta_0 = x_0^2 - y_0^m = x_0^2 + |y_0|^m \leq 2 \max\{x_0^2, |y_0|^m\}$)

$$L_0 \geq 2\beta_0^{\frac{1}{2}} = 2\beta_0\beta_0^{-\frac{1}{2}} \geq \sqrt{2} \frac{|P(\omega(t_0))|}{\max\{x_0, |y_0|^{\bar{m}}\}} = \sqrt{2} \delta_0,$$

which, by Lemma 3.10, yields $y_0 \geq c(\sqrt{2})\epsilon$, a contradiction. Therefore, we must have $\omega_2(t) < 0$ for all $t \in J_0$. This implies $\omega_1(t) < P(\omega(t))^{1/2} \leq \beta_0^{1/2}$ for all $t \in J_0$. Let $s_1, s_2 \in I_\omega$ be the

maximal interval containing t_0 such that $\omega_2(t) < 0$ for all $t \in (s_1, s_2)$. By obstruction (v) of Lemma 3.7, we infer that

$$\max_{t \in J_0} Q(\omega(t)) = \max_{t \in J_0} |Q(\omega(t))| > \max_{t \in I_\omega \setminus J_0} |Q(\omega(t))| \geq \max_{t \in (s_1, s_2) \setminus J_0} Q(\omega(t)),$$

which implies

$$Q(\omega(s_1)), Q(\omega(s_2)) \leq \max_{t \in (s_1, s_2)} Q(\omega(t)) = \max_{t \in J_0} Q(\omega(t)) \leq 4\beta_0 \max_{t \in J_0} \omega_1(t) \leq 4\beta_0^{\frac{3}{2}}. \quad (3.13)$$

Consequently, we have $\omega_1(s_1), \omega_1(s_2) \leq \beta_0^{1/2}$, and thus

$$L([\omega_1(s_1), \omega_1(s_2)]) = |\omega_1(s_1) - \omega_1(s_2)| \leq \beta_0^{1/2}.$$

Since $L(\omega|_{(s_1, s_2)}) \geq 2|y_0| \geq 2\beta_0^{1/2}$, it follows that

$$\Lambda := \frac{L(\omega|_{(s_1, s_2)}) - L([\omega_1(s_1), \omega_1(s_2)])}{4} \geq \frac{\beta_0^{\frac{1}{2}}}{4}. \quad (3.14)$$

Set $p = \bar{\omega}(\epsilon) + (\Lambda, 0)$ and consider the curve η formed by the concatenation of $[\bar{\omega}(\epsilon), p]$, ∂D , and $[p, \bar{\omega}(\epsilon)]$, where D is a closed disc containing p on its boundary such that $\min_{x \in D} Q(x) = Q(p)$ and $L(\partial D) = 2\Lambda$. We will conclude by obstruction (ii) of Lemma 3.7. Let $\alpha := \omega|_{[s_1, s_2]} * [\omega(s_2), \omega(s_1)]$. Observe that

$$L(\alpha) = 4\Lambda + 2L([\omega(s_2), \omega(s_1)]) \leq 4\Lambda + 2\beta_0^{\frac{1}{2}} \leq 12\Lambda,$$

where in the last inequality we used (3.14). By Lemma 3.1 (ii), and using (3.13), we deduce that

$$|A(\alpha)| \leq \frac{L(\alpha)^2}{4\pi} \max_{x \in \mathcal{E}(\alpha)} |Q(x)| \leq 144\Lambda^2 \max_{t \in (s_1, s_2)} |Q(\omega(t))| \leq 144\Lambda^2 \beta_0^{\frac{3}{2}}.$$

On the other hand, by construction, we have $|A(\eta)| \geq \Lambda^2 Q(p)/\pi$, where

$$Q(p) = 4(\epsilon^{\bar{m}} + \Lambda) \left((\epsilon^{\bar{m}} + \Lambda)^2 - \epsilon^m \right) = 4\Lambda(\epsilon^{\bar{m}} + \Lambda)(2\epsilon^{\bar{m}} + \Lambda) \geq 8\Lambda\epsilon^m \geq 2\epsilon^m \beta_0^{\frac{1}{2}}.$$

This shows that $|A(\alpha)| \leq |A(\eta)|$ for sufficiently small $\epsilon > 0$ (since $\beta_0 \leq \beta = o(\epsilon^m)$ by Lemma 3.5 (v)). Lemma 3.7 (iv) provides an obstruction, establishing that $y_0 \geq -\beta_0^{1/2}$.

Case $J_0 \subset I_1$: Since m is odd and $P(\omega(t_0)) < 0$, it follows that $y_0 > 0 \geq -\beta_0^{\frac{1}{2}}$. We now aim to prove that $x_0 \geq c\beta_0^{1/2}$ for some constant $c > 0$, provided that $\epsilon > 0$ is sufficiently small. In the following, for any $v, w \in \mathbb{R}^2 \setminus \{0\}$, let $\mathfrak{C}^+[v, w]$ denote the closed positive cone defined as the convex hull of the two half-lines in the directions v and w . Next, we define $c_0 := \sqrt{3}/3$ and, for $c \in (0, c_0)$,

$$U_c := \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 < 3\epsilon, 0 < x_1 < cx_2^{\bar{m}} \right\}.$$

The proof proceeds in three steps. We set $u := (1, 1)$, $v := (0, 1)$, and $w := (1, -1)$.

Step 1: We claim that for all $c \in (0, c_0)$, there exists $\epsilon_c > 0$ such that for any $\epsilon \in (0, \epsilon_c)$,

$$\langle \nabla Q(x), z \rangle < 0 \quad \forall x \in U_c, \forall z \in \mathfrak{C}^+[v, w] \setminus \{0\}. \quad (3.15)$$

To verify this, note that $\nabla Q(x) = 4(3x_1^2 - x_2^m, -mx_1x_2^{m-1})$. First, the inequality $\langle \nabla Q(x), v \rangle < 0$ follows directly. Additionally, we have $\langle \nabla Q(x), w \rangle = 4(3x_1^2 + mx_1x_2^{m-1} - x_2^m)$. For $x_1 \in (0, cx_2^{\bar{m}})$, this yields $\langle \nabla Q(x), w \rangle \leq 4x_2^m(3c^2 + cmx_2^{\bar{m}-1} - 1)$, which is negative for $c < c_0$ and sufficiently small $x_2 > 0$. We conclude by linearity.

Step 2: We claim that for all $c \in (0, c_0)$, there exists $\epsilon_c > 0$ such that for any $\epsilon \in (0, \epsilon_c)$, $\text{spt}(\ell_0) \not\subset U_c$. To prove this, we proceed by contradiction, assuming that $\text{spt}(\ell_0) \subset U_c$ for some $c \in (0, c_0)$. Since $J_0 \subset I_1$, obstruction (iv) of Lemma 3.7, together with Lemma 3.5 (i), ensures that $Q \circ \ell_0 < 0$. We then define the following affine cones:

$$\mathfrak{C}_1 := \omega(s_0^-) + \mathfrak{C}^+[u, -w], \quad \mathfrak{C}_2 := \omega(s_0^-) + \mathfrak{C}^+[-v, -w], \quad \mathfrak{C}_3 := \omega(s_0^-) + \mathfrak{C}^+[-v, u].$$

Since the union of these cones is equal to \mathbb{R}^2 , at least one of the following cases must hold:

$$\text{spt}(\ell_0) \subset \mathfrak{C}_1, \quad (\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_2 \neq \emptyset \quad \text{or} \quad \begin{cases} (\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_2 = \emptyset \\ (\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_3 \neq \emptyset. \end{cases}$$

Our goal is to derive a contradiction in each of these cases.

In the case where $(\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_2 \neq \emptyset$, choose $t \in (s_0^-, s_0^+)$ such that $\omega(t) \in \mathfrak{C}_2$. Define $T^r : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as the translation by the vector $r\bar{v}$, where $\bar{v} := \omega(s_0^-) - \omega(t)$ and $r > 0$, and set $\eta^r := T^r \circ \ell_0$. By Lemma 3.8 (ii), the support of ℓ_0 encloses a strictly convex set that is contained in the cone $\omega(s_0^-) + \mathfrak{C}^+[\dot{\omega}(s_0^-), -\dot{\omega}(s_0^+)]$. Thus, the vector $-\bar{v}$ lies the interior of the cone $\mathfrak{C}^+[\dot{\omega}(s_0^-), -\dot{\omega}(s_0^+)]$. This shows that η^r intersects $\omega_{|[0, s_0^-]} * \omega_{|[s_0^+, L(\omega)]}$ for $r > 0$ sufficiently small. Since T^r is an isometry, we have $L(\eta^r) = L(\ell_0)$. Furthermore, by (3.15), $Q \circ \ell_0 < 0$, and since $\bar{v} \in \mathfrak{C}^+[v, w]$ (because $\omega(t) \in \mathfrak{C}_2$), we have $|A(\eta^r)| > |A(\ell_0)|$ for $r > 0$ sufficiently small. This violates obstruction (i) in Lemma 3.7, leading to a contradiction.

In the case where $\text{spt}(\ell_0) \subset \mathfrak{C}_1$, consider the rotation $T_\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with angle ϕ around the point $\omega(s_0^-)$, and define $\eta_\phi := T_\phi \circ \ell_0$. By construction, since T_ϕ is an isometry fixing $\omega(s_0^-)$, the set $\text{spt}(\eta_\phi)$ intersects the support of $\omega_{|[0, s_0^-]} * \omega_{|[s_0^+, L(\omega)]}$, and $L(\eta_\phi) \leq L(\ell_0)$. Next, recalling that $\text{spt}(\ell_0)$ encloses a convex set, for any $x \in \mathcal{E}(\ell_0)$, the vector $x - \omega(s_0^-)$ belongs to $\mathfrak{C}^+[-w, u]$. Therefore, we have

$$-(x - \omega(s_0^-))^\perp \in \mathfrak{C}^+[-w, u] \subset \mathfrak{C}^+[v, w] \quad \forall x \in \mathcal{E}(\ell_0),$$

where $(v_1, v_2)^\perp = (-v_2, v_1)$. Thus, by (3.15), it follows that $\langle \nabla Q(x), -(x - \omega(s_0^-))^\perp \rangle < 0$ for all $x \in \mathcal{E}(\ell_0)$. Since $\frac{\partial T_\phi}{\partial \phi}(0, \cdot)$ is the rotation of angle $\pi/2$ at the origin, one then deduces that

$$\begin{aligned} \left. \frac{d}{d\phi} \right|_{\phi=0} \int \int_{\mathcal{E}(T_\phi \circ \ell_0)} Q(x) dx_1 dx_2 &= \left. \frac{d}{d\phi} \right|_{\phi=0} \int \int_{\mathcal{E}(T_\phi \circ \ell_0)} Q(T_\phi(x)) dx_1 dx_2 \\ &= \int \int_{\mathcal{E}(T_\phi \circ \ell_0)} \langle \nabla Q(x), (x - \omega(s_0^-))^\perp \rangle dx_1 dx_2 > 0. \end{aligned}$$

This implies that $|A(\eta^r)| > |A(\ell_0)|$ for small $\phi < 0$. Obstruction (i) in Lemma 3.7 leads again to a contradiction.

We now address the case where $(\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_2 = \emptyset$ and $(\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_3 \neq \emptyset$. Since $\dot{\omega}(\tau_1)$ points toward the set $\{P < 0\}$ at $\omega(\tau_1)$, we must have $\theta(\tau_1) \in (\pi/4, 3\pi/2) + 2k\pi$ for some $k \in \mathbb{Z}$. Without loss of generality, we assume $k = 0$. If $\lambda < 0$, then, by (2.3), θ is increasing on I_1 , thus by Lemma 3.8 (viii), we have $\theta(s_0^-) \in (\dot{\theta}(\tau_1), 3\pi/2) \subset (\pi/4, 3\pi/2)$. Since the support of ℓ_0 encloses a convex set contained within the cone $\omega(s_0^-) + \mathfrak{C}^+[\dot{\omega}(s_0^-), -\dot{\omega}(s_0^+)]$ with $\text{ang}(\dot{\omega}(s_0^-), -\dot{\omega}(s_0^+)) \in (0, \pi)$ and $(\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_2 = \emptyset$, we deduce that the set $\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}$ does not intersect the affine cone \mathfrak{C}_3 , leading to a contradiction. Therefore, since $\lambda \neq 0$ by Lemma 3.5 (vii), we must have $\lambda > 0$. By (2.3), θ is decreasing on I_1 . We claim that there exists a sequence $(\ell_k)_{k \in \mathbb{N}}$ of simple loops of $\omega|_{I_1}$, associated with a sequence of intervals $(J_k = [s_k^-, s_k^+])_{k \in \mathbb{N}}$, such that $(s_k)_{k \in \mathbb{N}}$ is increasing and the following properties hold for each $k \in \mathbb{N}$:

$$\begin{cases} (a) \exists t \in \text{int}(J_k) \text{ such that } \omega(t) \in \omega(s_k^-) + \mathfrak{C}^+[-v, u], \\ (b) \forall s \in \text{int}(J_k), \omega(s) \in U_c \setminus (\omega(s_k^-) + \mathfrak{C}^+[-v, -w]). \end{cases} \quad (3.16)$$

This claim contradicts the analyticity of ω . The base case for $k = 0$ is satisfied by ℓ_0 , given our assumptions: $\text{spt}(\ell_0) \subset U_c$, $(\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_2 = \emptyset$, and $(\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_3 \neq \emptyset$. Now assume that there exists a loop ℓ_k for some $k \in \mathbb{N}$ satisfying (3.16). We start by proving that

$$\theta(s_k^+) \in \left(\frac{\pi}{2}, \frac{5\pi}{4} \right] \pmod{2\pi}. \quad (3.17)$$

We prove (3.17) by contradiction. Without loss of generality, we assume that $\theta(s_k^+)$ lies in $(-\pi, \pi]$. We then consider separately the three cases where $\theta(s_k^+) \in (-\pi/4, \pi/2)$, $\theta(s_k^+) \in (-3\pi/4, -\pi/4]$, and $\theta(s_k^+) = \pi/2$.

- If $\theta(s_k^+) \in (-\pi/4, \pi/2)$, then $\omega(s_k^+ - s) \in \omega(s_k^-) + \mathfrak{C}^+[-v, -w]$ for sufficiently small $s > 0$, which contradicts property (b) in (3.16).
- If $\theta(s_k^+) \in (-3\pi/4, -\pi/4]$, then the set $\text{spt}(\ell_k) \setminus \{\omega(s_k^-)\}$ is contained within the interior of the affine cone $\omega(s_k^-) + \mathfrak{C}^+[-w, -\dot{\omega}(s_k^+)]$. This follows from property (b) satisfied by ℓ_k , the strict convexity of the region enclosed by its support (see Lemma 3.8 (ii)), and the fact that $\lambda > 0$, which implies that ℓ_k is negatively oriented. Consequently, $\text{spt}(\ell_k) \setminus \{\omega(s_k^-)\}$ is entirely contained within the interior of the affine cone $\omega(s_k^-) + \mathfrak{C}^+[-w, u]$, contradicting property (a) in (3.16).
- If $\theta(s_k^+) = \pi/2$, then since θ is monotone decreasing, a translation of $\text{spt}(\ell_k)$ in the direction $(0, 1)$ intersects the support of $\omega|_{(s_k^+, \tau_2]}$. This contradicts obstruction (ii) in Lemma 3.7.

Therefore, the proof of (3.17) is complete, and as a consequence, we have

$$\omega(s_k^+ + s) \in \omega(s_k^+) + \mathfrak{C}^+[\dot{\omega}(s_k^+), v] \subset U_c \quad \forall s > 0 \text{ small.}$$

By obstruction (iii) of Lemma 3.7, we have $\text{spt}(\omega|_{(s_k^+, \tau_2]}) \cap \{\omega(s_k^+) + t(0, 1) \mid t \geq 0\} = \emptyset$. Consequently, since $\omega(\tau_2) \notin \omega(s_k^+) + \mathfrak{C}^+[\dot{\omega}(s_k^+), v]$, the intersection $\text{spt}(\omega|_{(s_k^+, \tau_2]}) \cap \{\omega(s_k^+) + t\dot{\omega}(s_k^+) \mid t \geq 0\}$ must be nonempty. By Lemma 3.4, it follows that $\omega|_{(s_k^+, \tau_2]}$ contains a loop within U_c . We define ℓ_{k+1} as the first simple loop of $\omega|_{(s_k^+, \tau_2]}$. The fact that ℓ_{k+1} satisfies (3.16) can be demonstrated using the same arguments employed to show that $(\text{spt}(\ell_0) \setminus \{\omega(s_0^-)\}) \cap \mathfrak{C}_2 \neq \emptyset$ and $\text{spt}(\ell_0) \subset \mathfrak{C}_1$ are not possible. This completes the proof of Step 2.

Step 3: We complete the proof of Lemma 3.11 by proving that $x_0 \geq \beta_0^{1/2}/2$, provided that $\epsilon > 0$ is sufficiently small. Consider the curves

$$\Upsilon := \left\{ x \in \mathbb{R}^2 \mid 2x_1 = x_2^m, x_2 \geq 0 \right\} \quad \text{and} \quad \Xi_\rho := \left\{ x \in \mathbb{R}^2 \mid P(x) = -\rho, 2x_1 \in \left[0, \rho^{\frac{1}{2}} \right] \right\},$$

for $\rho > 0$. The intersection $\Upsilon \cap \Xi_1$ is empty because a point (x_1, x_2) in the intersection must satisfy $4x_1^2 = x_2^m = x_1^2 + 1$, which implies $x_1 = \sqrt{3}/3 > 1/2$. Moreover, for every $\rho \in (0, 1]$, the map $\Delta_\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $\Delta_\rho(x) := (\rho^{-1/2}x_1, \rho^{-1/m}x_2)$ is $\rho^{-1/2}$ -Lipschitz, and it satisfies $\Upsilon = \Delta_\rho(\Upsilon)$ and $\Xi_{\rho, 1} = \Delta_\rho(\Xi_{\rho, \rho})$. Hence, denoting for every $\rho > 0$ by $\text{dist}(\Upsilon, \Xi_\rho)$ the infimum of $|x - x'|$ for $x \in \Upsilon$ and $x' \in \Xi_\rho$, we have

$$\mu := \text{dist}(\Upsilon, \Xi_1) > 0 \quad \text{and} \quad \text{dist}(\Upsilon, \Xi_\rho) \geq \mu\rho^{\frac{1}{2}} \quad \forall \rho \in (0, 1]. \quad (3.18)$$

Set $c := 1/2 \in (0, c_0)$ and suppose, by contradiction, that $x_0 < \sqrt{\beta_0}/2$. Then, $\omega(t_0) = (x_0, y_0) \in \text{spt}(\ell) \cap \Xi_{\beta_0} \cap U_c$ (because $\beta_0 = y_0^m - x_0^2 > 4x_0^2$). However, Υ contains the boundary of U_c within the open set $\{x_1 > 0\}$ and, by Step 2, $\text{spt}(\ell_0)$ is not entirely contained in U_c for $\epsilon \in (0, \epsilon_c)$. Therefore, we may assume that $\text{spt}(\ell_0) \cap \Upsilon \neq \emptyset$. From (3.18), we deduce that

$$L_0 \geq \text{dist}(\Upsilon, \Xi_{\beta_0}) \geq \mu\beta_0^{\frac{1}{2}} \geq \mu \frac{\beta_0}{y_0^m} = \mu \frac{\beta_0}{\max\{x_0, |y_0|^m\}} = \mu\delta_0,$$

where we used the relations $\beta_0 = -P(\omega(t_0)) = y_0^m - x_0^2 \leq y_0^m$ and $y_0^m \geq x_0^2$. By Lemma 3.10, the inequality above implies $y_0 \geq c(\mu)\epsilon$. Using $x_0 < \sqrt{\beta_0}/2$ and Lemma 3.5 (v), we finally obtain

$$c(\mu)\epsilon \leq y_0 = (x_0^2 + \beta_0)^{\frac{1}{m}} \leq \left(\frac{5}{4}\right)^{\frac{1}{m}} \beta_0^{\frac{1}{m}} \leq \left(\frac{5C}{4}\right)^{\frac{1}{m}} \epsilon^{\frac{3}{2} - \frac{1}{m}},$$

which leads to a contradiction for sufficiently small $\epsilon > 0$, as $m \geq 5$. This completes the proof of Lemma 3.11. \square

Our next lemma is the following.

Lemma 3.12. *By taking $\epsilon_0 > 0$ smaller and $C > 0$ larger, if necessary, as prescribed in Lemma 3.5, it follows that for any $\epsilon \in (0, \epsilon_0)$, the inequality $|\lambda|\beta_0^2 \leq C$ holds.*

Proof. Set $M := \max\{|y_0|^m, \sqrt{\beta_0}\}$. There exists $\sigma = \pm 1$ such that $\beta_0 = \sigma(x_0^2 - y_0^m) > 0$. Therefore, we have $x_0^2 = \sigma\beta_0 + y_0^m \leq \beta_0 + |y_0|^m \leq 2M^2$. If $M = \sqrt{\beta_0}$, then Lemma 3.11 implies $x_0 \geq c\sqrt{\beta_0}$. However, if $M = |y_0|^m$, there are two cases to consider. If $y_0^m \geq 2\beta_0$, then we have $x_0^2 = \sigma\beta_0 + y_0^m \geq y_0^m/2 = M^2/2$. If $y_0^m < 2\beta_0$, since $\sigma\beta_0 + y_0^m \geq 0$ implies $|y_0|^m \leq 2\beta_0$, Lemma 3.11 gives $x_0 \geq c\sqrt{\beta_0} \geq cM/\sqrt{2}$. In conclusion, by setting $c_1 := \min\{1, c\}/\sqrt{2}$ and $c_2 := \sqrt{2}$, and by recalling that $\beta_0 \leq M^2$, we obtain

$$c_1 M \leq x_0 \leq c_2 M \quad \text{and} \quad \frac{\beta_0}{c_2 M} \leq \frac{\beta_0}{x_0} \leq \frac{\beta_0}{c_1 M} \leq \frac{M}{c_1}. \quad (3.19)$$

Given a parameter $a \in (0, 1]$ to be fixed later, we consider the interval $I_a := [t_0 - a\beta_0/x_0, t_0]$. Using the bounds $|\dot{\omega}_1| \leq 1$ and (3.19), we note that we have

$$\left(c_1 - \frac{a}{c_1}\right) M \leq x_0 - \frac{a\beta_0}{x_0} \leq \omega_1(t) \leq x_0 + \frac{a\beta_0}{x_0} \leq \left(c_2 + \frac{a}{c_1}\right) M \quad \forall t \in I_a \cap [0, L(\omega)]. \quad (3.20)$$

Thus, assuming $a \leq c_1^2$, we have $I_a \subset [0, L(\omega)]$, and, using $|\dot{\omega}_2| \leq 1$ and (3.19), we obtain

$$|\omega_2(t)| \leq |y_0| + \frac{a\beta_0}{x_0} \leq M^{\frac{2}{m}} + \frac{aM}{c_1} \leq 2M^{\frac{2}{m}} \quad \forall t \in I_a. \quad (3.21)$$

Using (3.20) and (3.21), we can bound the derivative of $P(t) = P(\omega(t))$ on I_a as follows:

$$|\dot{P}(t)| \leq 2\omega_1(t) + m|\omega_2(t)|^{m-1} \leq C_1 M + C_2 M^{2 - \frac{2}{m}} \leq CM \quad \forall t \in I_a, \quad (3.22)$$

where $C_1 := 2(c_1 + c_2)$, $C_2 := m2^{m-1}$ and $C = C_1 + C_2$, provided $\epsilon > 0$ is small enough. We now fix $a \in (0, c_1^2]$ such that $c_1 - a/c_1 \geq 1/(2c_1)$ and $aC/c_1 \leq 1/2$. From (3.20) and (3.22), it follows that

$$\omega_1(t) \geq \frac{M}{2c_1} \quad \text{and} \quad |P(t)| \geq \beta_0 - \frac{a\beta_0}{x_0} CM \geq \beta_0 - \frac{aC\beta_0}{c_1} \geq \frac{\beta_0}{2} \quad \forall t \in I_a. \quad (3.23)$$

From Lemma 3.8 (ii), we have $\int_{\ell_0} |\dot{\theta}(t)| dt < 2\pi$. Moreover, Lemma 3.8 (viii) implies that $\int_{\tau}^{s_0^-} |\dot{\theta}(t)| dt \leq 2\pi$, where $\tau = \tau_0$ if $J_0 \subset I_0$ and $\tau = \tau_1$ if $J_0 \subset I_1$. If $\tau = \tau_0$, we have

$$\int_{I_a} |\dot{\theta}(t)| dt \leq \int_0^{t_0} |\dot{\theta}(t)| dt \leq \int_0^{s_0^-} |\dot{\theta}(t)| dt + \int_{s_0^-}^{s_0^+} |\dot{\theta}(t)| dt \leq 4\pi.$$

If $\tau = \tau_1$, the curve ω_{I_0} is injective. Using the Gauss-Bonnet formula (3.3) applied to the positively oriented closed simple curve $\eta : [0, \tau_1 + \tau'] \rightarrow \mathbb{R}^2$ formed by concatenating $\omega|_{[0, \tau_1]}$

with the segment $P_{[\omega(\tau_1), \omega(0)]}$ of $\{P = 0\}$, joining $\omega(\tau_1)$ to $\omega(0)$, reparametrized by arc length on an interval $[0, \tau']$, we obtain

$$2\pi = \int_0^{\tau_1} \kappa(t) dt + \int_0^{\tau'} \bar{\kappa}(t) dt + \delta_0 + \delta_1,$$

where κ and $\bar{\kappa}$ are the signed curvature of ω and $P_{[\omega(\tau_1), \omega(0)]}$, respectively, and $\delta_0, \delta_1 \in [-\pi, \pi]$ represent the discontinuities of curvature of η at $t = 0$ and $t = \tau_1$. Given that $\tau' \leq 1$ and $|\bar{\kappa}| \leq 1$ for sufficiently small $\epsilon > 0$, recalling (3.1) gives

$$\int_{I_a} |\dot{\theta}(t)| dt \leq \int_0^\tau |\dot{\theta}(t)| dt + \int_\tau^{t_0} |\dot{\theta}(t)| dt = \left| \int_0^\tau \kappa(t) dt \right| + \int_\tau^{t_0} |\dot{\theta}(t)| dt \leq 8\pi + 1.$$

Finally, using (2.3), (3.19) and (3.23), we conclude that

$$8\pi + 1 \geq \int_{I_a} 4|\lambda|\omega_1(t)|P(t)| dt \geq \frac{a\beta_0}{x_0} \left(4|\lambda| \frac{M}{2c_1} \frac{\beta_0}{2} \right) \geq \frac{|\lambda|\beta_0^2}{c_1c_2},$$

which completes the proof. \square

We can now state the lemma required to end the proof of Proposition 3.9.

Lemma 3.13. *There is $c_0 > 0$ such that $L_0 \geq c\delta_0$.*

Proof. If $L_0 \geq x_0$, the result follows immediately from (3.12), because $L_0 \geq x_0 \geq c^2\beta_0/x_0 \geq c^2\delta_0$. We now address the case where $L_0 < x_0$. By Lemma 3.8 (ii), (2.3), and $|J_0| = L_0$, we obtain

$$\begin{aligned} \pi &\leq \int_{J_0} \left| \dot{\theta}(t) \right| dt = |\lambda| \int_{J_0} |Q(\omega(t))| dt \leq |\lambda| L_0 \max_{t \in J_0} |Q(\omega(t))| \\ &\leq 4|\lambda| L_0 \beta_0 \max_{t \in J_0} \omega_1(t) \leq 4|\lambda| L_0 \beta_0 (x_0 + L_0). \end{aligned}$$

Using $L_0 < x_0$ and Lemma 3.12, it follows that

$$L_0 \geq \frac{\pi}{4|\lambda|\beta_0(x_0 + L_0)} \geq \frac{\pi}{8|\lambda|\beta_0 x_0} \geq \frac{\pi\beta_0}{8Cx_0} \geq \frac{\pi\beta_0}{8C \max\{x_0, |y_0|^{\bar{m}}\}} = \frac{\pi}{8C} \delta_0,$$

which completes the proof of the lemma. \square

Proof of Proposition 3.9. Lemmas 3.13 and 3.10 imply that $y_0 = \omega_2(t_0) \geq c(c_0)\epsilon$. From Lemma 3.5 (viii), we have $L_0 = o(\epsilon^{\bar{m}})$. Then $\omega_2(s_0^-) \geq c(c_0)\epsilon/2$ for $\epsilon > 0$ small enough. The conclusion follows from Lemma 3.6. \square

3.7 End of proof of Proposition 2.2

Assertions (i) and (ii) have already been proven in Lemma 3.5. To prove (iii) we argue by contradiction and assume that $\lambda > 0$. We begin by applying several assertions from Lemma 3.8. By (iii) and (vii), if $\omega|_{I_0}$ admits a loop, it must be unique. Therefore, by (ix), we have $\omega_2(\tau_1) \leq \omega_2(0) = 0$, which contradicts (i) (note that $P(\omega(\tau_1)) = 0$). Hence, $\omega|_{I_0}$ is injective. From this, we conclude that the domain enclosed by $\omega|_{I_0}$ and the curve $\{P = 0, x_1 \geq 0\}$ is convex. Hence the maximum of P is attained on $\text{spt}(\omega_{[0, \tau_1]})$. A simple computation shows that the maximum of P over a segment $[0, (t^{\bar{m}}, t)]$, for $t > 0$, is given by $\alpha t^{\bar{m}}$ for some constant $\alpha > 0$. Hence, by Lemma 3.5 (v), we obtain

$$\alpha\omega_2(\tau_1)^{\bar{m}} \leq \max_{s \in [0, \tau_1]} P(\omega(s)) \leq \beta \leq C \epsilon^{3\bar{m}-1}.$$

By Lemma 3.8 (iv) and (v), we have $1 \in \mathcal{I}^-$, and $\omega|_{I_1}$ contains at least one loop. Furthermore, by Proposition 3.9 and Lemma 3.8 (vii), it contains exactly one loop, provided $\epsilon > 0$ is sufficiently small, specifically $\epsilon < \epsilon(c)$, where c is given by Proposition 3.9. Therefore, by Lemma 3.8 (ix), we must have $\omega_2(\tau_1) \geq \omega_2(\tau_2)$. Using the above inequality, we deduce

$$\omega_2(\tau_2) \leq \omega_2(\tau_1) \leq \left(\frac{C}{\alpha}\right)^{\frac{1}{m}} \epsilon^{\frac{3}{2} - \frac{1}{m}},$$

which, for sufficiently small $\epsilon > 0$, contradicts the lower bound $\omega_2(\tau_2) \geq c\epsilon$ provided by Proposition 3.9.

Before we proceed with the proof of the remaining assertions, we note that by Proposition 3.9, the upper bound on L_0 given by Lemma 3.5 (viii), the lower bound on L_0 provided by Lemma 3.13, and the inequality $\max\{x_0, |y_0|^{\bar{m}}\} \leq C\epsilon^{\bar{m}}$ (which follows by Lemma 3.5 (iv)), we may assume that

$$y_0 \geq c\epsilon \quad \text{and} \quad c\beta_0\epsilon^{-\bar{m}} \leq L_0 \leq C\beta^{1-\frac{1}{m}}, \quad (3.24)$$

for some constants $c, C > 0$ and sufficiently small $\epsilon > 0$. We further claim that we may also assume

$$|\lambda|\beta_0^2 \geq c. \quad (3.25)$$

Consider a time interval $[t_1, t_2] \subset J_0$ where the times t_1 and t_2 will be chosen later. By (2.3) and since ω_1 and ω_2 do not vanish on J_0 according to Proposition 3.9, we have for any $t \in [t_1, t_2]$,

$$\begin{aligned} \frac{d}{dt} \left\{ 2\lambda P(\omega(t))^2 - 2\sin(\theta(t)) - m \cos(\theta(t)) \frac{\omega_2(t)^{m-1}}{\omega_1(t)} \right\} &= -m \cos(\theta(t)) \frac{d}{dt} \left\{ \frac{\omega_2(t)^{m-1}}{\omega_1(t)} \right\} \\ &= -m \cos(\theta(t)) \frac{\omega_2(t)^{m-1}}{\omega_1(t)} \left(\cos(\theta(t)) + (m-1) \sin(\theta(t)) \frac{\omega_1(t)}{\omega_2(t)} \right). \end{aligned}$$

From Proposition 3.9 and Lemma 3.5 (iv), it follows that the absolute value of the right-hand side of the above equality is less than C/ϵ for some constant $C > 0$ and sufficiently small $\epsilon > 0$. Integrating this equality over $[t_1, t_2] \subset J_0$, we obtain

$$|2\lambda (P(\omega(t_2))^2 - P(\omega(t_1))^2) - 2(\sin(\theta(t_2)) - \sin(\theta(t_1)))| \leq |D| + \frac{CL_0}{\epsilon}, \quad (3.26)$$

where

$$D = m \left(\cos(\theta(t_2)) \frac{\omega_2(t_2)^{m-1}}{\omega_1(t_2)} - \cos(\theta(t_1)) \frac{\omega_2(t_1)^{m-1}}{\omega_1(t_1)} \right). \quad (3.27)$$

From Proposition 3.9 and Lemma 3.5 (iv), it follows that $|D| \leq C\epsilon^{\bar{m}-1}$. Additionally, from Lemma 3.5 (v) and (viii), we have $L_0\epsilon^{-1} \leq C^{2-1/m}\epsilon^{\bar{m}-1}$. Using (3.26), we deduce that by taking $C > 0$ larger if necessary,

$$|\lambda (P(\omega(t_2))^2 - P(\omega(t_1))^2)| \geq |\sin(\theta(t_2)) - \sin(\theta(t_1))| - C\epsilon^{\bar{m}-1}.$$

By Lemma 3.8 (ii), the times t_1 and t_2 can be chosen so that $|\sin(\theta(t_2)) - \sin(\theta(t_1))| \geq 1$. Therefore, by choosing $\epsilon_0 > 0$ small enough, we obtain

$$|\lambda (P(\omega(t_2))^2 - P(\omega(t_1))^2)| \geq 1/2.$$

The inequality (3.25) follows by noting that $|P(\omega(t_2))^2 - P(\omega(t_1))^2| \leq 2\beta_0^2$. Before returning to the proof of the remaining assertions, we also need the following result.

Lemma 3.14. *Let $\bar{P} := P \circ \omega : [0, L(\omega)] \rightarrow \mathbb{R}$ and $t_* \in [0, L(\omega)]$ such that $\bar{P}(t_*) \in (0, \beta_0)$, $\dot{\bar{P}}(t_*) = 0$, $\ddot{\bar{P}}(t_*) \leq 0$, and $\sin \theta(t_*) > 0$. Then $|\lambda|P(\omega(t_*))^{1+1/\bar{m}} \leq m(m-1)/8$.*

Proof. Set $x_* = (x_*, y_*) := \omega(t_*)$, $\beta_* := P(x_*)$ and $\theta_* := \theta(t_*)$. We have

$$\begin{cases} \dot{\bar{P}}(t_*) = 2x_* \cos \theta_* - my_*^{m-1} \sin \theta_* = 0 \\ \ddot{\bar{P}}(t_*) = 2 \cos^2 \theta_* - 2x_* \dot{\theta}(t_*) \sin \theta_* - m(m-1)y_*^{m-2} \sin^2 \theta_* - my_*^{m-1} \dot{\theta}(t_*) \cos \theta_* \leq 0, \end{cases} \quad (3.28)$$

with $\dot{\theta}(t_*) = 4\lambda x_* \beta_*$. Since $\beta_* = x_*^2 - y_*^m > 0$, we have $y_* \leq x_*^{2/m}$ (recall that $x_* > 0$), so the inequality $\ddot{\bar{P}}(t_*) \leq 0$ yields

$$\begin{aligned} 4\lambda x_* \beta_* (-2x_* \sin \theta_* - my_*^{m-1} \cos \theta_*) &\leq -2 \cos^2 \theta_* + m(m-1)y_*^{m-2} \sin^2 \theta_* \\ &\leq m(m-1)y_*^{m-2} \leq m(m-1)x_*^{2-4/m}, \end{aligned} \quad (3.29)$$

where, by the relation given by $\dot{\bar{P}}(t_*) = 0$, the left-hand side satisfies

$$4\lambda x_* \beta_* (-2x_* \sin \theta_* - my_*^{m-1} \cos \theta_*) = -8\lambda x_*^2 \beta_* \sin \theta_* (1 + \cot^2 \theta_*) \geq 8\lambda x_*^2 \beta_*, \quad (3.30)$$

because $\lambda < 0$ and $\sin(\theta_*) > 0$. Then, combining (3.29) and (3.30), and using $x_*^2 = \beta_* + y_*^m$, we infer that

$$|\lambda|P(\omega(t_*))^{1+\frac{1}{\bar{m}}} = |\lambda|\beta_*^{1+\frac{1}{\bar{m}}} \leq |\lambda|\beta_* (\beta_* + y_*^m)^{\frac{2}{\bar{m}}} = |\lambda|\beta_* x_*^{\frac{4}{\bar{m}}} \leq \frac{m(m-1)}{8}.$$

□

Let us now prove assertions (iv) and (v). By (3.9), there is $\bar{i} \in \{0, 1\}$ such that $J_0 \subset I_{\bar{i}}$. Using Proposition 3.9, Lemma 3.6, and obstruction (viii) of Lemma 3.7 we deduce that ω_{I_i} is injective, for all $i \neq \bar{i}$. As a consequence, if $J_0 \subset I_0$ then, by Lemma 3.8 (iv) and obstruction (iv) of Lemma 3.7, we have $I_\omega = I_0$. Moreover, by Lemma 3.8 (vii), ℓ_0 is the unique loop of ω . Next, we show that the case $J_0 \subset I_1$ cannot occur. In this case, there are two possibilities: either $I_\omega = I_0 \cup I_1$ with $\omega|_{I_0}$ injective, or $I_\omega = I_0 \cup I_1 \cup I_2$ with both $\omega|_{I_0}, \omega|_{I_2}$ injective. Set $\beta_+ := \max_{t \in I_\omega} P$. For $i = 0, 2$ (or only $i = 0$ in the first case), consider the simple closed curve η defined as the concatenation of $\omega|_{I_i}$ with the curve $\tilde{\omega} : [0, \tau := \tau_{i+1} - \tau_i] \rightarrow \mathbb{R}^2$, corresponding to the segment of $\bar{\omega}$ connecting $\omega(\tau_{i+1})$ to $\omega(\tau_i)$. Since $\lambda < 0$, Lemma 3.8 (ix) implies that $\omega(\tau_{i+1}) \geq \omega(\tau_i)$, and, by (2.3) and (3.1), the signed curvature κ of ω on (τ_i, τ_{i+1}) is negative. Consequently, the curve η is positively oriented, $\omega(\tau_{i+1}) \geq \omega(\tau_i)$, the oriented angles $\delta_i = \text{ang}(\dot{\omega}(\tau_{i+1}), \dot{\tilde{\omega}}(0))$ and $\delta_{i+1} = \text{ang}(\dot{\tilde{\omega}}(\tau), \dot{\omega}(\tau_i))$ lie in $(0, \pi)$, $\theta(\tau_{i+1}) < \theta(t) < \theta(\tau_i)$ for all $t \in (\tau_i, \tau_{i+1})$, and, by the Gauss-Bonnet formula (3.3), we have

$$2\pi - \delta_i - \delta_{i+1} = \int_{\tau_i}^{\tau_{i+1}} \kappa(t) dt + \int_{\tau_i}^{\tau_{i+1}} \tilde{\kappa}(t) dt,$$

where $\tilde{\kappa}$ denotes the signed curvature of $\tilde{\omega}$. Since $\kappa \leq 0$, $\tilde{\kappa}$ tends to 0 as $\epsilon \rightarrow 0$, and $\dot{\tilde{\omega}}$ approaches the vertical vector $(0, -1)$ as $\epsilon \rightarrow 0$, we deduce that δ_i, δ_{i+1} are close to π for sufficiently small $\epsilon > 0$, which implies that $\sin(\theta(t)) \geq 1/2$ for all $t \in I_i$. Therefore, by applying Lemma 3.14 and (3.25), we obtain a constant $D > 0$ such that

$$\beta_+ \leq D\beta_0^{\frac{2m}{m+2}}. \quad (3.31)$$

Next, by applying (2.5) from Proposition 2.1 to the curve ν defined as the concatenation of $\omega|_{[0, s_0^-]}$ with $\omega|_{[s_0^+, L(\omega)]}$, we obtain

$$L(\omega) = L(\nu) + L_0 \geq L(\bar{\omega}) - C\beta_+^{1-\frac{1}{m}} + L_0 \geq L(\bar{\omega}) - CD^{1-\frac{1}{m}}\beta_0^{\frac{2(m-1)}{m+2}} + L_0.$$

Combining this inequality with $L(\omega) \leq L(\bar{\omega})$ and the lower bound on L_0 obtained in (3.24), we get

$$\beta_0 \epsilon^{-\bar{m}} \leq E \beta_0^{\frac{2m}{m+2}} \left(1 - \frac{1}{m}\right) = E \beta_0^{\frac{2(m-1)}{m+2}} \leq E \beta_0,$$

for some constant E and $\epsilon > 0$ small. This leads to a contradiction, thereby concluding the proof that $I_\omega = I_0$. In summary, ℓ_0 is the unique loop ℓ of ω . It satisfies (iv) as a consequence of (3.24), except for the last property ($\beta = \max_{t \in J_\ell} |P(\omega(t))|$) (v) holds with β_0 instead of β in the lower bound for $L(\ell)$, and (vi) is satisfied by (3.25) and the relation $\beta \geq \beta_0$.

To prove (vii), we consider a time $t_* \in (\tau_0, \tau_1) \setminus J_0$ where P attains a local maximum (recall that $I_\omega = I_0 = [\tau_0, \tau_1]$). By Lemma 3.14, the result follows if we show that $\sin(\theta(t_*)) > 0$. Suppose, by contradiction, that $\sin(\theta(t_*)) \leq 0$. Then necessarily, by (3.28), $\cos(\theta(t_*)) < 0$ and $\sin(\theta(t_*)) \leq 0$, with $\sin(\theta(t_*)) = 0$ only if $\omega_2(t_*) = 0$. Let H_0 denote the half-line $A_0 + [0, +\infty)(0, -1)$ and let H denote the half-line $\omega(t_*) + [0, +\infty)\dot{\omega}(t_*)$. Now, consider the planar curve Υ defined by

$$\Upsilon := H_0 \cup \text{spt}(\omega|_{[0, t_*]}) \cup H.$$

Let U denote the connected component of $\mathbb{R}^2 \setminus \Upsilon$ that contains $\omega(t + \sigma)$ for small $\sigma > 0$. Then, $A_\epsilon = \omega(\tau_1) = L(\omega) \notin \bar{U}$. Therefore, the minimum $s_* \in (t_*, \tau_1]$ of $t \in (t_*, \tau_1]$ such that $\omega(t) \in \partial U$ is well-defined. We claim that $\omega|_{[t_*, s_*]}$ has a loop contained in U . Indeed, by Lemma 3.5 (i), $\omega(s_*) \notin H_0$. Since $t_* \notin J_0$, we cannot have $\omega(t) \in \text{spt}(\omega|_{[0, t_*]})$. Therefore, $\omega(s_*) \in H$, and by Lemma 3.4, $\omega|_{[t_*, s_*]}$ contains a loop that coincides with ℓ_0 , the unique loop of ω . Furthermore, $\text{spt}(\ell_0) \subset U$. We now claim that

$$P(\omega(t_*)) \geq \max\{P(x) \mid x \in \bar{U}, x_2 \geq 0\}.$$

Since $\text{spt}(\ell_0) \subset U$, we have $\omega(s_0^-) \in U$, so the maximum of x_2 on \bar{U} is greater than $c\epsilon$ (by (3.24)). Moreover, since $\sin \theta(t_*) \leq 0$, the maximum of x_2 on \bar{U} is attained on $\text{spt}(\omega|_{[0, t_*]})$. Therefore, by Lemma 3.6 we infer that $\omega_2(t_*) > 0$, for some $C > 0$. In particular, from the formula for $\dot{P}(t_*) = 0$ given in (3.28), we conclude that $\cos \theta(t_*) < 0$ and that U is bounded and convex. Since the set of $x \in \mathbb{R}^2$ such that $P(x) > P(\omega(t_*))$, $x_2 \geq 0$, and $x_1 > 0$ is convex and externally tangent to U at $\omega(t_*)$, our claim is proved. If $\omega_1(s) \leq \omega_1(t)$ for all $s \in J_0$, then the claim implies that $Q(\omega(s)) = 4\omega_1(s)P(\omega(s)) \leq Q(\omega(t_*))$ for all $s \in J_0$, which leads to a contradiction by obstruction (v) of Lemma 3.7. Therefore, there exists $\bar{s} \in J_0$ such that $\omega(\bar{s}) \in U \cap \{x_1 > \omega_1(t)\}$. Let $\mu > 0$ be such that $\omega(\bar{s}) - (0, \mu) \in \partial U$. Since $\cos(\theta(t_*)) < 0$, we have $x_1 < \omega_1(t_*)$ on $H \setminus \{\omega(t_*)\}$, which implies that $\omega(\bar{s}) - (0, \mu) \in \text{spt}(\omega|_{[0, t_*]})$, that is, $\text{spt}(\omega|_{[0, t_*]}) + (0, \mu) \in \ell_0$. We conclude by obstruction (iii) of Lemma 3.7.

To complete the proof of (v), it remains to show that the lower bound for $L(\ell)$ holds with β in place of β_0 and that the last property is satisfied. If the maximum of P is attained outside $J_\ell = J_0$, then by (vii), $|\lambda|\beta^{1+1/\bar{m}}$ is bounded, which contradicts (3.25) and (vi) for sufficiently small $\epsilon > 0$.

To prove (viii), we apply the Gauss-Bonnet formula to the concatenated curve η , which consists of $\omega|_{[0, s_0^-]}$, $\omega|_{[s_0^+, L(\omega)]}$, the line segment $[A_\epsilon, (0, \epsilon)]$, and the line segment $[(0, \epsilon), A_0]$. The integral of the signed curvature of η is bounded by $3\pi + \pi/2 \leq 4\pi$. Moreover, the total curvature of the loop is bounded by 2π . We can thus conclude easily.

A Proof of Proposition 2.1

Let $\epsilon, \rho > 0$ be fixed. It follows from classical results of calculus of variations with constraints that the curve minimizing the length among all Lipschitz curves $\zeta : [0, 1] \rightarrow \mathbb{R}^2$ satisfying (2.4)

is the concatenated curve ν_ϵ^ρ (reparametrized on $[0, 1]$) defined as follows (see Figure 2):

$$\nu_\epsilon^\rho := T_0 * \Gamma_\rho([t_0, t_1]) * T_1, \quad \text{with } T_0 := [A_0, \Gamma_\rho(t_0)] \quad \text{and} \quad T_1 := [\Gamma_\rho(t_1), A_\epsilon]. \quad (\text{A.1})$$

Here, $t_0 = t_0(\rho)$ is defined as the unique $t_0 \geq 0$ for which the line segment $(A_0, \Gamma_\rho(t_0))$ is tangent to $\Gamma_\rho([0, +\infty))$ at $\Gamma_\rho(t_0)$ and $t_1 = t_1(\rho, \epsilon) \geq 0$ is the unique $t_1 \geq 0$ such that the line segment $(A_\epsilon, \Gamma_\rho(t_1))$ is tangent to $\Gamma_\rho([0, +\infty))$ at $\Gamma_\rho(t_1)$. If the segment $[A_0, A_\epsilon]$ intersects $\Gamma_\rho([0, +\infty))$, then t_0 and t_1 are well-defined; otherwise we set $t_0 = t_1 := 0$.

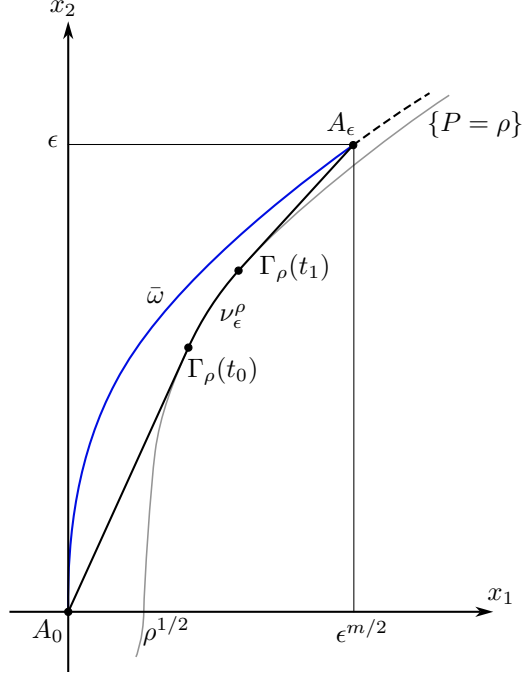


Figure 2: The curve ν_ϵ^ρ in black

Hence proving (2.5) is equivalent to prove

$$L(\nu_\epsilon^\rho) \geq L(\bar{\omega}_\epsilon) - C(K)\rho^{1-\frac{1}{m}}. \quad (\text{A.2})$$

Let us fix $K > 0$ and assume that $\rho < K\epsilon^{3\bar{m}-1}$ (recall that $\bar{m} = m/2$). The unique $t_0 \geq 0$ such that the line $(A_0, \Gamma_\rho(t_0))$ is tangent to $\Gamma_\rho([0, +\infty))$ at $\Gamma_\rho(t_0)$ must satisfy $f_\rho(t_0) = t_0 f'_\rho(t_0)$ and the unique $t_1 \geq 0$ such that the line $(A_\epsilon, \Gamma_\rho(t_1))$ is tangent to $\Gamma_\rho([0, +\infty))$ at $\Gamma_\rho(t_1)$ must satisfy $\epsilon^{\bar{m}} - f_\rho(t_1) = (\epsilon - t_1)f'_\rho(t_1)$ with $f'_\rho(t) = mt^{m-1}/(2f_\rho(t))$. From these equations, we deduce that

$$\rho = (\bar{m} - 1)t_0^m \quad \text{and} \quad (t_1^m + \rho)^{\frac{1}{2}} = \epsilon^{\bar{m}} - \sigma_1(\epsilon - t_1) \quad \text{with} \quad \sigma_1 = f'_\rho(t_1). \quad (\text{A.3})$$

The first equation implies that $t_0 = c\rho^{1/m}$ for some $c > 0$. Using the assumption on ρ , it follows that $t_0 = o(\epsilon)$ as $\epsilon \rightarrow 0$. From (A.1), we have

$$L(\nu_\epsilon^\rho) = L_1 + L_2 + L_3 \quad \text{with} \quad L_1 := L(T_0), \quad L_2 := L(\Gamma_\rho([t_0, t_1])), \quad L_3 := L(T_1). \quad (\text{A.4})$$

We now proceed to derive lower bounds for L_1, L_2 , and L_3 . For L_1 , we have

$$L_1 = |(f_\rho(t_0), t_0)| = \sqrt{t_0^2 + (t_0^m + \rho)} = \sqrt{t_0^2 + \bar{m}t_0^m} = t_0 + \frac{\bar{m}}{2}t_0^{m-1} + o(t_0^{m-1}) \quad (\text{A.5})$$

by applying the Taylor expansion of $\sqrt{1+u}$ near $u = 0$. For L_2 , we note that, on the one hand,

$$L_2 = \int_{t_0}^{t_1} \sqrt{1 + \bar{m}^2 t^{2m-2} (t^m + \rho)^{-1}} dt \geq \int_{t_0}^{t_1} \sqrt{1 + \bar{m}^2 t^{m-2}} dt = L(\bar{\omega}|_{[t_0, t_1]}).$$

On the other hand, we have

$$L(\bar{\omega}|_{[0, t_0]}) = t_0 + C t_0^{m-1} + o(t_0^{m-1}),$$

using (3.4), where $C = m^2/(8(m-1))$, and

$$L(\bar{\omega}|_{[t_1, \epsilon]}) = \int_{t_1}^{\epsilon} \sqrt{1 + \bar{m}^2 t^{m-2}} dt \leq (\epsilon - t_1) \sqrt{1 + \bar{m}^2 \epsilon^{m-2}}.$$

Combining these, we find that

$$L_2 \geq L(\bar{\omega}) - t_0 - C t_0^{m-1} - (\epsilon - t_1) \sqrt{1 + \bar{m}^2 \epsilon^{m-2}} + o(t_0^{m-1}). \quad (\text{A.6})$$

Finally, for L_3 , using (A.3), we have

$$L_3 = \sqrt{(\epsilon - t_1)^2 + \left(\epsilon^{\bar{m}} - (t_1^m + \rho)^{\frac{1}{2}}\right)^2} = (\epsilon - t_1) \sqrt{1 + \sigma_1^2}. \quad (\text{A.7})$$

By combining (A.4)-(A.7) and defining $C' := \bar{m}/2 - C$, we obtain

$$L(\nu_\epsilon^\rho) \geq L(\bar{\omega}) + C' t_0^{m-1} + (\epsilon - t_1) [\varphi(\sigma_1) - \varphi(\bar{m}\epsilon^{\bar{m}-1})] + o(t_0^{m-1}),$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is defined as $\varphi(\lambda) := \sqrt{1 + \lambda^2}$ for $\lambda \in \mathbb{R}$. Since by (A.3) $t_0^{m-1} = c\rho^{1-1/m}$ for some $c > 0$, proving (A.2) reduces to showing that the term $\Delta := (\epsilon - t_1)(\varphi(\sigma_1) - \varphi(\bar{m}\epsilon^{\bar{m}-1}))$ can be bounded from below by $-C\rho^{1-1/m}$ for some $C > 0$, provided that $\epsilon > 0$ is sufficiently small. Using the convexity of φ , we have

$$\varphi(\sigma_1) - \varphi(\bar{m}\epsilon^{\bar{m}-1}) \geq \varphi'(\bar{m}\epsilon^{\bar{m}-1}) (\sigma_1 - \bar{m}\epsilon^{\bar{m}-1}) \geq -\varphi'(\bar{m}\epsilon^{\bar{m}-1}) |\sigma_1 - \bar{m}\epsilon^{\bar{m}-1}|, \quad (\text{A.8})$$

where $\varphi'(\bar{m}\epsilon^{\bar{m}-1}) = \bar{m}\epsilon^{\bar{m}-1} + o(\epsilon^{\bar{m}-1})$, and, by (A.3), $\Delta_1 := \sigma_1 - \bar{m}\epsilon^{\bar{m}-1}$ can be expressed as

$$\Delta_1 = \bar{m} \left[\frac{t_1^{m-1}}{(t_1^m + \rho)^{\frac{1}{2}}} - \epsilon^{\bar{m}-1} \right] = \bar{m}\epsilon^{\bar{m}-1} \left[(1 - \xi)^{\bar{m}-1} \left(1 + \frac{\alpha}{(1 - \xi)^m} \right) - 1 \right], \quad (\text{A.9})$$

where we have introduced $\alpha = \rho/\epsilon^m$ and $\xi = 1 - t_1/\epsilon$. From (A.3), it follows that α, ξ satisfy

$$F(\alpha, \xi) = 0 \quad \text{with} \quad F(\alpha, \xi) := 1 - [(1 - \xi)^m + \alpha]^{\frac{1}{2}} - \bar{m}\xi(1 - \xi)^{m-1} [(1 - \xi)^m + \alpha]^{-\frac{1}{2}}.$$

We note that $F(0, 0) = 0$, $\frac{\partial F}{\partial \alpha}(0, 0) = -1/2$, and the only solution of $F(0, \xi) = 0$ with $\xi \in [0, 1]$ is $\xi = 0$. By the implicit function theorem, this ensures the existence of $\delta_\alpha, \delta_\xi > 0$ and a smooth function $\varphi : (-\delta_\xi, \delta_\xi) \rightarrow \mathbb{R}$ such that, for any solution (α, ξ) of $F(\alpha, \xi) = 0$ with $|\alpha| < \delta_\alpha$, we have $|\xi| < \delta_\xi$ and $\alpha = \varphi(\xi)$. Since $\rho < K e^{3\bar{m}-1}$ implies $\rho/\epsilon^m < \delta_\alpha$ for sufficiently small $\epsilon > 0$, we conclude that $\alpha = \rho/\epsilon^m$ and $\xi = 1 - t_1/\epsilon$ satisfy $\alpha = \varphi(\xi)$ for small $\epsilon > 0$, with ξ tending to 0 as $\epsilon \rightarrow 0$. Furthermore, since $\varphi'(0) = 0$ and $\varphi''(0) \neq 0$, there exists $c > 1$ such that $\xi^2/c \leq \alpha \leq c\xi^2$ for sufficiently small $\epsilon > 0$. Consequently, by (A.8)-(A.9), there is $c' > 0$ such that

$$\Delta \geq -\epsilon\xi (\bar{m}\epsilon^{\bar{m}-1} + o(\epsilon^{\bar{m}-1})) \bar{m}\epsilon^{\bar{m}-1} ((\bar{m} - 1)\xi + o(\xi)) \geq -c'\epsilon^{m-1}\xi^2,$$

for ϵ sufficiently small. Finally, we note that $\epsilon^{m-1}\xi^2 \leq c\epsilon^{m-1}\alpha = c\rho\epsilon^{-1} \leq c\rho^{1-\frac{2}{3m-2}} \leq c\rho^{1-\frac{1}{m}}$ (recall that $m \geq 5$), completing the proof of (A.2), and hence the one of (2.5).

It remains to prove (2.6). The first and second derivatives of $f_\rho(\tau) = (\tau^m + \rho)^{\frac{1}{2}}$, defined for $\tau \geq 0$, are given by

$$f'_\rho(\tau) = \bar{m}\tau^{\bar{m}-1} \left(1 + \frac{\rho}{\tau^m}\right)^{-\frac{1}{2}} \quad \text{and} \quad f''_\rho(\tau) = \bar{m}\tau^{\bar{m}-2} \left(1 + \frac{\rho}{\tau^m}\right)^{-\frac{1}{2}} \left[(m-1) - \bar{m} \left(1 + \frac{\rho}{\tau^m}\right)^{-1} \right].$$

These derivatives satisfy the bounds

$$0 \leq f'_\rho(\tau) \leq \bar{m}\tau^{\bar{m}-1} \quad \text{and} \quad 0 \leq f''_\rho(\tau) \leq \bar{m}(\bar{m}-1)\tau^{\bar{m}-2} \quad \forall \tau \geq 0. \quad (\text{A.10})$$

Since both f_ρ and $\varphi(\lambda) = \sqrt{1 + \lambda^2}$ are increasing on $[0, +\infty)$, the length of Γ_ρ restricted to $[t, s]$, with $s \geq t \geq 0$, satisfies

$$L(\Gamma_\rho|_{[t,s]}) = \int_t^s \sqrt{1 + f'_\rho(\tau)^2} d\tau \leq (s-t)\sqrt{1 + f'_\rho(s)^2}. \quad (\text{A.11})$$

Furthermore, by the convexity of φ and f_ρ , we have

$$\begin{aligned} L([\Gamma_\rho(t), \Gamma_\rho(s)]) &= (s-t)\varphi\left(\frac{f_\rho(s) - f_\rho(t)}{s-t}\right) \\ &\geq (s-t)\left[\varphi(f'_\rho(s)) + \varphi'(f'_\rho(s))\left(\frac{f_\rho(s) - f_\rho(t)}{s-t} - f'_\rho(s)\right)\right] \\ &= (s-t)\sqrt{1 + f'_\rho(s)^2} + \frac{(s-t)f'_\rho(s)}{\sqrt{1 + f'_\rho(s)^2}}(f'_\rho(t) - f'_\rho(s)), \end{aligned}$$

which, by the mean value theorem, is equal to

$$(s-t)\sqrt{1 + f'_\rho(s)^2} + \frac{(s-t)^2 f'_\rho(s)}{\sqrt{1 + f'_\rho(s)^2}} f''_\rho(u),$$

for some $u \in [t, s]$. Then, (2.6) follows by combining the inequality above with (A.10)-(A.11).

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